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FRACTIONAL LOGARITHMIC DOUBLE PHASE PROBLEMS: QUALITATIVE ANALYSIS IN THE ANISOTROPIC CASE*

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Abstract. This paper is concerned with the study of elliptic differential problems involving fractional variable exponent double phase operators with logarithmic perturbation $(-\Delta)^s_{\mathcal{H}}$ generated by $\mathcal{H}(x,y,t)=[\frac{t^{p(x,y)}}{p(x,y)}+\mu(x,y)\frac{t^{q(x,y)}}{q(x,y)}]\log(e+\alpha t)$. In the first part, we study fractional double phase elliptic inclusions with a generalized multivalued mapping and a maximal monotone operator which is formulated by the convex subdifferential of the indicator function to a convex set. Based on the sub-supersolution method along with truncation techniques and nonsmooth analysis we show an existence result and give an application construction such a pair of sub-supersolution. Additionally, under lattice conditions, we establish the compactness and the directedness of the solution set within a pair of sub- and supersolutions. In the second part, we consider a type of fractional Kirchhoff double phase problems governed by the operator $(-\Delta)^s_{\mathcal{H}}$. Applying variational methods, the Poincaré–Miranda existence theorem together with the quantitative deformation lemma, we prove a multiplicity result which says that the problem has at least a positive solution, a negative solution, and a sign-changing solution.

Key words. fractional logarithmic double phase operator, multivalued problem, sub-supersolution method, nonsmooth analysis, Kirchhoff-type problem, variational methods

MSC codes. 35R11, 35J15, 35R70, 49J52, 58E50, 74G40, 76A05

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1. Introduction.

In this paper, we study different problems involving the variable exponent fractional double phase operator with logarithmic perturbation given by

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$$
\begin{aligned}
(-\Delta)^s_{\mathcal{H}} u(x) &:= C_{N,s,p,q} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \mathcal{H}'\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{\mathrm{d}y}{|x-y|^{N+s}}\\
&= C_{N,s,p,q}\,\mathrm{PV}\int_{\mathbb{R}^N} \mathcal{H}'\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{\mathrm{d}y}{|x-y|^{N+s}},
\end{aligned}
\tag{1.1}
$$

where $0<s<1$, $0<C_{N,s,p,q}\in\mathbb{R}$ depending on N,s,p,q, PV represents the Cauchy principal value, and $B_\varepsilon(x):=\{z\in\mathbb{R}^N: |z-x|<\varepsilon\}$. Here, the function $\mathcal{H}\colon \mathbb{R}^N\times\mathbb{R}^N\times[0,\infty)\to[0,\infty)$ is defined by

$$
\mathcal{H}(x,y,t)=\left[\frac{t^{p(x,y)}}{p(x,y)}+\mu(x,y)\frac{t^{q(x,y)}}{q(x,y)}\right]\log(e+\alpha t)
\tag{1.2}
$$

for all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ and for all $t\geq 0$, and $\mathcal{H}'(x,y,\cdot)$ denotes the right derivative of $\mathcal{H}(x,y,\cdot)$ at t. And $\alpha\geq 0$, $p,q\in C(\mathbb{R}^N\times\mathbb{R}^N)$ are symmetric functions (that is, $p(x,y)=p(y,x)$, $q(x,y)=q(y,x)$),

$$
1<p(x,y)<\frac{N}{s}\quad\text{and}\quad p(x,y)\leq q(x,y)\quad\text{for all } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N,
$$

$0\leq\mu(\cdot,\cdot)\in L^1(\mathbb{R}^N\times\mathbb{R}^N)$ fulfills $\mu(x,y)=\mu(y,x)$, and

$$
\Omega_1:=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N: p(x,y)<q(x,y)\}\nsubseteq\Omega_0:=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N: \mu(x,y)=0\}.
$$

We note that the symmetry of p, q, and μ ensures the well-definedness of the norm. Specifically, the definition of the Gagliardo seminorm (see (2.2), (2.4), and (1.2)) involves double integrals. The symmetry guarantees that the integrand remains invariant under the exchange of variables ($x\leftrightarrow y$), which ensures that the integral is uniquely defined and thus induces a valid norm. We emphasize that throughout this work, the fractional order $s\in(0,1)$ is fixed. The term "variable exponent" refers to the spatial dependence of the exponents $p(x,y)$ and $q(x,y)$ governing the growth of the nonlinearity, and not to a variable order of differentiation $s(x)$. We concentrate on the qualitative analysis of problems driven by the operator (1.1), while the term "qualitative" refers to the study of the existence, multiplicity, and general properties of related solution sets.

Nowadays, there are numerous works on the so-called double phase problems, which are widely used in applications such as population dynamics, non-Newtonian fluids, material science, and quantum mechanics. Such problems appeared for the first time in Zhikov [82] in the study of elasticity and is represented by the double phase energy functional given by

$$
\phi\mapsto\int_\Omega\left(|\nabla\phi|^p+\mu(x)|\nabla\phi|^q\right)\mathrm{d}x.
\tag{1.3}
$$

Such a type of functionals was used for describing mathematical models of strongly anisotropic materials and it also plays a role in the study of the Lavrentiev phenomenon; see the works by Zhikov [82, 83]. To be more precise, the energy functional (1.3) is able to describe the geometric properties for the mixture of two materials with power hardening exponents p and q, which exhibits ellipticity in the gradient of order q in the domain $\Omega_{>0}:=\{x\in\Omega\colon \mu(x)>0\}$ and of order p in the domain $\Omega_{=0}:=\{x\in\Omega\colon \mu(x)=0\}$.

After the outstanding works of Zhikov, many impressive works have been carried out on double phase problems. The related double phase operator to the functional (1.3) is given by the form

$$\Delta_{\mathcal{Z}}\phi = \operatorname{div}\left(\frac{\mathcal{Z}'(x,|\nabla\phi|)}{|\nabla\phi|}\nabla\phi\right), \tag{1.4}$$

where $\mathcal{Z}$ is a certain N-function to be specified later in our case. The operator (1.4) has been discussed in the paper by Crespo-Blanco et al. [23] with the choice

$$\mathcal{Z}(x,\phi) = \frac{\phi^{p(x)}}{p(x)} + \mu(x)\frac{\phi^{q(x)}}{q(x)}, \tag{1.5}$$

concerning the properties of the related Musielak–Orlicz Sobolev spaces and the related logarithmic operator. Vetro and Zeng [70] considered the double phase operator with log L-perturbation generated by the N-function

$$\mathcal{Z}(\phi) = \left[\phi^p + \mu(x)\phi^q\right]\log(e+\phi), \tag{1.6}$$

while the case with variable exponents has been studied by Lu, Vetro, and Zeng [49], where

$$\mathcal{Z}(x,\phi) = \left[\phi^{p(x)} + \mu(x)\phi^{q(x)}\right]\log(e+\phi). \tag{1.7}$$

A different logarithmic double phase operator than the ones in (1.6) and (1.7) has been recently introduced by Arora, Crespo-Blanco, and Winkert [3, 4], where the N-function has the form

$$\mathcal{Z}(x,\phi) = \phi^{p(x)} + \mu(x)\phi^{q(x)}\log(e+\phi).$$

Also, the study of fractional double phase operators can be found in the work by de Albuquerque et al. [26], who considered a class of fractional operators involving the fractional variable exponent double phase operator with logarithmic perturbation as given in (1.1). We point out that fractional order problems have a compelling theoretical framework and several practical applications that can be widely used in fluid mechanics, conformal geometry, probability, molecular dynamics, obstacle problems, optimization, and image processing; see, for example, the works by Bahrouni, Rădulescu, and Repovš [8], Benci et al. [11], Bertoin [12], Cabré and Tan [15], Chen, Li, and Ma [22], and Charkaoui and Ben-loghfyry [21]. For more works concerning double phase or fractional double phase elliptic or parabolic problems, we refer the reader to Ambrosio [2], Bhakta and Mukherjee [13], Guarnotta, Livrea, and Winkert [36], Liu and Dai [47], Zeng et al. [74], Zeng, Rădulescu, and Winkert [77], and Zhang, Zhang, and Rădulescu [79]. Moreover, there are many papers dealing with the regularity of local minimizers for double phase problems; see, for instance, Beck and Mingione [10], Byun, Ok, and Song [14], Fuchs and Mingione [34], Marcellini [50, 51], and Prasad and Tewary [61]; see also the references therein. We also highlight notable works on fractional diffusion equations, particularly those by Zheng and Wang [81] concerning well-posedness and smoothing properties, Zheng and Wang [80] establishing optimal-order error estimates via fully discretized finite element approximations, and Qiu and Zheng [64] developing numerical analysis for high-order methods.

Double phase problems arise in various real-world applications across multiple disciplines. In 2000, in order to model the reaction-diffusion systems, Benci et al. [11] studied the equation

$$-\Delta_p u - \Delta_q u + q(x)|u|^{p-2}u + w(x)|u|^{q-2}u = \lambda f(x)|u|^{\gamma-2}u.$$

In this model, the double-diffusion term $-\Delta_p u - \Delta_q u$ describes composite diffusion processes occurring in biological tissues or chemical reactors. The reaction term

$w(x)|u|^{q-2}u$ represents source or absorption effects relevant to chemical kinetics, while the right-hand side $\lambda f(x)|u|^{\gamma-2}u$ accounts for external forcing or self-interaction phenomena observed in nonlinear optics or elementary particle models. In 2019, Bahrouni, Rădulescu, and Repovš [8] considered double phase models in transonic flow and the related energy functional is given as

$$E(u) = \int_\Omega \frac{G(x,y)\,|\nabla_x u|^{G(x,y)} + |x|^\gamma\,|\nabla_y u|^{G(x,y)}}{G(x,y)}\,\mathrm{d}z.$$

Here, the term $|\nabla_x u|^{G(x,y)}$ models nonlinear diffusion in the x-direction with a spatially varying exponent $G(x,y)$, simulating transport through heterogeneous media. The second term $|\nabla_y u|^{G(x,y)}$ introduces a degenerate weight in the y-direction, capturing anisotropic behavior near the degeneracy set. This functional effectively models composite materials consisting of two constituents and finds applications in analyzing shock wave formation and propagation in transonic flows. The logarithmic perturbation in function (1.2) has significant applications, particularly in the theory of plasticity with logarithmic hardening. Moreover, mathematical models with logarithmic perturbations are widely employed in ecological modeling, population dynamics, and quantum mechanics. For instance, Engen and Lande [30] developed a stochastic species abundance model that generates the lognormal distribution commonly observed in community ecology. Their model is defined by the diffusion process

$$m(x) = \left[r + \frac{1}{2}\left(\frac{x}{\sigma_d^2}\right) + \frac{1}{2}\sigma_e^2\right] x - xg(x),$$

where x represents species abundance, r denotes the intrinsic growth rate, σ_e^2 is the environmental variance, σ_d^2 is the demographic variance, and $g(x) = \ln(x+\varepsilon)$ represents the density regulation function of Gompertz type, with $\varepsilon = \frac{\sigma_e^2}{\sigma_d^2}$. In quantum gravity theory, Zloshchastiev [84] proposed a quantum wave equation with logarithmic nonlinearity

$$\left[\hat{H} - \beta^{-1}\ln\left(\Omega|\Psi|^2\right)\right]\Psi = 0.$$

In recent years, nonlocal models have demonstrated powerful capabilities in mathematical biology, particularly in describing biological processes with memory effects, anomalous diffusion, and long-range interactions. Pal and Melnik [56] discussed the following time-fractional reaction–diffusion equation

$$D^\alpha u = d\Delta u + au^2\,(1 - b\phi * u) - cu,$$

which describes the evolution of tumor cell density. Here, $D^\alpha u$ denotes the Caputo fractional derivative capturing memory effects, $\phi * u$ is a convolution term representing spatial nonlocal interactions, and the nonlinear term $u^2(1-b\phi*u)$ models the coupling between cell proliferation and resource competition. Wang and Yang [71] investigated the following variable-coefficient conservative fractional elliptic equation:

$$-D\left(K(x)\cdot {}_0D_x^{-\beta}Du\right) = f(x), \quad x\in(0,1), \quad u(0)=u_l, \quad u(1)=u_r,$$

where $K(x)$ is the diffusivity coefficient and ${}_0D_x^{-\beta}Du$ is the left-sided Riemann–Liouville fractional integral. This model describes anomalous diffusion with spatial heterogeneity, such as transport in porous media with variable permeability.

We now discuss potential applications of problems involving the operator (1.1).

- Del-Castillo-Negrete, Carreras, and Lynch [27] employed the following fractional diffusion equation:

$$\partial_t^{\beta} P(x,t) = \chi \partial_{|x|}^{\alpha} P(x,t)$$

  in order to model nondiffusive transport of tracer particles in plasmas. Here, $\alpha \in (0,2)$ controls spatial nonlocality (e.g., jump length distributions in Lévy flights), $\beta \in (0,1)$ characterizes temporal memory effects (e.g., particle trapping), and χ is the diffusion coefficient representing turbulence intensity. The fractional double phase problem with variable exponents and logarithmic perturbation (see (1.1) and (1.2)) offers potential advancements in this field. The double phase structure enables simultaneous capture of fast and slow transport modes, which is valuable for optimizing multiscale transport in fusion devices. The logarithmic perturbation can simulate nonlinear energy dissipation processes (e.g., turbulent cascades), enhancing the description of high-energy particle behavior. Moreover, the variable exponents $p(x,y)$ and $q(x,y)$ can dynamically adapt to plasma inhomogeneities.
- Metzler and Klafter [52] derived a fractional Fokker–Planck equation from continuous time random walks (CTRWs) of the form

$$\partial_t^{\beta} W(x,t) = -\partial_x [v(x) W(x,t)] + D \partial_{|x|}^{\alpha} W(x,t).$$

  This model simulates anomalous diffusion in complex media (e.g., polymers or biological tissues), where v is the drift velocity, D is the diffusion coefficient, $\alpha < 2$ reflects long-range jumps, and $\beta < 1$ reflects non-Markovian behavior due to particle trapping. The fractional double phase problem driven by (1.1) has promising applications in this context. The double phase term can distinguish between different relaxation modes (e.g., segmental motion versus whole-chain dynamics in polymer systems), while the logarithmic perturbation improves the fitting of nonexponential relaxation data, such as time-dependent behaviors in protein folding or colloidal systems.

On the one hand, by employing the sub-supersolution method combining the theory of nonsmooth analysis as well as truncation techniques, we will show existence results of weak solutions for elliptic inclusion problems concerning the fractional double phase operator with variable exponents and logarithmic perturbation defined by (1.1). More precisely, we are going to find $u \in K$ satisfying

$$0 \in (-\Delta)^s_{\mathcal{H}} \omega + \partial I_K(\omega) + \mathcal{F}(\omega) \text{ in } W_0^{s,\mathcal{H}}(\Omega)^*, \tag{1.8}$$

where K is a closed convex subset of $W_0^{s,\mathcal{H}}(\Omega)$ (see section 2), I_K is the indicator function of K, and ∂I_K is the subdifferential of I_K while $\mathcal{F}$ is a lower order multivalued operator. Note that the elliptic inclusion problem (1.8) possesses a lower multivalued operator $\mathcal{F}$ generated by a multivalued function satisfying some proper assumptions given in section 3. As we know, problems involving multivalued terms have wide application in practical problems such as frictional contact problems with multivalued constitutive laws; see Panagiotopoulos [57, 58] as well as Carl and Le [16] for more information. Another characteristic of (1.8) is the appearance of constraint set K, which has the form

$$K = \left\{ \omega \in W_0^{s,\mathcal{H}}(\Omega) \colon \omega(x) \geq \pi(x) \text{ a.e. in } \Omega \right\},$$

with $\pi\colon \Omega \to \mathbb{R}$ being an obstacle function. Generally, problems involving constraint sets as K are called obstacle problems. The study of obstacle problems goes back to the research of Stefan [67], who studied the temperature distribution in a homogeneous medium going through a phase change, typically a block of ice with the temperature of zero submerged in water. The research of obstacle problems attracted much attention since the famous work by Lions [46]. The study of obstacle problems can be broadly utilized in the research of physics, biology, and financial mathematics; see Duvaut and Lions [29], Rodrigues [65], Zeng et al. [74], and Zeng et al. [75]; see also the references therein.

Our proof of the existence of a solution for problem (1.8) is based on the sub-supersolution method inspired by the work of Carl, Le, and Winkert [17], who considered multivalued variational inequalities for variable exponent double phase problems: Find $\omega \in K$ satisfying

$$0 \in A\omega + \partial I_K(\omega) + \mathcal{F}(\omega) + \mathcal{F}_\Gamma(\omega),$$

where A is a variable exponent double phase operator formulated by (1.4) with $\mathcal{Z}$ given in (1.5). Also, Liu, Lu, and Vetro [48] studied the following double phase elliptic inclusion: Find $\omega \in K$ such that

$$0 \in A\omega + \partial I_K(\omega) + \mathcal{F}(\omega) + \mathcal{F}_\Gamma(\omega),$$

where A is a double phase operator with logarithmic perturbation defined by (1.4) with $\mathcal{Z}$ given in (1.7). We point out that the proof for the existence of solutions to problem (1.8) with the sub-supersolution method is new, even for $\alpha = 0$, that is, without the logarithmic perturbation.

On the other hand, with the use of variational methods, the Poincaré–Miranda existence theorem, and the quantitative deformation lemma, we will show the existence and multiplicity of weak solutions for the following fractional variable exponent perturbed double phase problem of Kirchhoff type:

$$\begin{cases} \psi(\tilde{I}_{s,\mathcal{H}})\,(-\Delta)^s_{\mathcal{H}}\, u = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.9}$$

for $u \in \widetilde{W}^{s,\mathcal{H}}_0(\Omega)$ (see section 2), where $\psi(t) = \theta_1 + \theta_2 t^{\varsigma-1}$ for $t \in \mathbb{R}$, with $\theta_1 \geq 0$, $\theta_2 > 0$, $\varsigma \geq 1$,

$$\tilde{I}_{s,\mathcal{H}}(u) := \int_Q \mathcal{H}(x,y,|D_s u(x,y)|)\,\mathrm{d}\nu,$$

$f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $Q := \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega)$, with $C\Omega = \mathbb{R}^N \setminus \Omega$,

$$\mathrm{d}\nu : \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}, \quad \text{and} \quad D_s u(x,y) := \frac{u(x)-u(y)}{|x-y|^s}.$$

Problem (1.9) is a kind of Kirchhoff problem which is developed from the model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.10}$$

with ρ, ρ_0, h, E, and L being constants. Kirchhoff [40] first proposed (1.10), which generalized the classical D'Alembert's wave equation by describing the effects of the changes of the length for the strings during the vibrations. The term $\frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,\mathrm{d}x$ in (1.10) represents the average additional tension across the entire string due to the vibration, where the integral $\int_0^L \frac{1}{2}(\frac{\partial u}{\partial x})^2\,\mathrm{d}x$ calculates the total elongation of the string beyond its rest length L. Studying for a Kirchhoff problem became an attractive topic after the work by Lions [45], who constructed an abstract framework for problems of Kirchhoff type. We point out that if $\theta_1 = 0$, then problem (1.9) is a degenerate Kirchhoff-type problem, and if $\theta_1 > 0$, (1.9) is a nondegenerate Kirchhoff-type problem. Note that the degenerate case is widely applied; for example, it can be used to describe the transverse oscillations of a stretched string. Moreover, nonlocal Kirchhoff parabolic problems can be utilized to model kinds of biological systems, for example the population density considered by Ghergu and Rădulescu [35]. More results concerning the basic theories and practical applications to Kirchhoff-type problems can be found in the works by Arosio and Panizzi [6], Carrier [18, 19], D'Ancona and Spagnolo [25], and Tang and Chen [68].

For more information with respect to double phase Kirchhoff problems, we mention that Fiscella and Pinamonti [32] researched the following double phase problem of Kirchhoff type:

$$\begin{cases} -M\left[\int_\Omega\left(\frac{|\nabla\omega|^p}{p}+\mu(x)\frac{|\nabla\omega|^q}{q}\right)\mathrm{d}x\right]\Delta_{\mathcal{Z}}\omega = f(x,\omega) & \text{in } \Omega,\\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{\mathcal{Z}}$ is given by (1.4) with $\mathcal{Z}$ defined in (1.5) and $M\colon [0,\infty)\to[0,\infty)$ is a continuous function satisfying proper conditions. The authors prove the existence of a nontrivial weak solution by using the mountain pass structure of the problem. Furthermore, based on variational tools, the Poincaré–Miranda existence theorem, and the quantitative deformation lemma, Crespo-Blanco, Gasiński, and Winkert [24] recently obtained the existence two constant sign solutions as well as a sign-changing solution of the degenerate Kirchhoff double phase problem

$$\begin{cases} -\psi\left[\int_\Omega\left(\frac{|\nabla\omega|^p}{p}+\mu(x)\frac{|\nabla\omega|^q}{q}\right)\mathrm{d}x\right]\Delta_{\mathcal{Z}}\omega = f(x,\omega) & \text{in } \Omega,\\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{\mathcal{Z}}$ is the operator given by (1.4) with $\mathcal{Z}$ defined in (1.5). Moreover, we also refer the reader to the contributions by Arora et al. [5], Cen, Vetro, and Zeng [20], Fiscella and Pinamonti [32], Gupta and Dwivedi [37], and Ho and Winkert [39] concerning details and methods for double phase Kirchhoff problems which we used as inspiration for our work. To the best of our knowledge, the existence results of constant sign and sign-changing weak solutions of problem (1.5) have not been established yet for such a general operator. We also mention some famous studies involving fractional Kirchhoff-type problems carried out recently. For instance, the existence results related to fractional problems of Kirchhoff type can be found in Fiscella, Pucci, and Zhang [33], Molica Bisci and Rădulescu [54], Pucci, Xiang, and Zhang [62, 63] and Xiang, Rădulescu, and Zhang [73].

Note that the operator (1.1) which appears in the problems (1.8) and (1.9) contains several interesting special cases, which we list below:

(i) if $\alpha = 0$, $\mu = 0$ in $\mathcal{H}$ (i.e. $\mathcal{H}(x,y,\phi) = \phi^{p(x,y)}$), then the operator (1.1) becomes the classical fractional $p(\cdot)$-Laplacian;

(ii) if $\alpha = 0$ and $1 < p(\cdot) \equiv p$, $1 < q(\cdot) \equiv q$ (i.e., $\mathcal{H}(x,y,\phi) = \phi^p + \mu(x,y)\phi^q$), then the operator (1.1) becomes the fractional constant exponent double phase operator;

(iii) if $\alpha = 0$ (i.e., $\mathcal{H}(x,y,\phi) = \phi^{p(x,y)} + \mu(x,y)\phi^{q(x,y)}$), then the operator (1.1) becomes the fractional variable exponent double phase operator without logarithmic perturbation;

(iv) if $1 < p(\cdot) \equiv p$ and $1 < q(\cdot) \equiv q$ (i.e., $\mathcal{H}(x,y,\phi) = [\phi^p + \mu(x,y)\phi^q]\log(e + \alpha\phi)$), then the operator (1.1) becomes the perturbed fractional double phase operator with constant exponents.

This paper is organized as follows. In section 2, we recall some basic properties of the fractional double phase operator (1.1) and the associated fractional Musielak–Orlicz Sobolev spaces. In subsection 3.1, we concentrate on establishing the existence results of weak solutions to problem (1.8), whereby the proof is mainly based on the sub-supersolution method. Also, an application will be given in subsection 3.2. Moreover, section 4 deals with the proof of the existence of weak solutions to problem (1.9) by employing variational methods, among others. To be more precise, we show the existence of two constant solutions of (1.9) in subsection 4.1 and the existence of a least energy sign-changing solution of (1.9) in subsection 4.2.

2. Preliminaries. In this section, we introduce the fractional Musielak–Sobolev spaces with respect to the function $\mathcal{H}$ defined by (1.2) and recall preliminary results that are essential for the proofs of our existence theorems given in sections 3 and 4. Throughout the paper, we denote by C a positive constant that will change from line to line and by C_r a constant depending on the parameter r.

First, we give the basic assumptions on the data:

(H1) $p,q \in C(\mathbb{R}^N \times \mathbb{R}^N)$ such that for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$

$$1 < \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) \le \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) < \frac{N}{s} \quad \text{and} \quad p(x,y) \le q(x,y)$$

with

$$\Omega_1 := \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : p(x,y) < q(x,y)\}$$
$$\subsetneq \Omega_0 := \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \mu(x,y) = 0\}$$

and $p(x,y) = p(y,x)$, $q(x,y) = q(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $0 \le \mu(\cdot,\cdot) \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ and $\mu(x,y) = \mu(y,x)$ for a.a. $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

We introduce the notations

$$p_s^*(x,y) = \frac{Np(x,y)}{N - sp(x,y)}, \quad p_- := \inf_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y), \quad \text{and} \quad p_+ := \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y).$$

Similarly, we can define q_-, q_+ as above.

Throughout this work, we denote by $M(\Omega)$ the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. Under the hypotheses of (H1), the function h fulfills (φ_1)–(φ_3) (see Appendix A) with $\ell = p_-$ and $m = q_+ + 1$, while $\mathcal{H}$ given in (1.2) is a locally integrable N-function (see Appendix A). Moreover, we introduce the function $\widehat{\mathcal{H}}\colon \Omega \times [0,\infty) \to [0,\infty)$ given by

$$\widehat{\mathcal{H}}(x,t) := \int_0^t \hat{h}(x,\tau)\,\mathrm{d}\tau,$$

where $\hat{h}(x,t) := h(x,x,t)$ for a.a. $(x,t) \in \Omega \times [0,\infty)$. According to the definitions concerning the Musielak–Orlicz spaces and the fractional Musielak–Sobolev spaces introduced in Appendix A, we can give the definition of the modular function related to $\widehat{\mathcal{H}}$ by

$$\rho_{\widehat{\mathcal{H}}}(u)=\int_\Omega \widehat{\mathcal{H}}(x,|u|)\,\mathrm{d}x,$$

whereby the corresponding Musielak–Orlicz space is given as

$$L^{\widehat{\mathcal{H}}}(\Omega)=\{u\in M(\Omega)\colon \rho_{\widehat{\mathcal{H}}}(\lambda u)<+\infty \text{ for some } \lambda>0\},$$

equipped with the Luxemburg norm

$$\|u\|_{\widehat{\mathcal{H}}}=\inf\left\{\lambda>0\colon \rho_{\widehat{\mathcal{H}}}\left(\frac{u}{\lambda}\right)\le 1\right\}. \tag{2.1}$$

In addition, the fractional Musielak–Orlicz Sobolev space $W^{s,\mathcal{H}}(\Omega)$ is formulated as

$$W^{s,\mathcal{H}}(\Omega):=\left\{u\in L^{\widehat{\mathcal{H}}}(\Omega)\colon \rho_{s,\mathcal{H}}(\lambda u)<\infty \text{ for some } \lambda>0\right\},$$

where

$$\rho_{s,\mathcal{H}}(u):=\int_\Omega\int_\Omega \mathcal{H}(x,y,|D_s u(x,y)|)\,\mathrm{d}\nu \quad \text{for } s\in(0,1). \tag{2.2}$$

Note that $\mathrm{d}\nu$ is a regular Borel measure on $\Omega\times\Omega$. We point out that $W^{s,\mathcal{H}}(\Omega)$ is endowed with the norm

$$\|u\|_{s,\mathcal{H}}:=\|u\|_{\widehat{\mathcal{H}}}+[u]_{s,\mathcal{H}}, \tag{2.3}$$

with $[\,\cdot\,]_{s,\mathcal{H}}$ being the $(s,\mathcal{H})$-Gagliardo seminorm defined by

$$[u]_{s,\mathcal{H}}:=\inf\left\{\lambda>0\colon \rho_{s,\mathcal{H}}\left(\frac{u}{\lambda}\right)\le 1\right\}. \tag{2.4}$$

It is well known that the Luxemburg norm (2.1) possesses positive definiteness and positive homogeneity and satisfies the triangle inequality. That is for all $u,v\in L^{\widehat{\mathcal{H}}}(\Omega)$, the following hold:

- positive definiteness: $\|u\|_{\widehat{\mathcal{H}}}\ge 0$, $\|u\|_{\widehat{\mathcal{H}}}=0\Leftrightarrow u=0$;
- positive homogeneity: $\|\lambda u\|_{\widehat{\mathcal{H}}}=\lambda\|u\|_{\widehat{\mathcal{H}}}$ for all $\lambda\in\mathbb{R}$ (or $\mathbb{C}$);
- triangle inequality: $\|u+v\|_{\widehat{\mathcal{H}}}\le\|u\|_{\widehat{\mathcal{H}}}+\|v\|_{\widehat{\mathcal{H}}}$.

In addition, the Gagliardo seminorm (2.4) fulfills the following conditions: for all $u,v\in W^{s,\mathcal{H}}(\Omega)$, the following hold:

- nonnegativity: $[u]_{s,\mathcal{H}}\ge 0$;
- positive homogeneity: $[\lambda u]=\lambda[u]$ for all $\lambda\in\mathbb{R}$ (or $\mathbb{C}$);
- triangle inequality: $[u+v]_{s,\mathcal{H}}\le[u]_{s,\mathcal{H}}+[v]_{s,\mathcal{H}}$.

Note that $[u]=0$ does not imply $u=0$ pointwise, but only that $u=c$ for some $c\in\mathbb{R}$. Hence, $[u]_{s,\mathcal{H}}$ is a seminorm. From the above conclusions, we see that the norm defined by (2.3) satisfies positive homogeneity and the triangle inequality. Moreover, $\|u\|_{s,\mathcal{H}}\ge 0$, and $\|u\|_{s,\mathcal{H}}=\|u\|_{\widehat{\mathcal{H}}}+[u]_{s,\mathcal{H}}=0$ if and only if $u=0$. Therefore, the norm $\|u\|_{s,\mathcal{H}}$ is well-defined. Furthermore, we introduce

$$W_0^{s,\mathcal{H}}(\Omega)=\left\{u\in W^{s,\mathcal{H}}(\mathbb{R}^N)\colon u=0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega\right\},$$

which is a closed subspace of $W^{s,\mathcal{H}}(\Omega)$. Since $\mathcal{H}$ satisfies (φ_1)–(φ_3), we infer from de Albuquerque et al. [26] that $L^{\widehat{\mathcal{H}}}(\Omega)$ and $W_0^{s,\mathcal{H}}(\Omega)$ are separable and reflexive Banach spaces.

In this paper, we denote by $X \hookrightarrow Y$ the continuous embedding from the space X into the space Y while the compact embedding is denoted by $X \hookrightarrow\hookrightarrow Y$. In Appendix A, we give the definition of a Young function. Referring to the work by Alberico et al. [1, Theorem 8.1], we get the following continuous embedding result for the space $W^{s,Y}(\Omega)$.

THEOREM 2.1. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with Lipschitz boundary and* $0 < s < 1$. *If* Y *is a Young function satisfying conditions* (A.2) *and* $Y_{\frac{N}{s}}$ *is given by* (A.3), *then*

$$W^{s,Y}(\Omega) \hookrightarrow L^{Y_{\frac{N}{s}}}(\Omega),$$

and the embedding is optimal.

It is easy to infer that $W_0^{s,Y}(\Omega) \hookrightarrow W^{s,Y}(\Omega) \hookrightarrow L^{Y_{\frac{N}{s}}}(\Omega)$ under the hypotheses of Theorem 2.1. Moreover, from Example 8.3 in Alberico et al. [1], we see that if we take

$$Y := t^{p_-}\log(e+\alpha t) + \mu(x)t^{q_-}\log(e+\alpha t),$$

then

$$Y_{\frac{N}{s}} \sim Y^* := t^{(p_-)_s^*}\log^{\frac{(p_-)_s^*}{N}}(e+\alpha t) + \mu(x)^\gamma t^{(q_-)_s^*}\log^{\frac{(q_-)_s^*}{N}}(e+\alpha t)$$

for $1 \le p_-, q_- < \frac{N}{s}$, for all $t \ge 0$, and $\gamma > 0$. Hence, if $1 < r(x) \le (p_-)_s^*$ for all $x \in \overline{\Omega}$, then

$$W_0^{s,\mathcal{H}}(\Omega) \hookrightarrow W_0^{s,Y} \hookrightarrow L^{Y_{\frac{N}{s}}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega).$$

By [1, Theorem 9.1], the following compact embedding holds.

PROPOSITION 2.2. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with Lipschitz boundary, and let* $s \in (0,1)$. *Assume that* Y *is a Young function satisfying conditions* (A.2) *and* $Y_{\frac{N}{s}}$ *is given by* (A.3). *If* G *is a Young function satisfying* $G \ll Y_{\frac{N}{s}}$, *then there holds that*

$$W^{s,Y}(\Omega) \hookrightarrow\hookrightarrow L^G(\Omega).$$

Furthermore, $W_0^{s,Y}(\Omega) \hookrightarrow W^{s,Y}(\Omega) \hookrightarrow\hookrightarrow L^G(\Omega)$.

So, if $1 < r(x) < (p_-)_s^*$ for all $x \in \overline{\Omega}$, then

$$W_0^{s,\mathcal{H}}(\Omega) \hookrightarrow W_0^{s,Y} \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega).$$

Let X be a given Banach space and X^* be the dual space of X. We introduce the following notation:

$$\mathcal{K}(X^*) = \{U \subset X^* : U \neq \emptyset, U \text{ is closed and convex}\}.$$

Next, we recall some results in the theory for operators of monotone type.

DEFINITION 2.3. *Let* X *be a reflexive Banach space and its dual space be denoted by* X^*; *we denote the duality pairing by* $\langle\cdot,\cdot\rangle$. *Then, for an operator* $A\colon X \to X^*$, *we say the following:*

(i) *A satisfies the* (S_+)*-property if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$ *imply* $u_n \to u$ *in* X*;*

(ii) *A is monotone (strictly monotone) if* $\langle Au - Av, u - v\rangle \ge 0$ (>0) *for all* $u, v \in X$ *such that* $u \ne v$*;*

(iii) *A is coercive if there exists a function* $g\colon [0,\infty) \to \mathbb{R}$ *with* $\lim_{t\to\infty} g(t) = \infty$ *such that*

$$\frac{\langle Au, u\rangle}{\|u\|_X} \ge g(\|u\|_X) \quad \text{for all } u \in X;$$

(iv) *A is pseudomonotone if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$ *imply* $\langle Au, u_n - u\rangle \le \liminf_{n\to+\infty}\langle Au_n, u_n - u\rangle$ *for all* $v \in X$.

By applying Lemma 3.10 and Lemma 3.11 of de Albuquerque et al. [26], we have some useful properties of the energy functional given by

$$I_{s,\mathcal{H}}(u) = \rho_{s,\mathcal{H}}(u) := \int_\Omega\int_\Omega \mathcal{H}(x,y,|D_s u(x,y)|)\,\mathrm{d}\nu$$

as well as its Gâteaux derivative $\mathcal{J}$.

PROPOSITION 2.4. *Let hypotheses* (H1) *be satisfied. Then* $I_{s,\mathcal{H}} \in C^1(W_0^{s,\mathcal{H}}(\Omega),\mathbb{R})$ *and the Gâteaux derivative* $\mathcal{J}$ *of* $I_{s,\mathcal{H}}$ *is formulated as*

$$\langle \mathcal{J}(u), v\rangle = \int_\Omega\int_\Omega \mathcal{H}'(x,y,|D_s u(x,y)|) D_s v(x,y)\,\mathrm{d}\nu$$

for all $u, v \in W_0^{s,\mathcal{H}}(\Omega)$*. Moreover,* $\mathcal{J}$ *is bounded, coercive, and monotone (hence pseudomonotone) and satisfies the* (S_+)*-property.*

In order to deal with the Kirchhoff problem in section 4, we consider new fractional Musielak–Orlicz spaces $\widetilde{W}^{s,\mathcal{H}}(\Omega)$ defined by

$$\widetilde{W}^{s,\mathcal{H}}(\Omega) := \left\{ u \in L^{\widehat{\mathcal{H}}}(\Omega)\colon \tilde{\rho}_{s,\mathcal{H}}(\lambda u) < \infty \text{ for some } \lambda > 0 \right\},$$

with

$$\tilde{\rho}_{s,\mathcal{H}}(u) := \int_Q \mathcal{H}(x,y,|D_s u(x,y)|)\,\mathrm{d}\nu$$

for $s \in (0,1)$ and $(x,y) \in Q$ with $Q := \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega)$, where $C\Omega = \mathbb{R}^N \setminus \Omega$. Moreover, we can define $\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ and the corresponding Gagliardo seminorm of $u \in \widetilde{W}^{s,\mathcal{H}}(\Omega)$ denoted by $[u]_{s,\mathcal{H},Q}$ is similar to $[u]_{s,\mathcal{H}}$. Note that by the definitions of $[\cdot]_{s,\mathcal{H},Q}$ and $[\cdot]_{s,\mathcal{H}}$, there holds that $[u]_{s,\mathcal{H}} \le [u]_{s,\mathcal{H},Q} < +\infty$ for $u \in W^{s,\mathcal{H}}(\Omega)$. Due to this fact and applying the properties of the function $\mathcal{H}$, we can verify that the corresponding fractional Musielak–Orlicz Sobolev spaces $\widetilde{W}^{s,\mathcal{H}}(\Omega), \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$, functional $\widetilde{I}_{s,\mathcal{H}} = \tilde{\rho}_{s,\mathcal{H}}$, and Gagliardo seminorm $[\cdot]_{s,\mathcal{H},Q}$ also possess the properties of $W^{s,\mathcal{H}}(\Omega), W_0^{s,\mathcal{H}}(\Omega)$, $I_{s,\mathcal{H}}$ and $[\cdot]_{s,\mathcal{H}}$, respectively.

Next, let $B_R(0) := \{u \in X\colon \|u\|_X < R\}$ be an open ball centered at 0 with radius $R > 0$. The following surjectivity result is taken from Le [43].

THEOREM 2.5. *Suppose that* X *is a real reflexive Banach space and* X^* *is the related dual space; let* $F\colon D(F) \subset X \to 2^{X^*}$ *be a maximal monotone operator, let* $G\colon D(G) = X \to 2^{X^*}$ *be a bounded multivalued pseudomonotone operator, and let* $L \in X^*$*. If we can find* $u_0 \in X$ *and* $R \ge \|u_0\|_X$ *such that* $D(F) \cap B_R(0) \ne \emptyset$ *and*

$$\langle \xi + \eta - L, u - u_0\rangle_{X^*\times X} > 0$$

for all $u \in D(F)$ with $\|u\|_X = R$, for all $\xi \in F(u)$ and for all $\eta \in G(u)$, then it holds that

$$F(u) + G(u) \ni L$$

possesses a solution in $D(F)$, that is, $F + G$ is surjective.

For $v \in \mathbb{R}$, we define $v^{\pm} = \max\{\pm v, 0\}$ and for $u \in W_0^{s,\mathcal{H}}(\Omega)$ we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. As in Proposition 2.2 of Lu, Vetro, and Zeng [49] we know that

$$u^{\pm} \in W_0^{s,\mathcal{H}}(\Omega).$$

For given $\mathcal{E} \in C^1(X)$, we define

$$K_{\mathcal{E}} = \{u \in X : \mathcal{E}'(u) = 0\}$$

as the critical set of $\mathcal{E}$. The functional $\mathcal{E}$ is said to satisfy the Cerami condition (C-condition) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ fulfilling $\{\mathcal{E}(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\, \mathcal{E}'(u_n) \to 0 \quad \text{as } n \to \infty$$

possesses a strongly convergent subsequence. In addition, we say that $\mathcal{E}$ satisfies the Cerami condition at the level $c \in \mathbb{R}$ (C_c-condition) if the above result holds for all sequences fulfilling $\mathcal{E}(u_n) \to c$ as $n \to \infty$ instead of all the bounded sequences.

Next, we recall a version of the mountain pass theorem; see Papageorgiou, Rădulescu, and Repovš [59, Theorem 5.4.6].

THEOREM 2.6 (mountain pass theorem). *Let X be a Banach space, and suppose $\mathcal{E} \in C^1(X)$, $u_0, u_1 \in X$ with $\|u_1 - u_0\| > \delta > 0$,*

$$\max\{\mathcal{E}(u_0), \mathcal{E}(u_1)\} \le \inf\{\mathcal{E}(u) : \|u - u_0\| = \delta\} = m_\delta,$$
$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \mathcal{E}(\gamma(t)) \text{ with } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\},$$

and $\mathcal{E}$ fulfills the C_c-condition. Then c is a critical value of $\mathcal{E}$ with $c \ge m_\delta$. Furthermore, if $c = m_\delta$, then we can find $u \in \partial B_\delta(u_0)$ such that $\mathcal{E}'(u) = 0$.

The following version of the quantitative deformation lemma is taken from the monograph by Willem [72, Lemma 2.3].

LEMMA 2.7 (quantitative deformation lemma). *Let X be a Banach space, let $\mathcal{E} \in C^1(X;\mathbb{R})$, let $\emptyset \ne S \subseteq X$, let $c \in \mathbb{R}$, and let $\varepsilon, \delta > 0$ such that for all $u \in \mathcal{E}^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$ there holds that $\|\mathcal{E}'(u)\|_* \ge 8\varepsilon/\delta$, where $S_r = \{u \in X : d(u,S) = \inf_{u_0\in S}\|u - u_0\| < r\}$ for any $r > 0$. Then one can find $\eta \in C([0,1] \times X; X)$, fulfilling the following:*

(i) *$\eta(t,u) = u$ if $t = 0$ or if $u \notin \mathcal{E}^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$;*
(ii) *$\mathcal{E}(\eta(1,u)) \le c - \varepsilon$ for all $u \in \mathcal{E}^{-1}((-\infty, c + \varepsilon]) \cap S$;*
(iii) *$\eta(t,\cdot)$ is a homeomorphism of X for all $t \in [0,1]$;*
(iv) *$\|\eta(t,u) - u\| \le \delta$ for all $u \in X$ and $t \in [0,1]$;*
(v) *$\mathcal{E}(\eta(\cdot,u))$ is decreasing for all $u \in X$;*
(vi) *$\mathcal{E}(\eta(t,u)) < c$ for all $u \in \mathcal{E}^{-1}((-\infty, c]) \cap S_\delta$ and $t \in (0,1]$.*

Finally, we recall the Poincaré–Miranda existence theorem, which is a generalization of the intermediate value property. This result is named after Poincaré [60] (who conjectured it in 1883) and Miranda [53] (who established that it is equivalent to the Brouwer fixed point theorem). We refer to Kulpa [42] for an elementary proof.

THEOREM 2.8 (Poincaré–Miranda existence theorem). *Let* $U = [-t_1, t_1] \times \cdots \times [-t_N, t_N]$ *with* $t_i > 0$ *for* $i \in 1, \ldots, N$, *and let* $d \colon U \to \mathbb{R}^N$ *be continuous. If for each* $i \in \{1, \ldots, N\}$ *there holds that*

$$\begin{aligned} d_i(a) &\le 0 \quad \text{when } a \in U \text{ and } a_i = -t_i, \\ d_i(a) &\ge 0 \quad \text{when } a \in U \text{ and } a_i = t_i, \end{aligned}$$

then there exists at least one zero point of d *in* U.

3. Sub-supersolution method. In this section, based on the sub-supersolution method along with the nonsmooth calculus analysis, we study the following problem: Find $u \in K$ satisfying

$$0 \in (-\Delta)^s_{\mathcal{H}} u + \partial I_K(u) + \mathcal{F}(u) \quad \text{in } W_0^{s,\mathcal{H}}(\Omega)^*, \tag{3.1}$$

with $W_0^{s,\mathcal{H}}(\Omega)^*$ being the dual space of $W_0^{s,\mathcal{H}}(\Omega)$, K being a closed subset of $W_0^{s,\mathcal{H}}(\Omega)$, and I_K being the indicator function of K, while ∂I_K represents the subdifferential of I_K in the sense of convex analysis. Moreover, $\mathcal{F}$ is a lower order multivalued operator which is generated by $f \colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$. We establish the main existence results in subsection 3.1, and the related applications are given in subsection 3.2.

First, we introduce the definitions of a weak solution, a weak subsolution, and a weak supersolution to problem (3.1).

DEFINITION 3.1. *We say that* $u \in K$ *is a weak solution of problem* (3.1) *if there exist* $\tau \in C(\Omega)$ *satisfying* $1 < \tau(x) < (p_-)^*_s$ *for all* $x \in \overline{\Omega}$ *and* $\vartheta \in L^{\tau'(\cdot)}(\Omega)$ *satisfying* $\vartheta(x) \in \mathcal{F}(u)(x) := f(x, u(x))$ *for a.a.* $x \in \Omega$ *such that*

$$\int_\Omega \int_\Omega \frac{\mathcal{H}'(x, |D_s u|)}{|D_s u|} D_s u \cdot D_s(v - u) \, \mathrm{d}\nu + \int_\Omega \vartheta (v - u) \, \mathrm{d}x \ge 0$$

for all $v \in K$.

DEFINITION 3.2. *We say that* $\underline{u} \in W_0^{s,\mathcal{H}}(\Omega)$ *is a subsolution of problem* (3.1) *if there exist* $\tau \in C(\Omega)$ *satisfying* $1 < \tau(x) < (p_-)^*_s$ *for all* $x \in \overline{\Omega}$ *and a function* $\underline{\vartheta} \in L^{\tau'(\cdot)}(\Omega)$ *such that the following hold:*

(i) $\underline{u} \vee K \subset K$;
(ii) $\underline{\vartheta}(x) \in f(x, \underline{u}(x))$ *for a.a.* $x \in \Omega$;
(iii)

$$\int_\Omega \int_\Omega \frac{\mathcal{H}'(x, |D_s \underline{u}|)}{|D_s \underline{u}|} D_s \underline{u} \cdot D_s(v - \underline{u}) \, \mathrm{d}\nu + \int_\Omega \underline{\vartheta} (v - \underline{u}) \, \mathrm{d}x \ge 0$$

for all $v \in \underline{u} \wedge K$.

DEFINITION 3.3. *We say that* $\overline{u} \in W_0^{s,\mathcal{H}}(\Omega)$ *is a supersolution of problem* (3.1) *if there exist* $\tau \in C(\Omega)$ *satisfying* $1 < \tau(x) < (p_-)^*_s$ *for all* $x \in \overline{\Omega}$ *and a function* $\overline{\vartheta} \in L^{\tau'(\cdot)}(\Omega)$ *such that the following hold:*

(i) $\overline{u} \wedge K \subset K$;
(ii) $\overline{\vartheta}(x) \in f(x, \overline{u}(x))$ *for a.a.* $x \in \Omega$;
(iii)

$$\int_\Omega \int_\Omega \frac{\mathcal{H}'(x, |D_s \overline{u}|)}{|D_s \overline{u}|} D_s \overline{u} \cdot D_s(v - \overline{u}) \, \mathrm{d}\nu + \int_\Omega \overline{\vartheta} (v - \overline{u}) \, \mathrm{d}x \ge 0$$

for all $v \in \overline{u} \vee K$.

3.1. Existence results. We suppose the following hypotheses:

(H2) $f\colon \Omega\times\mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ is a graph measurable function, and $f(x,\cdot)\colon \mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ is upper semicontinuous for a.a. $x\in\Omega$.

(H3) There exist $\tau\in C(\Omega)$ with $1<\tau(x)<(p_-)_s^*$ for all $x\in\overline{\Omega}$, $\beta\geq 0$, and a nonnegative function $\alpha_\Omega\in L^{\tau'(\cdot)}(\Omega)$ such that

$$\sup\{|\vartheta|\colon \vartheta\in f(x,t)\}\leq \alpha_\Omega(x)+\beta_\Omega|t|^{\tau(x)-1}\quad \text{for a.a. } x\in\Omega \text{ and for all } t\in\mathbb{R}.$$

(H4) Let $\underline{u}$ and $\overline{u}$ be a pair of sub- and supersolutions of (3.1) such that $\underline{u}\leq\overline{u}$, and for $\tau\in C(\Omega)$ satisfying $1<\tau(x)<(p_-)_s^*$ for all $x\in\overline{\Omega}$ and some function $\gamma_\Omega\in L^{\tau'(\cdot)}(\Omega)$, it holds that

$$\sup\{|\vartheta|\colon \vartheta\in f(x,t)\}\leq\gamma_\Omega(x)\quad \text{for a.a. } x\in\Omega \text{ and for all } t\in[\underline{u},\overline{u}].$$

By hypotheses (H2), we see that $i_{\tau(\cdot)}\colon W_0^{s,\mathcal{H}}(\Omega)\to L^{\tau(\cdot)}(\Omega)$ is compact. The adjoint operator is denoted by $i^*_{\tau(\cdot)}: L^{\tau'(\cdot)}(\Omega)\to W_0^{s,\mathcal{H}}(\Omega)^*$. For any $u\in M(\Omega)$, we define

$$\tilde{f}(u)=\{\vartheta\in M(\Omega)\colon \vartheta(x)\in f(x,u(x)) \text{ for a.a. } x\in\Omega\}$$

as the set of measurable selections of $f(\cdot,u)$, which is nonempty due to (H1).

Due to (H2), for every $u\in L^{\tau(\cdot)}(\Omega)$, we get $\tilde{f}(u)\subset L^{\tau'(\cdot)}(\Omega)$. Furthermore, we employ the mappings $\tilde{f}: L^{\tau(\cdot)}(\Omega)\to L^{\tau'(\cdot)}(\Omega)$ with $u\mapsto\tilde{f}(u)$ and $\mathcal{F}=i^*_{\tau(\cdot)}\tilde{f}i_{\tau(\cdot)}$: $W_0^{s,\mathcal{H}}(\Omega)\to 2^{W_0^{s,\mathcal{H}}(\Omega)^*}$, that is, $\mathcal{F}(u)=\{\hat{\vartheta}\in W_0^{s,\mathcal{H}}(\Omega)^*\colon \hat{\vartheta}\in\tilde{f}(u)\}$.

Arguing as in the proof of Proposition 3.1 in Carl, Le, and Winkert [17], we deduce the following proposition.

PROPOSITION 3.4. *Let* (H1), (H2), *and* (H3) *be satisfied. Then* $\mathcal{F}=i^*_{\tau(\cdot)}\tilde{f}i_{\tau(\cdot)}$ *is a bounded and pseudomonotone mapping from* $W_0^{s,\mathcal{H}}(\Omega)$ *to* $\mathcal{K}(W_0^{s,\mathcal{H}}(\Omega)^*)$.

Next, we are ready to show the existence results with respect to problem (3.1) if it possesses a pair of sub- and supersolutions.

THEOREM 3.5. *Let hypotheses* (H1), (H2), *and* (H4) *be satisfied, and assume that* $\underline{u}$ *is a subsolution of* (3.1) *and* $\overline{u}$ *is a supersolution of* (3.1). *Then there exists a solution* u^* *of problem* (3.1) *fulfilling*

$$\underline{u}\leq u^*\leq\overline{u}\quad \text{in } \Omega.$$

Proof. Let τ, $\underline{u}$, and $\overline{u}$ fulfill (H2). and let $\underline{\vartheta},\overline{\vartheta}$ be the functions mentioned in Definitions 3.2 and 3.3 with respect to $\underline{u}$ and $\overline{u}$, respectively. Furthermore, we consider the truncation function $F_t\colon\Omega\times\mathbb{R}\to 2^{\mathbb{R}}$ as

$$F_t(x,v)=\begin{cases}\{\underline{\vartheta}(x)\} & \text{if } v<\underline{u}(x),\\ f(x,v) & \text{if } \underline{u}(x)\leq v\leq\overline{u}(x),\\ \{\overline{\vartheta}(x)\} & \text{if } v>\overline{u}(x).\end{cases}$$

By assumptions (H2) and (H4), we deduce that F_t satisfies (H2). Furthermore, by its definition and condition (H4) it follows that

$$\sup\{|\phi|\colon \phi\in F_t(x,v)\}\leq\gamma_\Omega(x)+|\underline{\vartheta}(x)|+|\overline{\vartheta}(x)|\quad \text{for a.a. } x\in\Omega \text{ and for all } v\in\mathbb{R},$$

where $\gamma_\Omega+|\underline{\vartheta}|+|\overline{\vartheta}|\in L^{\tau'(\cdot)}(\Omega)$. Therefore, F_t fulfills (H3) with $\beta_\Omega=0$ and $\alpha_\Omega(x)=\gamma_\Omega(x)+|\underline{\vartheta}(x)|+|\overline{\vartheta}(x)|$. According to Proposition 3.4, we know that $i^*_{\tau(\cdot)}\tilde{F}_t i_{\tau(\cdot)}\colon W_0^{s,\mathcal{H}}(\Omega)\to\mathcal{K}(W_0^{s,\mathcal{H}}(\Omega)^*)$ is bounded and pseudomonotone.

Next, we consider the following auxiliary problem: Find $u^* \in K$ and $\vartheta \in L^{\tau'(\cdot)}(\Omega)$ satisfying

$$\vartheta(x) \in F_t(x, u^*(x)) \quad \text{for a.a. } x \in \Omega, \tag{3.2}$$

$$\langle \mathcal{J}u^*, v - u^* \rangle + \int_\Omega \vartheta(v - u^*)\, \mathrm{d}x \geq 0 \quad \text{for all } v \in K. \tag{3.3}$$

Note that inequality (3.3) means finding $u \in K$ such that

$$\left\langle \mathcal{J}u^* + \tilde{\vartheta}, v - u^* \right\rangle \geq 0 \quad \text{for all } v \in K,$$

with $\tilde{\vartheta} = i^*_{\tau(\cdot)} \vartheta i_{\tau(\cdot)} \in [i^*_{\tau(\cdot)} \tilde{F}_t i_{\tau(\cdot)}](u^*)$. More explicitly, one needs to find $u \in D(\partial I_K)$, $\xi \in \partial I_K(u)$, and

$$\tilde{\vartheta} = i^*_{\tau(\cdot)} \vartheta i_{\tau(\cdot)} \in \left[i^*_{\tau(\cdot)} \tilde{F}_t i_{\tau(\cdot)}\right](u^*),$$

fulfilling

$$\mathcal{A}(u^*, \xi, \tilde{\vartheta}) := \mathcal{J}u^* + \xi + \tilde{\vartheta} = 0 \quad \text{in } W_0^{s,\mathcal{H}}(\Omega)^*.$$

Since ∂I_K is maximal monotone and

$$\mathcal{J} + i^*_{\tau(\cdot)} \tilde{F}_t i_{\tau(\cdot)} \colon W_0^{s,\mathcal{H}}(\Omega) \to 2^{W_0^{s,\mathcal{H}}(\Omega)^*}$$

is bounded and pseudomonotone, according to Le [43, Corollary 2.3], we only need to check the following coercivity condition: there exists $u_0 \in K$ satisfying

$$\lim_{\substack{[u^*]_{s,\mathcal{H}} \to \infty \\ u^* \in K}} \left[\inf_{\substack{\xi \in \partial I_K(u^*) \\ \tilde{\vartheta} \in \left[i^*_{\tau(\cdot)} \tilde{F}_t i_{\tau(\cdot)}\right](u^*)}} \left\langle \mathcal{A}\left(u, \xi, \tilde{\vartheta}\right), u^* - u_0 \right\rangle \right] = \infty. \tag{3.4}$$

Indeed, for any fixed $u_0 \in K$, for all $u \in K$, and for every $\xi \in (\partial I_K)(u^*)$ it holds that $0 = I_K(u_0) - I_K(u^*) \geq \langle \xi, u_0 - u^* \rangle$, which implies $\langle \xi, u^* - u_0 \rangle \geq 0$. Thus, to verify (3.4) means verifying the following condition:

$$\inf_{\tilde{\vartheta} \in \left[i^*_{\tau(\cdot)} \tilde{F}_t i_{\tau(\cdot)}\right](u^*)} \left\langle \hat{\mathcal{A}}\left(u, \tilde{\vartheta}\right), u^* - u_0 \right\rangle \to \infty \tag{3.5}$$

as $[u^*]_{s,\mathcal{H}} \to \infty$ with $u^* \in K$, where

$$\hat{\mathcal{A}}(u^*, \tilde{\vartheta}) := \mathcal{J}u^* + \tilde{\vartheta}$$

and $\tilde{\vartheta} = i^*_\tau \vartheta i_\tau \in [i^*_{\tau(\cdot)} \tilde{F}_t i_{\tau(\cdot)}](u^*)$ with $\vartheta \in \tilde{F}_t(u^*)$. By (H4), we calculate that

$$\begin{aligned} \left|\left\langle \tilde{\vartheta}, u^* - u_0 \right\rangle\right| &\leq \left(\|\gamma_\Omega\|_{\tau'} + \|\underline{\vartheta}\|_{\tau'} + \|\overline{\vartheta}\|_{\tau'}\right)\left(\|u^*\|_\tau + \|u_0\|_\tau\right) \\ &\leq C\left(\|u^*\|_\tau + 1\right) \\ &\leq C([u^*]_{s,\mathcal{H}} + 1). \end{aligned} \tag{3.6}$$

Note that the potential functional

$$
\begin{aligned}
I_{s,\mathcal{H}}(u^*) &= \int_\Omega\int_\Omega \left[|D_s u^*|^{p(x,y)} + \mu(x,y)|D_s u^*|^{q(x,y)}\right]\log(e+\alpha|D_s u^*|)\,\mathrm{d}\nu \\
&= \int_\Omega\int_\Omega \mathcal{H}(x,y,|D_s u^*(x,y)|)\,\mathrm{d}\nu
\end{aligned}
$$

of $\mathcal{J}$ is convex, and it fulfills

$$
\langle \mathcal{J}u^*, u^*-u_0\rangle \geq I_{s,\mathcal{H}}(u^*) - I_{s,\mathcal{H}}(u_0) = I_{s,\mathcal{H}}(u^*) - C. \tag{3.7}
$$

Combining (3.6) and (3.7), we get

$$
\begin{aligned}
&\left\langle \mathcal{J}u^* + \tilde{\vartheta}, u^* - u_0 \right\rangle \\
&\quad \geq \int_\Omega\int_\Omega \left[(|D_s u^*|^{p(x,y)} + \mu(x,y)|D_s u^*|^{q(x,y)})\log(e+\alpha|D_s u^*|)\right]\mathrm{d}\nu - C([u^*]_{s,\mathcal{H}}+1) \\
&\quad = \int_\Omega\int_\Omega \mathcal{H}(x,|D_s u^*|)\,\mathrm{d}\nu - C([u^*]_{s,\mathcal{H}}+1)
\end{aligned} \tag{3.8}
$$

for any $u\in K$, $\tilde{\vartheta}\in [i^*_{\tau(\cdot)}\tilde{F}_t i_{\tau(\cdot)}](u^*)$. The coercivity of $\mathcal{J}$ yields

$$
\lim_{[u^*]_{s,\mathcal{H}}\to\infty} \frac{1}{[u^*]_{s,\mathcal{H}}} \int_\Omega\int_\Omega \mathcal{H}(x,|D_s u^*|)\,\mathrm{d}\nu = \infty.
$$

From this and (3.8), it follows (3.5). Hence, according to Le [43, Corollary 2.3], there exist u^*, ϑ satisfying (3.2) and (3.3).

Next, we check that

$$
\underline{u} \leq u^* \leq \overline{u} \quad \text{in } \Omega. \tag{3.9}
$$

Testing (3.3) with $v = \underline{u}\vee u^* := u^* + (\underline{u}-u^*)^+ \in K$, we obtain

$$
\left\langle \mathcal{J}u^*, (\underline{u}-u^*)^+\right\rangle + \int_\Omega \vartheta\,(\underline{u}-u^*)^+\,\mathrm{d}x \geq 0. \tag{3.10}
$$

Then we choose $v = \underline{u} - (\underline{u}-u^*)^+ = \underline{u}\wedge u^* \in \underline{u}\wedge K$ in Definition 3.2 to find

$$
-\left\langle \mathcal{J}\underline{u}, (\underline{u}-u^*)^+\right\rangle - \int_\Omega \underline{\vartheta}\,(\underline{u}-u^*)^+\,\mathrm{d}x \geq 0. \tag{3.11}
$$

Inequalities (3.10) and (3.11) yield

$$
\left\langle \mathcal{J}u^* - \mathcal{J}\underline{u}, (\underline{u}-u^*)^+\right\rangle + \int_\Omega (\vartheta-\underline{\vartheta})\,(\underline{u}-u^*)^+\,\mathrm{d}x \geq 0.
$$

Utilizing the strictly monotonicity of $\mathcal{J}$, we arrive at

$$
\begin{aligned}
&\left\langle \mathcal{J}u^* - \mathcal{J}\underline{u}, (\underline{u}-u^*)^+\right\rangle \\
&\quad = \int_{\{x\in\Omega:\ \underline{u}(x)\geq u^*(x)\}}\int_{\{y\in\Omega:\ \underline{u}(y)\geq u^*(y)\}} \left(\frac{\mathcal{H}'(x,|D_s u^*|)}{|D_s u^*|}D_s u^* - \frac{\mathcal{H}'(x,|D_s \underline{u}|)}{|D_s \underline{u}|}D_s \underline{u}\right) \\
&\qquad \cdot D_s\,(\underline{u}-u^*)\,\mathrm{d}\nu \leq 0.
\end{aligned}
$$

Note that for any $x\in\Omega$ satisfying $\underline{u}(x) > u^*(x)$ there holds that $\vartheta(x)\in\{\underline{\vartheta}(x)\}$ (namely, $\vartheta(x)=\underline{\vartheta}(x)$). Therefore,

$$
\int_\Omega (\vartheta-\underline{\vartheta})\,(\underline{u}-u^*)^+\,\mathrm{d}x = \int_{\{x\in\Omega:\,\underline{u}(x)>u^*(x)\}} (\vartheta-\underline{\vartheta})\,(\underline{u}-u^*)\,\mathrm{d}x = 0.
$$

We infer that $(\underline{u}-u^*)^+=0$, and thus $u^*(x)\geq \underline{u}(x)$ for a.a. $x\in\Omega$. Analogously, one can verify that $u^*(x)\leq \overline{u}(x)$ for a.a. $x\in\Omega$. Furthermore, from (3.9), we infer that $F_t(x,u^*(x))=f(x,u^*(x))$ for a.a. $x\in\Omega$. This shows that u^* solves problem (3.1). ☐

Similar to the proof of Theorem 4.1 in Liu, Lu, and Vetro [48], one can obtain the following result concerning the solution set $\mathcal{S}$ within a pair of sub-supersolutions $\underline{u}$ and $\overline{u}$ such that $\underline{u}\leq\overline{u}$.

THEOREM 3.6. *Let hypotheses* (H1)*,* (H2)*, and* (H4) *be satisfied. Then the following hold:*

(a) *$\mathcal{S}$ is compact in $W_0^{s,\mathcal{H}}(\Omega)$.*

(b) *Under the lattice conditions*

$$\mathcal{S}\wedge K\subset K \quad \text{and} \quad \mathcal{S}\vee K\subset K, \tag{3.12}$$

the following hold:

(i) *$u\in\mathcal{S}$ is a subsolution of problem* (3.1) *and at the same time a supersolution of* (3.1)*.*

(ii) *$\mathcal{S}$ is directed both downward and upward; that is, for all $u_1,u_2\in\mathcal{S}$, there exist $v_1,v_2\in\mathcal{S}$ fulfilling*

$$v_1\leq\min\{u_1,u_2\} \quad \text{and} \quad v_2\geq\max\{u_1,u_2\}.$$

(c) *If conditions* (3.12) *hold, then there exist $s_1,s_2\in\mathcal{S}$ such that $s_1\leq u\leq s_2$ for all $u\in\mathcal{S}$.*

3.2. Applications. In this subsection, we are going to apply the results of subsection 3.1 to the elliptic inclusion problem (3.1). To this end, suppose that assumptions (H1) and (H2) are fulfilled. We can rewrite the multivalued function f as

$$f(x,t)=[f_1(x,t),f_2(x,t)]$$

for all $(x,t)\in\Omega\times\mathbb{R}$, where $f_i(x,s), i=1,2$, are single-valued functions. Due to (H1) and (H2), it is not hard to check that for $i=1,2$, $x\mapsto f_i(x,u(x))$ are measurable for any $u\in M(\Omega)$. Moreover, $s\mapsto f_1(x,s)$ is a single-valued lower semicontinuous function and $s\mapsto f_2(x,s)$ is a single-valued upper semicontinuous function. Furthermore, we assume the following conditions on $f_i (i=1,2)$ to guarantee the existence of sub- and supersolutions:

(Hf) Let $a_i\in L^{\tau'(\cdot)}(\Omega)$, $i=1,2$, fulfill

$$f_1(x,t)\leq a_1(x) \quad \text{and} \quad f_2(x,t)\geq a_2(x)$$

for a.a. $x\in\Omega$ and for all $t\in\mathbb{R}$.

Next, we suppose that $u_i\in W_0^{s,\mathcal{H}}(\Omega)$, $i=1,2$, fulfill

$$\begin{cases}\mathcal{J}u_i=-a_i & \text{in } \Omega,\\ u_i=0 & \text{on } \mathbb{R}^N\setminus\Omega.\end{cases} \tag{3.13}$$

According to Zeng et al. [76], in which boundedness results of weak solutions to elliptic inclusions driven by the operator $\mathcal{J}$ have been established, we know that $u_i\in L^\infty(\Omega)$.

Example 3.7. Let $K=W_0^{s,\mathcal{H}}(\Omega)$; then problem (3.1) becomes the multivalued elliptic problem

$$\begin{aligned}(-\Delta)^s_{\mathcal{H}}u+f(x,u)\ni 0 \quad &\text{in } \Omega,\\ u=0 \quad &\text{on } \mathbb{R}^N\setminus\Omega.\end{aligned} \tag{3.14}$$

Applying Theorem 3.5, we get the following result.

THEOREM 3.8. *Let hypotheses* (H1), (H2), *and* (Hf) *be satisfied. Then, for* $C_1 \geq 0$ *large enough, there exists at least a solution* u^* *of* (3.14) *fulfilling* $u_1(x) \leq u^*(x) \leq u_2(x) + C_1$ *in* Ω.

Proof. With a view to Theorem 3.5, we only need to check the validity of the definition of weak sub- and supersolutions of problem (3.14) (see Definitions 3.2 and 3.3). We claim that $\underline{u} := u_1$ is a weak subsolution of (3.14) and $\overline{u} := u_2 + C_1$ is a weak supersolution of (3.14).

First, we show that $\underline{u} := u_1$ is a weak subsolution of (3.14). We know that $W_0^{s,\mathcal{H}}(\Omega)$ satisfies the lattice conditions, and thus $u_1 \vee K \subset K$. By setting $\underline{\vartheta}(x) = f_1(x, u_1(x))$, we get $\underline{\vartheta} \in L^{\tau'(\cdot)}(\Omega)$ (note that $1 < \tau(x) < (p_-)_s^*$ for all $x \in \overline{\Omega}$) and $\underline{\vartheta}(x) \in f(x, u_1(x))$, so $\underline{u}$ fulfills Definition 3.2(ii). It remains to verify (iii), that is,

$$\langle \mathcal{J} u_1, v - u_1 \rangle + \int_\Omega \underline{\vartheta}\,(v - u_1)\,\mathrm{d}x \geq 0 \quad \text{for all } v \in u_1 \wedge K, \tag{3.15}$$

where

$$\langle \mathcal{J} u_1, v - u_1 \rangle = \int_\Omega \int_\Omega \frac{\mathcal{H}'(x, |D_s u_1|)}{|D_s u_1|} D_s u_1 \cdot D_s\,(v - u_1)\,\mathrm{d}\nu.$$

Note that $v \in u_1 \wedge K$ means $v = u_1 \wedge \psi = u_1 - (u_1 - \psi)^+$ for some $\psi \in K$. Then (3.15) is equivalent to

$$\left\langle \mathcal{J} u_1, (u_1 - \psi)^+ \right\rangle + \int_\Omega \underline{\vartheta}\,(u_1 - \psi)^+\,\mathrm{d}x \leq 0 \quad \text{for all } \psi \in K.$$

Combining the fact that $(u_1 - \psi)^+ \in \{v \in W_0^{s,\mathcal{H}}(\Omega) \colon v \geq 0\}$ with $\underline{\vartheta} = f_1(\cdot, u_1)$, we only need to show

$$\langle \mathcal{J} u_1, v \rangle + \int_\Omega f_1\,(x, u_1)\,v\,\mathrm{d}x \leq 0$$

for all $v \in W_0^{s,\mathcal{H}}(\Omega)$ such that $v \geq 0$. Hypotheses (Hf) and (3.13) yield $-a_1(x) + f_1(x, u_1) \leq 0$. Hence,

$$\langle \mathcal{J} u_1, v \rangle + \int_\Omega f_1\,(x, u_1)\,v\,\mathrm{d}x = \int_\Omega (-a_1(x) + f_1\,(x, u_1))\,v\,\mathrm{d}x \leq 0$$

for all $v \in W_0^{s,\mathcal{H}}(\Omega)$ with $v \geq 0$. This shows (iii) in Definition 3.2, and therefore $\underline{u} = u_1$ turns out to be a subsolution of problem (3.14).

Next, we will prove that $\overline{u} = u_2 + C_1$ satisfies the conditions of Definition 3.3 with $C_1 \geq 0$ large enough. Since u_2 is bounded and by (3.13), we see that $\overline{u} = u_2 + C_1 \in W^{s,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ solves problem

$$\begin{cases} \mathcal{J}\overline{u} = -a_2 & \text{in } \Omega, \\ \overline{u} = C_1 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.16}$$

Note that $\overline{u} = u_2 + C_1 \in W_0^{s,\mathcal{H}}(\Omega)$ and $K = W_0^{s,\mathcal{H}}(\Omega)$ satisfies the lattice conditions, and hence $\overline{u} \wedge K \subset W_0^{s,\mathcal{H}}(\Omega)$. Taking $\overline{\vartheta} = f_2(\cdot, \overline{u})$, we have $\overline{\vartheta} \in L^{\tau'}(\Omega)$ and $\overline{u}(x) \in f(x, \overline{u}(x))$. Now it remains to verify Definition 3.3(iii); namely, for all $v \in \overline{u} \vee K$ it holds that

$$\langle \mathcal{J}\overline{u}, v - \overline{u} \rangle + \int_\Omega \overline{\vartheta}(v - \overline{u})\,\mathrm{d}x \geq 0. \tag{3.17}$$

Since $v \in \overline{u} \vee K$ means $v = \overline{u} \vee \psi = \overline{u} + (\zeta - \overline{u})^+$ for some $\zeta \in K$, (3.17) can be rewritten as

$$\left\langle \mathcal{J}\overline{u}, (\zeta - \overline{u})^+ \right\rangle + \int_\Omega \overline{\vartheta}(\zeta - \overline{u})^+ \,\mathrm{d}x \ge 0 \quad \text{for all } \zeta \in K. \tag{3.18}$$

Due to $(\zeta - \overline{u})^+ \in \{v \in W_0^{s,\mathcal{H}}(\Omega) \colon v \ge 0\}$, inequality (3.18) can be written as

$$\langle \mathcal{J}\overline{u}, v\rangle + \int_\Omega \overline{\vartheta} v \,\mathrm{d}x \ge 0 \quad \text{for all } v \in W_0^{s,\mathcal{H}}(\Omega) \text{ with } v \ge 0.$$

Employing (Hf) and (3.16), we obtain $\overline{\vartheta} - a_2(x) = f_2(\cdot, \overline{u}) - a_2(x) \ge 0)$. Hence, it holds that

$$\langle \mathcal{J}\overline{u}, v\rangle + \int_\Omega \overline{\vartheta} v \,\mathrm{d}x = \int_\Omega (-a_2(x) + f_2(\cdot, \overline{u}))\, v \,\mathrm{d}x \ge 0$$

for all $v \in W_0^{s,\mathcal{H}}(\Omega)$ with $v \ge 0$. Therefore, $\overline{u} = u_2 + C_1$ is a weak supersolution of (3.14). Recalling that $u_i \in L^\infty(\Omega), i = 1, 2$, we take $C_1 \ge 0$ sufficiently large, satisfying $\underline{u} = u_1 \le u_2 + C_1 = \overline{u}$. Finally, Theorem 3.5 yields the assertion. ∎

By applying Theorem 3.6, we deduce some results concerning the solution set $\mathcal{S}$ of (3.14) within the order interval $[\underline{u}, \overline{u}]$.

COROLLARY 3.9. *Let hypotheses* (H1), (H2), *and* (Hf) *be satisfied. Then* $\mathcal{S} \subset W_0^{s,\mathcal{H}}(\Omega)$ *is compact, and there exist* $s_1, s_2 \in \mathcal{S}$ *such that* $s_1 \le u \le s_2$ *for all* $u \in \mathcal{S}$.

In addition, we deal with a multivalued obstacle problem with K defined as

$$K = \left\{ u \in W_0^{s,\mathcal{H}}(\Omega) \colon u(x) \ge \phi(x) \text{ a.e. in } \Omega \right\}. \tag{3.19}$$

We suppose the following assumptions on the obstacle function $\phi(\cdot)$:

(Hϕ) Let $\phi \in W_0^{s,\mathcal{H}}(\Omega)$ such that $\phi(x) \le c_\phi$ for a.a. $x \in \Omega$ with $c_\phi > 0$.

Example 3.10. If K is formulated by (3.19), then problem (3.1) can be represented as

$$\begin{aligned} (-\Delta)^s_{\mathcal{H}} u + f(x,u) &\ni 0 && \text{in } \Omega, \\ u(x) &\ge \phi(x) && \text{in } \Omega, \\ u(x) &= 0 && \text{on } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{3.20}$$

THEOREM 3.11. *Let hypotheses* (H1), (H2), (Hf), *and* (Hϕ) *be satisfied. Then, for* $C_2 \ge 0$ *sufficiently large, it holds that* $\underline{u} = u_1$ *and* $\overline{u} = u_2 + C_2$ *are sub- and supersolutions of problem* (3.20), *respectively. Thus,* (3.20) *possesses a solution* u^* *satisfying* $\underline{u} \le u^*(x) \le \overline{u}$ *in* Ω. *Moreover, the solution set* $\mathcal{S} \subseteq [\underline{u}, \overline{u}] \subset W_0^{s,\mathcal{H}}(\Omega)$ *of* (3.20) *is compact, and there exist* $s_1, s_2 \in \mathcal{S}$ *such that* $s_1 \le u \le s_2$ *for all* $u \in \mathcal{S}$.

Proof. As done in the proof of Theorem 3.8, we only need to show that $\underline{u} = u_1$ and $\overline{u} = u_2 + C_2$ are sub- and supersolutions of problem (3.20), respectively. Since $u_1 \vee K \subset K$, we see that $\underline{u}$ satisfies Definition 3.2(i). Moreover, due to that fact that $\overline{u} = u_2 + C_2 \in W_0^{s,\mathcal{H}}(\Omega)$ and $u_2 \in L^\infty(\Omega)$, by applying (Hϕ) we get $u_2 + C_2 \ge c_\phi \ge \phi$ for C_2 large enough. This implies $\overline{u} \wedge K \subset K$ and thus $\overline{u}$ satisfies Definition 3.3(i). The remaining proof is similar to the one of Theorem 3.8. ∎

4. Kirchhoff problem. In this section, we are interested in the existence of weak solutions to the problem

$$\begin{cases} \psi(\tilde{I}_{s,\mathcal{H}})\,(-\Delta)^s_{\mathcal{H}}\,u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases} \tag{4.1}$$

for $u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$, where $\psi(t) = \theta_1 + \theta_2 t^{\varsigma-1}$ for $t \in \mathbb{R}$ with $\theta_1 \geq 0, \theta_2 > 0$, $\varsigma \geq 1$, and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ being a Carathéodory function satisfying suitable assumptions; see (H6) below. To be more precise, we are going to show the existence of constant sign solutions of (4.1) in subsection 4.1 and a least energy sign-changing solution of (4.1) in subsection 4.2.

Clearly, weak solutions of (4.1) coincide with the critical points of the related energy functional $E\colon \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \to \mathbb{R}$ given by

$$E(u) = \Psi[\tilde{I}_{s,\mathcal{H}}(u)] - \int_\Omega F(x,u)\,\mathrm{d}x,$$

with $\Psi\colon [0,\infty) \to [0,\infty)$ formulated as

$$\Psi(t) = \int_0^t \psi(\tau)\,\mathrm{d}\tau = \theta_1 t + \frac{\theta_2}{\varsigma} t^{\varsigma}.$$

In addition, the truncated functionals $E_\pm\colon \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \to \mathbb{R}$ of E are defined by

$$E_\pm(u) = \Psi[\tilde{I}_{s,\mathcal{H}}(u)] - \int_\Omega F(x, \pm u^\pm)\,\mathrm{d}x.$$

Note that for $u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ satisfying $u^+ \neq 0 \neq u^-$, there hold that $E(u) > E(u^+) + E(-u^-)$, $\langle E'(u), u^+\rangle > \langle E'(u^+), u^+\rangle$, and $\langle E'(u), -u^-\rangle > \langle E'(-u^-), -u^-\rangle$. We point out that for seeking sign-changing solutions for the semilinear elliptic equation $-\Delta u + u = f(u)$, Bartsch and Weth [9] introduced the following type of constraint set:

$$\mathcal{N} = \left\{ u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)\colon u^\pm \neq 0, \left\langle E'(u), u^+ \right\rangle = \left\langle E'(u), -u^- \right\rangle = 0 \right\}.$$

Note that every sign-changing solution for problem (4.1) is contained in $\mathcal{N}$. After the work of Bartsch and Weth, several papers appeared using the same constraint set $\mathcal{N}$; see, for example, Liang and Rădulescu [44], Shuai [66], Tang and Chen [68], Tang and Cheng [69], and Zhang [78]. The following proofs for the existence of positive, negative, and sign-changing solutions to problem (4.1) using the Poincaré–Miranda existence theorem, the quantitative deformation lemma, and the the mountain pass theorem are mainly motivated by the works of Arora, Crespo-Blanco, and Winkert [4] and Crespo-Blanco, Gasiński, and Winkert [24].

Let us formulate the precise assumptions on the data of problem (4.1). First, note that κ is the constant such that the function

$$f^\varepsilon\colon [0,+\infty) \to [0,+\infty), \quad f^\varepsilon(t) = \frac{t^\varepsilon}{\log(e + \alpha t)}$$

is increasing for $\varepsilon \geq \kappa$ and is almost increasing for $0 < \varepsilon < \kappa$; see Lemma 3.1 in Arora, Crespo-Blanco, and Winkert [4] for more details.

(H5) Let $\psi\colon [0,\infty) \to [0,\infty)$ be a continuous function given by $\psi(t) = \theta_1 + \theta_2 t^{\varsigma-1}$ for $t \in \mathbb{R}$ with $\theta_1 \geq 0, \theta_2 > 0$, and let $\varsigma \geq 1$ satisfy $\varsigma q_+ < (p_-)^*_s$.

(H6) $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that the following hold:

(f1) There exist $r\in C_+(\overline{\Omega})$ satisfying $r_+<(p_-)_s^*$ and $C>0$ satisfying
$$|f(x,t)|\le C\left(1+|t|^{r(x)-1}\right)$$
for a.a. $x\in\Omega$ and for all $t\in\mathbb{R}$.

(f2) For some $0<\eta<1$, there holds that
$$\lim_{t\to\pm\infty}\frac{F(x,t)}{|t|^{\varsigma q_+ +\eta}}=+\infty \quad \text{uniformly for a.a. } x\in\Omega.$$

(f3)

(i) If $\theta_1>0$, then for $0<\eta<1$ there holds that
$$\lim_{t\to 0}\frac{f(x,t)}{|t|^{p_+-1+\eta}t}=0 \quad \text{uniformly for a.a. } x\in\Omega.$$

(ii) If $\theta_1=0$, then for $0<\eta<1$ there holds that
$$\lim_{t\to 0}\frac{f(x,t)}{|t|^{\varsigma p_+-1+\eta}t}=0 \quad \text{uniformly for a.a. } x\in\Omega.$$

(f4) For $F(x,t):=\int_0^t f(x,\tau)\,\mathrm{d}\tau$, the function
$$t\mapsto \overline{F}(x,t):=f(x,t)t-\varsigma q_+\left(1+\frac{\kappa}{p_-}\right)F(x,t)$$
is nonincreasing on $(-\infty,0]$ and nondecreasing on $[0,+\infty)$ for a.a. $x\in\Omega$. Moreover,
$$\lim_{t\to+\infty}\overline{F}(x,t)=+\infty \quad \text{uniformly for a.a. } x\in\Omega.$$

(f5) The function
$$t\mapsto\frac{f(x,t)}{|t|^{\varsigma(q_++1)-1}}$$
is strictly increasing on $(-\infty,0)$ and on $(0,+\infty)$ for a.a. $x\in\Omega$, and $\varsigma q_+<\varsigma(q_++1)<(p_-)_s^*$.

Remark 4.1.

(i) From (f1) and (f2), we deduce that $\varsigma q_+<r_-$. Moreover, by (H5) there holds that $\varsigma q_+<(p_-)_s^*$; then there exists $r\in C_+(\overline{\Omega})$ such that $\varsigma q_+<r_-\le r_+<(p_-)_s^*$.

(ii) From (f1) and (f2), we can find some constant $C>0$ satisfying $F(x,t)>-C$ for a.a. $x\in\Omega$ and for all $t\in\mathbb{R}$.

(iii) From (f1) and (f3), we deduce that $f(x,0)=0$, while from (f1) and (f3)(i) we see that for any $\varepsilon>0$ one can find some constant $C_\varepsilon>0$ satisfying
$$|F(x,t)|\le\frac{\varepsilon}{p_-}|t|^{p(x)}+C_\varepsilon|t|^{r(x)} \quad \text{for a.a. } x\in\Omega \text{ and for all } t\in\mathbb{R}. \tag{4.2}$$
Also, from (f1) and (f3)(ii), for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ satisfying
$$|F(x,t)|\le\frac{\varepsilon}{\varsigma p_-}|t|^{\varsigma p_++\varepsilon}+C_\varepsilon|t|^{r(x)} \quad \text{for a.a. } x\in\Omega \text{ and for all } t\in\mathbb{R}. \tag{4.3}$$

(iv) From (f1) and (f2), for any $\varepsilon>0$ there exists some constant $C_\varepsilon>0$ satisfying
$$F(x,t)\ge\frac{\varepsilon}{\varsigma q_+}|t|^{\varsigma q_+}\log^{\varsigma}(e+\alpha|t|)-C_\varepsilon \quad \text{for a.a. } x\in\Omega \text{ and for all } t\in\mathbb{R}.$$

According to [4, Lemma 3.1] we have the following lemma.

LEMMA 4.2. *Let $C > 1$, and let $g\colon [0,\infty) \to [0,\infty)$ be defined as $g(t) = \frac{\alpha t}{C(e+\alpha t)\log(e+\alpha t)}$. Then the maximum value of g is $\frac{\kappa}{C}$.*

4.1. Existence of constant sign solutions. We first start by showing that the truncated functionals E_+ and E_- fulfill the Cerami condition.

PROPOSITION 4.3. *Let hypotheses* (H1), (H5), *and* (f1)–(f4) *be satisfied. Then the functionals $E_\pm$ satisfy the Cerami condition.*

Proof. We first show that the functional E_+ satisfies the Cerami condition. For this purpose, let $\{u_n\}_{n\geq 1} \subseteq \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ be a sequence such that

$$|E_+(u_n)| \leq C_1 \quad \text{for all } n\in\mathbb{N} \text{ and for some } C_1 > 0, \tag{4.4}$$

$$\left(1 + [u_n]_{s,\mathcal{H},Q}\right) E'_+(u_n) \to 0 \quad \text{in } \widetilde{W}_0^{s,\mathcal{H}}(\Omega)^*. \tag{4.5}$$

By (4.5), there exists a sequence $\varepsilon_n \to 0^+$ fulfilling

$$\begin{aligned}
&\Bigg|\psi\left(\tilde{I}_{s,\mathcal{H}}(u_n)\right)\int_Q\Bigg(\left[\log(e+\alpha|D_s u_n|) + \frac{\alpha|D_s u_n|}{p(x,y)(e+\alpha|D_s u_n|)}\right]\\
&\quad\times |D_s u_n|^{p(x,y)-2}\cdot D_s u_n \cdot D_s v\\
&\quad+\mu(x,y)\left[\log(e+\alpha|D_s u_n|) + \frac{\alpha|D_s u_n|}{q(x,y)(e+\alpha|D_s u_n|)}\right]|D_s u_n|^{q(x,y)-2}\cdot D_s u_n\cdot D_s v\Bigg)\,\mathrm{d}\nu\\
&\quad-\int_\Omega f\left(x,u_n^+\right)v\,\mathrm{d}x\Bigg| \leq \frac{\varepsilon_n[v]_{s,\mathcal{H},Q}}{1+[u_n]_{s,\mathcal{H},Q}}
\end{aligned} \tag{4.6}$$

for all $v\in\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$. Setting $v = -u_n^- \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ in (4.6) and noting that $f(x,u_n^+)u_n^- = 0$, we obtain

$$\begin{aligned}
&\theta_2\left(\tilde{I}_{s,\mathcal{H}}(u_n^-)\right)^{\varsigma}\\
&\leq \psi\left(\tilde{I}_{s,\mathcal{H}}(u_n^-)\right)\int_Q\left\{\left[\log(e+\alpha|D_s u_n^-|) + \frac{\alpha|D_s u_n^-|}{p(x,y)(e+\alpha|D_s u_n^-|)}\right]|D_s u_n^-|^{p(x,y)}\right.\\
&\quad\left.+\mu(x,y)\left[\log(e+\alpha|D_s u_n^-|) + \frac{\alpha|D_s u_n^-|}{q(x,y)(e+\alpha|D_s u_n^-|)}\right]|D_s u_n^-|^{q(x,y)}\right\}\mathrm{d}\nu\\
&\leq \varepsilon_n \quad \text{for all } n\in\mathbb{N}.
\end{aligned}$$

Proposition A.7 then yields

$$-u_n^- \to 0 \quad \text{in } \widetilde{W}_0^{s,\mathcal{H}}(\Omega). \tag{4.7}$$

Next, we claim that one can find a constant $C > 0$ satisfying $[u_n^+]_{s,\mathcal{H},Q} \leq C$ for all $n\in\mathbb{N}$.

Conversely, suppose that

$$[u_n^+]_{s,\mathcal{H},Q} \to +\infty \quad \text{as } n\to+\infty.$$

Setting $y_n = \frac{u_n^+}{[u_n^+]_{s,\mathcal{H},Q}}$ for $n\in\mathbb{N}$ implies $[y_n]_{s,\mathcal{H},Q} = 1$ and $y_n \geq 0$ for all $n\in\mathbb{N}$. By the reflexivity of $\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$, there exists $0 \leq y \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ fulfilling

$$y_n \rightharpoonup y \quad \text{in } \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{r(x)}(\Omega). \tag{4.8}$$

Case 1: $y \neq 0$. The set $\Omega_+ := \{x \in \Omega \colon y(x) > 0\}$ has positive Lebesgue measure, so we get from (4.8)

$$u_n^+ \to +\infty \quad \text{for a.a. } x \in \Omega_+.$$

Taking $v = u_n^+ \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ in (4.6) and utilizing Lemma 4.2, we have

$$\begin{aligned}
&- \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) \Psi(\tilde{I}_{s,\mathcal{H}}(u_n^+)) + \int_\Omega f(x, u_n^+) u_n^+ \,\mathrm{d}x \\
&\le -\psi\left(\tilde{I}_{s,\mathcal{H}}\left(u_n^+\right)\right) \int_Q \left(\left[\log(e + \alpha|D_s u_n^+|) + \frac{\alpha|D_s u_n^+|}{p(x,y)(e + \alpha|D_s u_n^+|)}\right] |D_s u_n^+|^{p(x,y)} \right. \\
&\quad \left. + \mu(x,y) \left[\log(e + \alpha|D_s u_n^+|) + \frac{\alpha|D_s u_n^+|}{q(x,y)(e + \alpha|D_s u_n^+|)}\right] |D_s u_n^+|^{q(x,y)} \right) \mathrm{d}\nu \\
&\quad + \int_\Omega f\left(x, u_n^+\right) u_n^+ \,\mathrm{d}x \le \varepsilon_n
\end{aligned} \tag{4.9}$$

for all $n \in \mathbb{N}$. From (4.4) and (4.7), we can find $C_2 > 0$ such that

$$\varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) \Psi(\tilde{I}_{s,\mathcal{H}}(u_n^+)) - \int_\Omega \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) F(x, u_n^+) \,\mathrm{d}x \le C_2 \tag{4.10}$$

for all $n \in \mathbb{N}$. Adding (4.9) and (4.10) gives

$$\int_\Omega f(x, u_n^+) u_n^+ - \int_\Omega \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) F(x, u_n^+) \,\mathrm{d}x \le C_3$$

for all $n \in \mathbb{N}$ and for some $C_3 > 0$. This contradicts (f4).

Case 2: $y \equiv 0$. Take $\lambda \ge 1$, and define

$$v_n = \lambda y_n \quad \text{for all } n \in \mathbb{N}.$$

From (4.8), we get

$$v_n \to 0 \quad \text{in } \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \quad \text{and} \quad v_n \to 0 \quad \text{in } L^{r(x)}(\Omega).$$

Hence,

$$\int_\Omega F\left(x, v_n\right) \mathrm{d}x \to 0. \tag{4.11}$$

Next, let $t_n \in [0,1]$ be such that

$$E_+\left(t_n u_n^+\right) = \max\left\{E_+\left(t u_n^+\right) : t \in [0,1]\right\}. \tag{4.12}$$

Due to $[u_n^+]_{s,\mathcal{H},Q} \to \infty$, there exists $n_0 \in \mathbb{N}$ satisfying

$$0 < \frac{\lambda}{[u_n^+]_{s,\mathcal{H},Q}} \le 1 \quad \text{for all } n \ge n_0. \tag{4.13}$$

Then, applying (4.11) to (4.13), we get

$$\begin{aligned}
E_+(t_n u_n^+) &\ge \Psi(\tilde{I}(\lambda y_n)) - \int_\Omega F(x, v_n) \,\mathrm{d}x \\
&\ge \lambda^{\varsigma p_-} \Psi(\tilde{I}(y_n)) - \int_\Omega F(x, v_n) \,\mathrm{d}x \to +\infty
\end{aligned}$$

for λ large enough, which means

$$E_+(t_n u_n^+) \to +\infty \quad \text{as } n \to +\infty. \tag{4.14}$$

However, by (4.4) we see that for some $C_4 > 0$

$$E_+(0) = 0 \quad \text{and} \quad E_+(u_n^+) \le C_4 \tag{4.15}$$

for all $n \in \mathbb{N}$. From (4.14) and (4.15), one can find $n_1 \in \mathbb{N}$ such that

$$t_n \in (0,1) \quad \text{for all } n \ge n_1. \tag{4.16}$$

Thus, utilizing the chain rule along with (4.12) and (4.16) we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} E_+\left(t u_n^+\right)\Big|_{t=t_n} = \left\langle E_+'\left(t_n u_n^+\right), u_n^+\right\rangle \quad \text{for all } n \ge n_1,$$

which means

$$\begin{aligned}
&\psi\left(\tilde{I}_{s,\mathcal{H}}\left(t_n u_n^+\right)\right) \int_Q \left(\left[\log(e + \alpha|D_s(t_n u_n^+)|) + \frac{\alpha|D_s(t_n u_n^+)|}{p(x,y)(e+\alpha|D_s(t_n u_n^+)|)}\right] |D_s(t_n u_n^+)|^{p(x,y)}\right.\\
&\qquad \left. + \mu(x,y)\left[\log(e + \alpha|D_s(t_n u_n^+)|) + \frac{\alpha|D_s(t_n u_n^+)|}{q(x,y)(e+\alpha|D_s(t_n u_n^+)|)}\right] |D_s(t_n u_n^+)|^{q(x,y)}\right) \mathrm{d}\nu\\
&= \int_\Omega f\left(x, t_n u_n^+\right) t_n u_n \,\mathrm{d}x.
\end{aligned} \tag{4.17}$$

By (4.9), (4.10), (4.17), and hypotheses (f4), we arrive at

$$\begin{aligned}
&\varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) E_+(t_n u_n^+)\\
&= \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) \Psi(\tilde{I}_{s,\mathcal{H}}(t_n u_n^+)) - \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) \int_\Omega F(x, t_n u_n^+)\,\mathrm{d}x\\
&= \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) \Psi(\tilde{I}_{s,\mathcal{H}}(t_n u_n^+)) - \psi\left(\tilde{I}_{s,\mathcal{H}}\left(t_n u_n^+\right)\right) \int_Q \Bigg(\Big[\log(e + \alpha|D_s(t_n u_n^+)|)\\
&\quad + \frac{\alpha|D_s(t_n u_n^+)|}{p(x,y)(e+\alpha|D_s(t_n u_n^+)|)}\Big] |D_s(t_n u_n^+)|^{p(x,y)}\\
&\quad + \mu(x,y)\left[\log(e + \alpha|D_s(t_n u_n^+)|) + \frac{\alpha|D_s(t_n u_n^+)|}{q(x,y)(e+\alpha|D_s(t_n u_n^+)|)}\right] |D_s(t_n u_n^+)|^{q(x,y)}\Bigg) \mathrm{d}\nu\\
&\quad + \int_\Omega f\left(x, t_n u_n^+\right) t_n u_n \,\mathrm{d}x - \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) \int_\Omega F(x, t_n u_n^+)\,\mathrm{d}x\\
&\le \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) \Psi(\tilde{I}_{s,\mathcal{H}}(u_n^+)) - \psi\left(\tilde{I}_{s,\mathcal{H}}\left(u_n^+\right)\right) \int_Q \Bigg(\Big[\log(e + \alpha|D_s(u_n^+)|)\\
&\quad + \frac{\alpha|D_s(u_n^+)|}{p(x,y)(e+\alpha|D_s(u_n^+)|)}\Big] |D_s(u_n^+)|^{p(x,y)}\\
&\quad + \mu(x,y)\left[\log(e + \alpha|D_s(u_n^+)|) + \frac{\alpha|D_s(u_n^+)|}{q(x,y)(e+\alpha|D_s(u_n^+)|)}\right] |D_s(u_n^+)|^{q(x,y)}\Bigg) \mathrm{d}\nu\\
&\quad + \int_\Omega f\left(x, u_n^+\right) u_n \,\mathrm{d}x - \varsigma q_+ \left(1 + \frac{\kappa}{p_-}\right) \int_\Omega F(x, u_n^+)\,\mathrm{d}x\\
&\le C_5
\end{aligned}$$

for all $n \ge n_1$ and some $C_5 > 0$, which contradicts (4.14). This proves the claim.

From the claim and (4.7), it follows that $\{u_n\}_{n\geq 1}$ is bounded in $\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$. Thus, we can find $u\in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{r(x)}(\Omega). \tag{4.18}$$

Taking $v=u_n-u$ in (4.6), we have

$$\begin{aligned}
&\psi\left(\tilde{I}_{s,\mathcal{H}}(u_n)\right)\int_Q\Bigg(\left[\log(e+\alpha|D_su_n|)+\frac{\alpha|D_su_n|}{p(x,y)(e+\alpha|D_su_n|)}\right]\\
&\times|D_su_n|^{p(x,y)-2}\cdot D_su_n\cdot D_s(u_n-u)\\
&+\mu(x,y)\left[\log(e+\alpha|D_su_n|)+\frac{\alpha|D_su_n|}{q(x,y)(e+\alpha|D_su_n|)}\right]\\
&\times|D_su_n|^{q(x,y)-2}\cdot D_su_n\cdot D_s(u_n-u)\Bigg)\,\mathrm{d}\nu\\
&-\int_\Omega f\left(x,u_n^+\right)(u_n-u)\,\mathrm{d}x\leq\varepsilon_n[(u_n-u)]_{s,\mathcal{H},Q}.
\end{aligned}\tag{4.19}$$

Passing to the limes superior as $n\to\infty$ in (4.19) and applying (4.18) along with hypotheses (f1), we get

$$\limsup_{n\to\infty}\langle\mathcal{J}(u_n),u_n-u\rangle\leq 0.$$

Utilizing the (S$_+$)-property of the operator $\mathcal{J}$ (see Proposition 2.4), we infer that $u_n\to u$ in $\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$. Hence, E_+ satisfies the Cerami condition. In a similar way, one can show that E_- fulfills the Cerami condition as well. ∎

PROPOSITION 4.4. *Let hypotheses* (H1), (H5), *and* (f1)–(f4) *be satisfied. Then there exists* $\delta>0$ *satisfying* $E(u)>0$ *and* $E_\pm(u)>0$ *for* $0<[u]_{s,\mathcal{H},Q}<\delta$.

Proof. We only show that $E(u)>0$ for $0<[u]_{s,\mathcal{H},Q}<\delta$ with $\delta>0$ small enough. The remaining proofs for $E_\pm$ are very similar. Suppose that $[u]_{s,\mathcal{H},Q}<1$.

Case 1: $\boldsymbol{\theta_1>0}$. Then, by Propositions A.6, A.7, and 2.2 along with (A.1) and Remark 4.1(iii), we obtain

$$\begin{aligned}
E(u)&>\theta_1\tilde{I}_{s,\mathcal{H}}(u)-\int_\Omega F(x,u)\,\mathrm{d}x\\
&\geq\theta_1\tilde{I}_{s,\mathcal{H}}(u)-\frac{\varepsilon}{p_-}\rho_{p(\cdot)}(u)-C_\varepsilon\rho_{r(\cdot)}(u)\\
&\geq\theta_1\tilde{I}_{s,\mathcal{H}}(u)-\frac{\varepsilon p_+}{p_-}\int_\Omega\widehat{\mathcal{H}}(u)\,\mathrm{d}x-C_\varepsilon\rho_{r(\cdot)}(u)\\
&\geq\theta_1\tilde{I}_{s,\mathcal{H}}(u)-\frac{\lambda_1\varepsilon p_+}{p_-}\tilde{I}_{s,\mathcal{H}}(u)-C_\varepsilon\rho_{r(\cdot)}(u)\\
&\geq\left(\frac{\theta_1}{C_\eta}-\frac{\lambda_1\varepsilon p_+}{C_\eta p_-}\right)[u]_{s,\mathcal{H},Q}^{q_++\eta}-C_\varepsilon\max_{k\in\{r_+,r_-\}}\left\{C_{e1}^k[u]_{s,\mathcal{H},Q}^k\right\}\\
&\geq[u]_{s,\mathcal{H},Q}^{q_++\eta}\left(\frac{\theta_1}{C_\eta}-\frac{\lambda_1\varepsilon p_+}{C_\eta p_-}-C_\varepsilon\max\left\{C_{e1}^{r_-}[u]_{s,\mathcal{H},Q}^{r_--q_+-\eta},C_{e1}^{r_+}[u]_{s,\mathcal{H},Q}^{r_+-q_+-\eta}\right\}\right),
\end{aligned}$$

where $\eta,C_\eta>0$ and C_{e1} is the embedding constant of $\widetilde{W}_0^{s,\mathcal{H}}(\Omega)\hookrightarrow L^{r(\cdot)}(\Omega)$. Since $q_+<r_-$, we take $0<\eta<r_--q_+$, and if we let

$$[u]_{s,\mathcal{H},Q} \le \delta_1 := \min\left\{1, \left(\frac{\theta_1}{C_\eta C_\varepsilon C_{e1}^{r_-}} - \frac{\lambda_1 p_+ \varepsilon}{C_\eta C_\varepsilon C_{e1}^{r_-} p_-}\right)^{\frac{1}{r_- - q_+ - \eta}}, \right.$$
$$\left. \left(\frac{\theta_1}{C_\eta C_\varepsilon C_{e1}^{r_+}} - \frac{\lambda_1 p_+ \varepsilon}{C_\eta C_\varepsilon C_{e1}^{r_+} p_-}\right)^{\frac{1}{r_+ - q_+ - \eta}}\right\},$$

then $E(u) > 0$.

Case 2: $\boldsymbol{\theta_1 = 0}$. Then, by Propositions A.6, A.7, and 2.2 along with (A.1), Remark 4.1(iii), and $0 < \eta < \frac{\varepsilon}{\varsigma}$ (ε is the constant given in (4.3)), we get

$$\begin{aligned}
E(u) &= \Psi(\tilde{I}_{s,\mathcal{H}}(u)) - \int_\Omega F(x,u)\,\mathrm{d}x \\
&\ge \Psi(\tilde{I}_{s,\mathcal{H}}(u)) - \frac{\varepsilon}{\varsigma p_-}\int_\Omega |u|^{\varsigma p_+ + \varepsilon}\,\mathrm{d}x - C_\varepsilon \rho_{r(\cdot)}(u) \\
&\ge \frac{\theta_2}{\varsigma}(\tilde{I}_{s,\mathcal{H}}(u))^\varsigma - \frac{\varepsilon}{\varsigma p_-}\|u\|_{L^{\varsigma p_+ + \varepsilon}}^{\varsigma p_+ + \varepsilon} - C_\varepsilon \rho_{r(\cdot)}(u) \\
&\ge \frac{\theta_2}{\varsigma}(\tilde{I}_{s,\mathcal{H}}(u))^\varsigma - \frac{\varepsilon C_{e2}}{\varsigma p_-}[u]_{s,\mathcal{H},Q}^{\varsigma(p_+ + \frac{\varepsilon}{\varsigma})} - C_\varepsilon \rho_{r(\cdot)}(u) \\
&\ge \frac{\theta_2}{\varsigma}(\tilde{I}_{s,\mathcal{H}}(u))^\varsigma - \frac{\varepsilon C_{e2}}{\varsigma p_-}\left[(C_\eta)^{\frac{p_+ + \frac{\varepsilon}{\varsigma}}{p_+ + \eta}}\tilde{I}_{s,\mathcal{H}}(u)^{\frac{p_+ + \frac{\varepsilon}{\varsigma}}{p_+ + \eta}}\right]^\varsigma - C_\varepsilon \rho_{r(\cdot)}(u) \\
&\ge \left(\frac{\theta_2}{\varsigma} - \frac{\varepsilon C_{e2}}{\varsigma p_-}(C_\eta)^{\frac{p_+ + \frac{\varepsilon}{\varsigma}}{p_+ + \eta}}\right)(\tilde{I}_{s,\mathcal{H}}(u))^\varsigma - C_\varepsilon \rho_{r(\cdot)}(u) \\
&\ge \left(\frac{\theta_2}{\varsigma} - \frac{\varepsilon C_{e2}}{\varsigma p_-}(C_\eta)^{\frac{p_+ + \frac{\varepsilon}{\varsigma}}{p_+ + \eta}}\right)\left(\frac{1}{C_{\eta'}}\right)^\varsigma [u]_{s,\mathcal{H},Q}^{(q_+ + \eta')\varsigma} - C_\varepsilon \max_{k\in\{r_+, r_-\}}\left\{C_{e1}^k [u]_{s,\mathcal{H},Q}^k\right\} \\
&\ge [u]_{s,\mathcal{H},Q}^{(q_+ + \eta')\varsigma}\left[\left(\frac{\theta_2}{\varsigma} - \frac{\varepsilon C_{e2}}{\varsigma p_-}(C_\eta)^{\frac{p_+ + \frac{\varepsilon}{\varsigma}}{p_+ + \eta}}\right)\left(\frac{1}{C_{\eta'}}\right)^\varsigma \right. \\
&\qquad \left. - C_\varepsilon \max\left\{C_{e1}^{r_-}[u]_{s,\mathcal{H},Q}^{r_- - \varsigma(q_+ + \eta')}, C_{e1}^{r_+}[u]_{s,\mathcal{H},Q}^{r_+ - \varsigma(q_+ + \eta')}\right\}\right],
\end{aligned}$$

where $\eta, \eta', C_{\eta'} > 0$ and C_{e2} is the embedding constant of $\widetilde{W}_0^{s,\mathcal{H}}(\Omega) \hookrightarrow L^{\varsigma p_+ + \varepsilon}(\Omega)$.

Since $\varsigma q_+ < r_-$, we take $0 < \eta' < \frac{r_- - \varsigma q_+}{\varsigma}$, and if we let

$$[u]_{s,\mathcal{H},Q} \le \delta_2 := \min\left\{1, \left[\left(\frac{\theta_2}{\varsigma} - \frac{\varepsilon C_{e2}}{\varsigma p_-}(C_\eta)^{\frac{p_+ + \frac{\varepsilon}{\varsigma}}{p_+ + \eta}}\right)\left(\frac{1}{C_{\eta'}}\right)^\varsigma \cdot \frac{1}{C_\varepsilon C_{e1}^{r_-}}\right]^{\frac{1}{r_- - \varsigma(q_+ + \eta')}} \right.$$
$$\left. \left[\left(\frac{\theta_2}{\varsigma} - \frac{\varepsilon C_{e2}}{\varsigma p_-}(C_\eta)^{\frac{p_+ + \frac{\varepsilon}{\varsigma}}{p_+ + \eta}}\right)\left(\frac{1}{C_{\eta'}}\right)^\varsigma \cdot \frac{1}{C_\varepsilon C_{e1}^{r_+}}\right]^{\frac{1}{r_+ - \varsigma(q_+ + \eta')}}\right\},$$

then $E(u) > 0$. Finally, choosing $\delta = \min\{\delta_1, \delta_2\}$ completes the proof. □

PROPOSITION 4.5. *Let hypotheses* (H1), (H5), *and* (f1)–(f4) *be satisfied. Then, for* $0 \ne u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$, *there holds that* $E(tu) \to -\infty$ *as* $t \to \pm\infty$. *Moreover, if* $u \ge 0$ *a.e. in* Ω, *then* $E_\pm(tu) \to -\infty$ *as* $t \to \pm\infty$.

Proof. We only need to prove the assertion for E since under the case that $u \ge 0$ a.e. in Ω, we have $E_\pm(tu) = E(tu)$ for $\pm t > 0$.

Take any fixed $0 \ne u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$, and let $t > 1, \varepsilon > 0$. Note that

$$\log(e + ab) \le \log(e + a) + \log(e + b) \quad \text{for all } a, b > 0. \tag{4.20}$$

According to (4.20) and Remark 4.1(iv), we calculate that

$$\begin{aligned} E(tu) \leq \theta_1 \Bigg[& \frac{|t|^{p_+}}{p_-} \log(e+\alpha|t|) \int_Q |D_s u|^{p(x,y)} \,\mathrm{d}\nu + \frac{|t|^{p_+}}{p_-} \int_Q |D_s u|^{p(x,y)} \log(e+\alpha|D_s u|) \,\mathrm{d}\nu \\ & + \mu(x,y) \frac{|t|^{q_+}}{q_-} \log(e+\alpha|t|) \int_Q |D_s u|^{q(x,y)} \,\mathrm{d}\nu \\ & + \mu(x,y) \frac{|t|^{q_+}}{q_-} \int_Q |D_s u|^{q(x,y)} \log(e+\alpha|D_s u|) \,\mathrm{d}\nu \Bigg] \\ & + \frac{4\theta_2}{\varsigma} (\tilde{I}_{s,\mathcal{H}}(u))^{\varsigma} |t|^{\varsigma q_+} \log^{\varsigma}(e+\alpha|t|) - \frac{\varepsilon \|u\|_{L^{\varsigma q_+}}^{\varsigma q_+}}{\varsigma q_+} |t|^{\varsigma q_+} \log^{\varsigma}(e+\alpha|t|) + C_\varepsilon |\Omega|, \end{aligned}$$

which implies $E(tu) \to -\infty$ as $t \to \pm\infty$ for ε large enough. ∎

Now we are able to prove the existence of constant sign weak solutions of problem (4.1).

THEOREM 4.6. *Let hypotheses* (H1), (H5), *and* (f1)–(f4) *be satisfied. Then problem* (4.1) *possesses at least two nontrivial weak solutions* $u_1, u_2 \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ *satisfying*

$$u_1(x) \geq 0 \quad \textit{and} \quad u_2(x) \leq 0 \quad \textit{for a.a. } x \in \Omega.$$

Proof. According to Propositions 4.3, 4.4, and 4.5, we see that $E_\pm$ fulfill the conditions of the mountain pass theorem stated in Theorem 2.6. Hence, there exist $u_1, u_2 \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ fulfilling $E_+'(u_1) = 0$ and $E_-'(u_2) = 0$. Also it follows that

$$\begin{aligned} E_+(u_1) &\geq \inf_{[u]_{s,\mathcal{H},Q}=\delta} E_\pm(u) > 0 = E_+(0), \\ E_-(u_2) &\geq \inf_{[u]_{s,\mathcal{H},Q}=\delta} E_\pm(u) > 0 = E_-(0), \end{aligned}$$

and thus $u_1 \neq 0$ and $u_2 \neq 0$. Moreover, testing $E_+(u_1) = 0$ with $-u_1^-$ yields $\tilde{I}_{s,\mathcal{H}}(u_1^-) = 0$, which implies that $-u_1^- = 0$ a.e. in Ω. Hence, $u_1 \geq 0$ a.e. in Ω. Using similar arguments, we get $u_2 \leq 0$ a.e. in Ω, and the proof is finished. ∎

4.2. Existence of sign-changing solutions. As discussed before, any sign-changing solution of (4.1) belongs to the constraint set

$$\mathcal{N} = \left\{ u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \colon u^\pm \neq 0, \left\langle E'(u), u^+ \right\rangle = \left\langle E'(u), -u^- \right\rangle = 0 \right\}.$$

First, we will study properties of the set $\mathcal{N}$.

PROPOSITION 4.7. *Let hypotheses* (H1), (H5), *and* (f1)–(f5) *be satisfied, and let* $u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ *be such that* $u^\pm \neq 0$. *Then there exist* $\gamma_u, \beta_u > 0$ *satisfying* $\gamma_u u^+ - \beta_u u^- \in \mathcal{N}$. *Furthermore, if* $u \in \mathcal{N}$, *then for all* $s_1, s_2 > 0$ *there holds that*

$$E\left(s_1 u^+ - s_2 u^-\right) \leq E\left(u^+ - u^-\right) = E(u),$$

and the above inequality is strict if $(s_1, s_2) \neq (1,1)$.

Proof. We divide the proof into three steps.
Step 1. We prove the existence of $0 < \gamma_u, \beta_u < \infty$ such that $\gamma_u u^+ - \beta_u u^- \in \mathcal{N}$.

Due to (f5), for $t \in (0,1)$ and $|u(x)| > 0$ a.e. in Ω there holds that

$$\frac{f(x,tu)(tu)}{t^{(q_+ +1)\varsigma} |u|^{(q_+ +1)\varsigma}} \leq \frac{f(x,u)u}{|u|^{(q_+ +1)\varsigma}} \quad \text{for a.a. } x \in \Omega,$$

which implies that

$$f(x,tu)u \le t^{(q_+ +1)\varsigma -1} f(x,u)u \quad \text{for a.a. } x\in\Omega. \tag{4.21}$$

For $0<\gamma<1$ small enough and all $\beta>0$, by applying (4.20) and (4.21), we get that

$$\begin{aligned}
&\langle E'(\gamma u^+ - \beta u^-, \gamma u^+)\rangle \\
&= \Big[\theta_1 + \theta_2 (\tilde{I}_{s,\mathcal{H}}(\gamma u^+ - \beta u^-))^{\varsigma-1}\Big] \\
&\quad \times \int_Q \bigg(\bigg[\log(e+\alpha|D_s(\gamma u^+ - \beta u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta u^-)|}{p(x,y)(e+\alpha|D_s(\gamma u^+ - \beta u^-)|)}\bigg] \\
&\quad \times |D_s(\gamma u^+ - \beta u^-)|^{p(x,y)-2} \big(D_s(\gamma u^+ - \beta u^-)\big)\big(D_s(\gamma u^+)\big) \\
&\quad + \mu(x,y) \bigg[\log(e+\alpha|D_s(\gamma u^+ - \beta u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta u^-)|}{q(x,y)(e+\alpha|D_s(\gamma u^+ - \beta u^-)|)}\bigg] \\
&\quad \times |D_s(\gamma u^+ - \beta u^-)|^{q(x,y)-2} \big(D_s(\gamma u^+ - \beta u^-)\big)\big(D_s(\gamma u^+)\big) \bigg) \,\mathrm{d}\nu \\
&\quad - \int_\Omega f(x,\gamma u^+)\gamma u^+ \,\mathrm{d}x \\
&= \Big[\theta_1 + \theta_2 (\tilde{I}_{s,\mathcal{H}}(\gamma u^+ - \beta u^-))^{\varsigma-1}\Big] \\
&\quad \times \int_Q \bigg(\bigg[\log(e+\alpha|D_s(\gamma u^+ - \beta u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta u^-)|}{p(x,y)(e+\alpha|D_s(\gamma u^+ - \beta u^-)|)}\bigg] \\
&\quad \times |D_s(\gamma u^+ - \beta u^-)|^{p(x,y)-2} \bigg[\big(D_s(\gamma u^+)\big)^2 + \frac{2\gamma\beta u^+(x)u^-(y)}{|x-y|^{2s}}\bigg] \\
&\quad + \mu(x,y) \bigg[\log(e+\alpha|D_s(\gamma u^+ - \beta u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta u^-)|}{q(x,y)(e+\alpha|D_s(\gamma u^+ - \beta u^-)|)}\bigg] \\
&\quad \times |D_s(\gamma u^+ - \beta u^-)|^{q(x,y)-2} \bigg[\big(D_s(\gamma u^+)\big)^2 + \frac{2\gamma\beta u^+(x)u^-(y)}{|x-y|^{2s}}\bigg] \bigg) \,\mathrm{d}\nu \\
&\quad - \int_\Omega f(x,\gamma u^+)\gamma u^+ \,\mathrm{d}x \\
&\ge \Big[\theta_1 + \theta_2 (\tilde{I}_{s,\mathcal{H}}(\gamma u^+ - \beta u^-))^{\varsigma-1}\Big] \\
&\quad \times \int_Q \bigg(\bigg[\log(e+\alpha|D_s(\gamma u^+)|) + \frac{\alpha|D_s(\gamma u^+)|}{p(x,y)(e+\alpha|D_s(\gamma u^+)|)}\bigg] |D_s(\gamma u^+)|^{p(x,y)} \\
&\quad + \mu(x,y) \bigg[\log(e+\alpha|D_s(\gamma u^+)|) + \frac{\alpha|D_s(\gamma u^+)|}{q(x,y)(e+\alpha|D_s(\gamma u^+)|)}\bigg] |D_s(\gamma u^+)|^{q(x,y)} \bigg) \,\mathrm{d}\nu \\
&\quad - \int_\Omega f(x,\gamma u^+)\gamma u^+ \,\mathrm{d}x \\
&\ge \frac{\theta_2}{p_+^{\varsigma-1}} \gamma^{\varsigma p_+} \left(\int_Q |D_s(u^+)|^{p(x,y)} \,\mathrm{d}\nu\right)^{\varsigma} - \gamma^{\varsigma(q_+ +1)} \int_\Omega f(x,u^+)u^+ \,\mathrm{d}x > 0.
\end{aligned}$$

Analogously, for all $\gamma>0$ and $0<\beta<1$ small enough we have

$$\begin{aligned}
&\langle E'(\gamma u^+ - \beta u^-, -\beta u^-)\rangle \\
&= \Big[\theta_1 + \theta_2 (\tilde{I}_{s,\mathcal{H}}(\gamma u^+ - \beta u^-))^{\varsigma-1}\Big] \\
&\quad \times \int_Q \bigg(\bigg[\log(e+\alpha|D_s(\gamma u^+ - \beta u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta u^-)|}{p(x,y)(e+\alpha|D_s(\gamma u^+ - \beta u^-)|)}\bigg] \\
&\quad \times |D_s(\gamma u^+ - \beta u^-)|^{p(x,y)-2} \big(D_s(\gamma u^+ - \beta u^-)\big)\big(D_s(-\beta u^-)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\mu(x,y)\left[\log(e+\alpha|D_s(\gamma u^+-\beta u^-)|)+\frac{\alpha|D_s(\gamma u^+-\beta u^-)|}{q(x,y)(e+\alpha|D_s(\gamma u^+-\beta u^-)|)}\right]\\
&\times|D_s(\gamma u^+-\beta u^-)|^{q(x,y)-2}\left(D_s(\gamma u^+-\beta u^-)\right)\left(D_s(-\beta u^-)\right)\Bigg)\,\mathrm{d}\nu\\
&-\int_\Omega f(x,-\beta u^-)(-\beta u^-)\,\mathrm{d}x\\
=&\left[\theta_1+\theta_2(\tilde{I}_{s,\mathcal{H}}(\gamma u^+-\beta u^-))^{\varsigma-1}\right]\\
&\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(\gamma u^+-\beta u^-)|)+\frac{\alpha|D_s(\gamma u^+-\beta u^-)|}{p(x,y)(e+\alpha|D_s(\gamma u^+-\beta u^-)|)}\right]\\
&\times|D_s(\gamma u^+-\beta u^-)|^{p(x,y)-2}\left[\left(D_s(-\beta u^-)\right)^2+\frac{2\gamma\beta u^+(x)u^-(y)}{|x-y|^{2s}}\right]\\
&+\mu(x,y)\left[\log(e+\alpha|D_s(\gamma u^+-\beta u^-)|)+\frac{\alpha|D_s(\gamma u^+-\beta u^-)|}{q(x,y)(e+\alpha|D_s(\gamma u^+-\beta u^-)|)}\right]\\
&\times|D_s(\gamma u^+-\beta u^-)|^{q(x,y)-2}\left[\left(D_s(-\beta u^-)\right)^2+\frac{2\gamma\beta u^+(x)u^-(y)}{|x-y|^{2s}}\right]\Bigg)\,\mathrm{d}\nu\\
&-\int_\Omega f(x,-\beta u^-)(-\beta u^-)\,\mathrm{d}x\\
\geq&\left[\theta_1+\theta_2(\tilde{I}_{s,\mathcal{H}}(\gamma u^+-\beta u^-))^{\varsigma-1}\right]\\
&\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(\beta u^-)|)+\frac{\alpha|D_s(\beta u^-)|}{p(x,y)(e+\alpha|D_s(\beta u^-)|)}\right]|D_s(\beta u^-)|^{p(x,y)}\\
&+\mu(x,y)\left[\log(e+\alpha|D_s(\beta u^-)|)+\frac{\alpha|D_s(\beta u^-|}{q(x,y)(e+\alpha|D_s(\beta u^-)|)}\right]|D_s(\beta u^-)|^{q(x,y)}\Bigg)\,\mathrm{d}\nu\\
&-\int_\Omega f(x,-\beta u^-)(-\beta u^-)\,\mathrm{d}x\\
\geq&\frac{\theta_2}{p_+^{\varsigma-1}}\beta^{\varsigma p_+}\left(\int_Q|D_s(u^-)|^{p(x,y)}\,\mathrm{d}\nu\right)^\varsigma-\beta^{\varsigma(q_++1)}\int_\Omega f(x,-u^-)(-u^-)\,\mathrm{d}x>0.
\end{aligned}
$$

Thus, we deduce from the above inequalities that for all $\gamma,\beta>0$ there exists $t_1>0$ satisfying

$$
\left\langle E'\left(t_1u^+-\beta u^-\right),t_1u^+\right\rangle>0\quad\text{and}\quad\left\langle E'\left(\gamma u^+-t_1u^-\right),-t_1u^-\right\rangle>0. \tag{4.22}
$$

Next, we set $t_2>\max\{1,t_1\}$ and note that there exists $C_\eta>0$ such that

$$
\log(e+t)\leq C_\eta t^\eta\quad\text{for all } t>1 \text{ and } \eta>0. \tag{4.23}
$$

Then, by (4.20), hypotheses (f2), and (4.23), for $0<\beta<t_2$ and η, C_η, $C'_\eta>0$ it holds that

$$
\begin{aligned}
&\frac{\langle E'(t_2u^+-\beta u^-,t_2u^+)\rangle}{t_2^{\varsigma q_++\eta}}\\
&=\frac{\left[\theta_1+\theta_2(\tilde{I}_{s,\mathcal{H}}(t_2u^+-\beta u^-))^{\varsigma-1}\right]}{t_2^{\varsigma q_++\eta}}\\
&\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(t_2u^+-\beta u^-)|)+\frac{\alpha|D_s(t_2u^+-\beta u^-)|}{p(x,y)(e+\alpha|D_s(t_2u^+-\beta u^-)|)}\right]\\
&\times|D_s(t_2u^+-\beta u^-)|^{p(x,y)-2}\left(D_s(t_2u^+-\beta u^-)\right)\left(D_s(t_2u^+)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mu(x,y)\left[\log(e+\alpha|D_s(t_2u^+-\beta u^-)|)+\frac{\alpha|D_s(t_2u^+-\beta u^-)|}{q(x,y)(e+\alpha|D_s(t_2u^+-\beta u^-)|)}\right]\\
&\times|D_s(t_2u^+-\beta u^-)|^{q(x,y)-2}\left(D_s(t_2u^+-\beta u^-)\right)\left(D_s(t_2u^+)\right)\Bigg)\,\mathrm{d}\nu\\
&-\int_\Omega\frac{f(x,t_2u^+)t_2u^+}{t_2^{\varsigma q_++\eta}}\,\mathrm{d}x\\
\le{}&\frac{\left[\theta_1+\theta_2(\tilde{I}_{s,\mathcal{H}}(t_2u^+-\beta u^-))^{\varsigma-1}\right]}{t_2^{\varsigma q_++\eta}}\\
&\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(t_2u^+-\beta u^-)|)+\frac{\alpha|D_s(t_2u^+-\beta u^-)|}{p(x,y)(e+\alpha|D_s(t_2u^+-\beta u^-)|)}\right]\\
&\times|D_s(t_2u^+-\beta u^-)|^{p(x,y)-2}\left(D_s(t_2u^+-\beta u^-)\right)^2\\
&+\mu(x,y)\left[\log(e+\alpha|D_s(t_2u^+-\beta u^-)|)+\frac{\alpha|D_s(t_2u^+-\beta u^-)|}{q(x,y)(e+\alpha|D_s(t_2u^+-\beta u^-)|)}\right]\\
&\times|D_s(t_2u^+-\beta u^-)|^{q(x,y)-2}\left(D_s(t_2u^+-\beta u^-)\right)^2\Bigg)\,\mathrm{d}\nu\\
&-\int_\Omega\frac{f(x,t_2u^+)t_2u^+}{t_2^{\varsigma q_++\eta}}\,\mathrm{d}x\\
\le{}&\left[\frac{2\theta_1C_\eta}{t_2^{q_+(\varsigma-1)}}+2^\varsigma\theta_2C'_\eta(\tilde{I}_{s,\mathcal{H}}(u^+-u^-))^{\varsigma-1}\right]\\
&\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{p(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{p(x,y)}\\
&+\mu(x,y)\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{q(x,y)}\Bigg)\,\mathrm{d}\nu\\
&-\int_\Omega\frac{f(x,t_2u^+)}{(t_2u^+)^{\varsigma q_++\eta-1}}(u^+)^{\varsigma q_++\eta}\,\mathrm{d}x\\
<{}&0.
\end{aligned}
$$

Note that the last inequality holds for t_2 large enough. Similarly, for t_2 large enough and $0<\gamma<t_2$ there holds that

$$\frac{\langle E'(\gamma u^+-t_2u^-,-t_2u^-)\rangle}{t_2^{\varsigma q_++\eta}}<0.$$

By the above inequalities, we obtain

$$\left\langle E'\left(t_2u^+-\beta u^-\right),t_2u^+\right\rangle<0\quad\text{and}\quad\left\langle E'\left(\gamma u^+-t_2u^-\right),-t_2u^-\right\rangle<0, \tag{4.24}$$

with $0<\gamma,\beta<t_2$, and $t_2>0$ large enough. We define the mapping $T_u\colon[0,\infty)\times[0,\infty)\to\mathbb{R}^2$ by $T_u(\gamma,\beta)=(\langle E'(\gamma u^+-\beta u^-),\gamma u^+\rangle,\langle E'(\gamma u^+-\beta u^-),-\beta u^-\rangle)$. Thus, considering (4.22), (4.24), and Theorem 2.8 (Poincaré–Miranda existence theorem), one can find a pair $(\gamma_u,\beta_u)\in(0,\infty)\times(0,\infty)$ satisfying $T_u(\gamma_u,\beta_u)=(0,0)$, which indicates that $\gamma_uu^+-\beta_uu^-\in\mathcal{N}$.

Step 2. We show the uniqueness of the pair (γ_u,β_u) obtained in Step 1.

We claim that for every $u\in\mathcal{N}$ we have

$$\left\langle E'\left(\gamma u^+-\beta u^-\right),\gamma u^+\right\rangle<0 \text{ for } \gamma>1, 0<\beta<\gamma, \tag{4.25}$$

$$\left\langle E'\left(\gamma u^+-\beta u^-\right),\gamma u^+\right\rangle>0 \text{ for } \gamma<1, 0<\gamma<\beta, \tag{4.26}$$

$$\left\langle E'\left(\gamma u^+ - \beta u^-\right), -\beta u^-\right\rangle < 0 \text{ for } \beta > 1, 0 < \gamma < \beta, \tag{4.27}$$

$$\left\langle E'\left(\gamma u^+ - \beta u^-\right), -\beta u^-\right\rangle > 0 \text{ for } \beta < 1, 0 < \beta < \gamma. \tag{4.28}$$

First, we prove (4.25) by contradiction, that is, assume $\langle E'(\gamma u^+ - \beta u^-), \gamma u^+\rangle \geq 0$ for $\gamma > 1$ and $0 < \beta \leq \gamma$. For $\gamma > 1$ and due to $\log(e + Ca) \leq C \log(e + a)$ for all $C \geq 1$, it follows that

(4.29)

$$\begin{aligned}
0 &\leq \left\langle E'(\gamma u^+ - \beta u^-), \gamma u^+\right\rangle \\
&= \left[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(\gamma u^+ - \beta u^-))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(\gamma u^+ - \beta u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta u^-)|}{p(x,y)(e + \alpha|D_s(\gamma u^+ - \beta u^-)|)}\right] \\
&\quad \times |D_s(\gamma u^+ - \beta u^-)|^{p(x,y)-2} \left(D_s(\gamma u^+ - \beta u^-)\right)\left(D_s(\gamma u^+)\right) \\
&\quad + \mu(x,y)\left[\log(e + \alpha|D_s(\gamma u^+ - \beta u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta u^-)|}{q(x,y)(e + \alpha|D_s(\gamma u^+ - \beta u^-)|)}\right] \\
&\quad \times |D_s(\gamma u^+ - \beta u^-)|^{q(x,y)-2} \left(D_s(\gamma u^+ - \beta u^-)\right)\left(D_s(\gamma u^+)\right) \Bigg) \,\mathrm{d}\nu \\
&\quad - \int_\Omega f(x, \gamma u^+)\gamma u^+ \,\mathrm{d}x \\
&\leq \left[\theta_1 \gamma^{q_+ +1} + \theta_2 \gamma^{(q_+ +1)\varsigma}(\tilde{I}_{s,\mathcal{H}}(u))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{p(x,y)(e + \alpha|D_s(u)|)}\right] |D_s(u)|^{p(x,y)-2} D_s u D_s(u^+) \\
&\quad + \mu(x,y)\left[\log(e + \alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{q(x,y)(e + \alpha|D_s(u)|)}\right] |D_s(u)|^{q(x,y)-2} D_s u D_s(u^+) \Bigg) \,\mathrm{d}\nu \\
&\quad - \int_\Omega f(x, \gamma u^+)\gamma u^+ \,\mathrm{d}x.
\end{aligned}$$

However, for $u \in \mathcal{N}$ it holds that

(4.30)

$$\begin{aligned}
0 &= \left\langle E'(u), u^+\right\rangle \\
&= \left[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(u))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{p(x,y)(e + \alpha|D_s(u)|)}\right] |D_s(u)|^{p(x,y)-2} D_s u D_s(u^+) \\
&\quad + \mu(x,y)\left[\log(e + \alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{q(x,y)(e + \alpha|D_s(u)|)}\right] |D_s(u)|^{q(x,y)-2} D_s u D_s(u^+) \Bigg) \,\mathrm{d}\nu \\
&\quad - \int_\Omega f(x, u^+)u^+ \,\mathrm{d}x.
\end{aligned}$$

Dividing (4.29) by $\gamma^{\varsigma(q_+ +1)}$ and applying (4.30) along with hypotheses (f5), we obtain

$$\begin{aligned}
0 &< \int_\Omega \left(\frac{f\left(x, \gamma u^+\right)}{(\gamma u^+)^{\varsigma(q_+ +1)-1}} - \frac{f\left(x, u^+\right)}{(u^+)^{\varsigma(q_+ +1)-1}} \right) \left(u^+\right)^{\varsigma(q_+ +1)} \,\mathrm{d}x \\
&\leq \theta_1 \left(\frac{1}{\gamma^{(\varsigma-1)(q_+ +1)}} - 1 \right) \\
&\quad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{p(x,y)(e + \alpha|D_s(u)|)}\right] |D_s(u)|^{p(x,y)-2} D_s u D_s(u^+)
\end{aligned}$$

$$
\begin{aligned}
&+\mu(x,y)\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{q(x,y)-2}D_suD_s(u^+)\Bigg)\,\mathrm{d}\nu\\
&<0,
\end{aligned}
$$

which is a contradiction. So (4.25) holds true. With a similar argument, one can show (4.26).

Now we prove (4.28) by contradiction. Assume that $\langle E'(\gamma u^+-\beta u^-),-\beta u^-\rangle\le 0$ for $\beta<1$ and $0<\beta<\gamma$. For $\beta<1$, we have

$$
\begin{aligned}
0&\ge\left\langle E'(\gamma u^+-\beta u^-),-\beta u^-\right\rangle\\
&=\left[\theta_1+\theta_2(\tilde I_{s,\mathcal{H}}(\gamma u^+-\beta u^-))^{\varsigma-1}\right]\\
&\quad\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(\gamma u^+-\beta u^-)|)+\frac{\alpha|D_s(\gamma u^+-\beta u^-)|}{p(x,y)(e+\alpha|D_s(\gamma u^+-\beta u^-)|)}\right]\\
&\quad\times|D_s(\gamma u^+-\beta u^-)|^{p(x,y)-2}\left(D_s(\gamma u^+-\beta u^-)\right)\left(D_s(-\beta u^-)\right)\\
&\quad+\mu(x,y)\left[\log(e+\alpha|D_s(\gamma u^+-\beta u^-)|)+\frac{\alpha|D_s(\gamma u^+-\beta u^-)|}{q(x,y)(e+\alpha|D_s(\gamma u^+-\beta u^-)|)}\right]\\
&\quad\times|D_s(\gamma u^+-\beta u^-)|^{q(x,y)-2}\left(D_s(\gamma u^+-\beta u^-)\right)\left(D_s(-\beta u^-)\right)\Bigg)\,\mathrm{d}\nu\\
&\quad-\int_\Omega f(x,-\beta u^-)(-\beta u^-)\,\mathrm{d}x\\
&\ge\left[\theta_1\beta^{q_++1}+\theta_2\beta^{(q_++1)\varsigma}(\tilde I_{s,\mathcal{H}}(u))^{\varsigma-1}\right]\\
&\quad\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{p(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{p(x,y)-2}D_suD_s(-u^-)\\
&\quad+\mu(x,y)\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{q(x,y)-2}D_suD_s(-u^-)\Bigg)\,\mathrm{d}\nu\\
&\quad-\int_\Omega f(x,-\beta u^-)(-\beta u^-)\,\mathrm{d}x.
\end{aligned}\tag{4.31}
$$

However, for $u\in\mathcal{N}$, it holds that

$$
\begin{aligned}
0&=\left\langle E'(u),-u^-\right\rangle\\
&=\left[\theta_1+\theta_2(\tilde I_{s,\mathcal{H}}(u))^{\varsigma-1}\right]\\
&\quad\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{p(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{p(x,y)-2}D_suD_s(-u^-)\\
&\quad+\mu(x,y)\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{q(x,y)-2}D_suD_s(-u^-)\Bigg)\,\mathrm{d}\nu\\
&\quad-\int_\Omega f(x,-u^-)(-u^-)\,\mathrm{d}x.
\end{aligned}\tag{4.32}
$$

Dividing (4.31) by $\beta^{\varsigma(q_++1)}$ and applying (4.32) along with hypotheses (f5), we arrive at

$$
\begin{aligned}
0&>\int_\Omega\left(\frac{f\left(x,\beta u^+\right)}{(\beta u^+)^{\varsigma(q_++1)-1}}-\frac{f\left(x,u^+\right)}{(u^+)^{\varsigma(q_++1)-1}}\right)\left(u^+\right)^{\varsigma(q_++1)}\,\mathrm{d}x\\
&\ge\theta_1\left(\frac{1}{\beta^{(\varsigma-1)(q_++1)}}-1\right)\\
&\quad\times\int_Q\Bigg(\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{p(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{p(x,y)-2}D_suD_s(-u^-)
\end{aligned}
$$

$$+\mu(x,y)\left[\log(e+\alpha|D_s(u)|)+\frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{q(x,y)-2}D_suD_s(-u^-)\Big)\,\mathrm{d}\nu$$
$$>0,$$

which is a contradiction, and thus (4.28) is satisfied. In a similar way, we can show (4.27). This shows the claim.

On the one hand, for any $u\in\mathcal{N}$, let (γ_1,β_1) be a pair such that $\alpha_1u^+-\beta_1u^-\in\mathcal{N}$. If $0<\gamma_1\le\beta_1$, then (4.26) and (4.27) indicate that $1\le\gamma_1\le\beta_1\le 1$, that is, $\gamma_1=\beta_1=1$. Moreover, if $0<\beta_1\le\gamma_1$, then (4.25) and (4.28) indicate that $1\le\beta_1\le\gamma_1\le 1$, that is, $\beta_1=\gamma_1=1$. Hence, if $u\in\mathcal{N}$, then $(\gamma_1,\beta_1)=(1,1)$ is the unique pair satisfying $\alpha_uu^+-\beta_uu^-\in\mathcal{N}$.

On the other hand, for any $u\notin\mathcal{N}$, let (γ_2,β_2) and (γ_3,β_3) be such that $\alpha_2u^+-\beta_2u^-\in\mathcal{N}$ and $\alpha_3u^+-\beta_3u^-\in\mathcal{N}$. This implies that

$$\alpha_3u^+-\beta_3u^-=\left(\frac{\gamma_3}{\gamma_2}\right)(\gamma_2u^+)-\left(\frac{\beta_3}{\beta_2}\right)(\beta_2u^-)\in\mathcal{N}. \tag{4.33}$$

Since $\alpha_2u^+-\beta_2u^-\in\mathcal{N}$, we see that $[(\frac{\gamma_3}{\gamma_2}),(\frac{\beta_3}{\beta_2})]=(1,1)$ is the unique pair fulfilling (4.33). So, we see that $\gamma_2=\gamma_3$ and $\beta_2=\beta_3$. This proves Step 2.

Step 3. Let $G_u\colon[0,\infty)\times[0,\infty)\to\mathbb{R}$ be defined by

$$G_u(\gamma,\beta)=E(\gamma u^+-\beta u^-).$$

We are going to show that the pair (γ_u,β_u) given in Step 1 is the unique maximum point of G_u.

First, we demonstrate that G_u has a maximum point. By the continuity of G_u, we see that there exists a maximum on $[0,1]\times[0,1]$. Then we may assume $\gamma\ge\beta\ge 1$, and then by (4.23), for $\eta,C_\eta,C'_\eta>0$ it follows that

(4.34)
$$\begin{aligned}
&\frac{G_u(\gamma,\beta)}{\gamma^{\varsigma q_++\eta}}\\
&\quad=\frac{E(\gamma u^+-\beta u^-)}{\gamma^{\varsigma q_++\eta}}\\
&\quad=\frac{\theta_1}{\gamma^{\varsigma q_++\eta}}\left[\int_Q\Big(|D_s(\gamma u^+-\beta u^-)|^{p(x,y)}\right.\\
&\qquad\left.+\mu(x,y)|D_s(\gamma u^+-\beta u^-)|^{q(x,y)}\Big)\log(e+\alpha|D_s(\gamma u^+-\beta u^-)|)\,\mathrm{d}\nu\right]\\
&\qquad+\frac{\theta_2}{\varsigma\gamma^{\varsigma q_++\eta}}\left[\int_Q\Big(|D_s(\gamma u^+-\beta u^-)|^{p(x,y)}\right.\\
&\qquad\left.+\mu(x,y)|D_s(\gamma u^+-\beta u^-)|^{q(x,y)}\Big)\log(e+\alpha|D_s(\gamma u^+-\beta u^-)|)\,\mathrm{d}\nu\right]^\varsigma\\
&\qquad-\int_\Omega\frac{F(x,\gamma u^+-\beta u^-)}{\gamma^{\varsigma q_++\eta}}\,\mathrm{d}x\\
&\quad\le\frac{2\theta_1C_\eta}{\gamma^{(\varsigma-1)q_+}}\left[\int_Q\Big(|D_su|^{p(x,y)}+\mu(x,y)|D_su|^{q(x,y)}\Big)\log(e+\alpha|D_su|)\,\mathrm{d}\nu\right]\\
&\qquad+\frac{2^\varsigma\theta_2C'_\eta}{\varsigma}\left[\int_Q\Big(|D_su|^{p(x,y)}\mu(x,y)|D_su|^{q(x,y)}\Big)\log(e+\alpha|D_su|)\,\mathrm{d}\nu\right]^\varsigma\\
&\qquad-\int_\Omega\left(\frac{F(x,\gamma u^+)}{(\gamma u^+)^{\varsigma q_++\eta}}(u^+)^{\varsigma q_++\eta}+\frac{F(x,-\beta u^-)}{|-\beta u^-|^{\varsigma q_++\eta}}\left(\frac{\beta}{\gamma}\right)^{\varsigma q_++\eta}(u^-)^{\varsigma q_++\eta}\right)\mathrm{d}x.
\end{aligned}$$

By (f2) and (4.34), we see that

$$\lim_{|(\gamma,\beta)|\to\infty} G_u(\gamma,\beta) = -\infty,$$

which means G_u possesses a maximum.

Next, we show that the maximum point of G_u is not on the boundary of $[0,\infty)\times[0,\infty)$. Conversely, we assume that $(0,\beta_*)$ with $\beta_*\geq 0$ is a maximum point for G_u. Recall the following inequalities:

$$C_\eta^{-1} c^\eta \log(e+\alpha t) \leq \log(e+\alpha ct) \quad \text{for all } \eta>0, t\geq 0, \text{ and } 0<c<1, \tag{4.35}$$

$$\log(e+\alpha ct) \leq C_\eta c^\eta \log(e+\alpha t) \quad \text{for all } \eta>0, t\geq 0, \text{ and } c>1. \tag{4.36}$$

For $0<\gamma<1$ and $\theta_1>0$, applying (4.35) we have

$$\begin{aligned}
&\frac{\partial G_u(\gamma,\beta_*)}{\partial\gamma}\\
&= \frac{E(\gamma u^+ - \beta_* u^-)}{\partial\gamma}\\
&= \left[\theta_1 + \frac{\theta_2}{\varsigma}(\tilde{I}_{s,\mathcal{H}}(\gamma u^+ - \beta_* u^-))^{\varsigma-1}\right]\\
&\quad\times \int_Q \Bigg(\left[\log(e+\alpha|D_s(\gamma u^+ - \beta_* u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta_* u^-)|}{p(x,y)(e+\alpha|D_s(\gamma u^+ - \beta_* u^-)|)}\right]\\
&\quad\times |D_s(\gamma u^+ - \beta_* u^-)|^{p(x,y)-2}\left(D_s(\gamma u^+ - \beta_* u^-)\right)\left(D_s(u^+)\right)\\
&\quad + \mu(x,y)\left[\log(e+\alpha|D_s(\gamma u^+ - \beta_* u^-)|) + \frac{\alpha|D_s(\gamma u^+ - \beta_* u^-)|}{q(x,y)(e+\alpha|D_s(\gamma u^+ - \beta_* u^-)|)}\right]\\
&\quad\times |D_s(\gamma u^+ - \beta_* u^-)|^{q(x,y)-2}\left(D_s(\gamma u^+ - \beta_* u^-)\right)\left(D_s(u^+)\right)\Bigg)\,\mathrm{d}\nu\\
&\quad - \int_\Omega f(x,\gamma u^+)\gamma u^+\,\mathrm{d}x\\
&\geq \frac{\theta_1\gamma^{p_+ +\eta-1}}{C_\eta}\int_Q \left[\log(e+\alpha|D_s u^+|) + \frac{\alpha|D_s u^+|}{p(x,y)(e+\alpha|D_s u^+|)}\right]|D_s u^+|^{p(x,y)}\,\mathrm{d}\nu\\
&\quad - \int_\Omega f(x,\gamma u^+)u^+\,\mathrm{d}x.
\end{aligned}$$

Dividing the above inequality by $\gamma^{p_+ +\eta-1}$, we get

$$\begin{aligned}
\frac{1}{\gamma^{p_+ +\eta-1}}\frac{\partial G_u(\gamma,\beta_*)}{\partial\gamma} &\geq \frac{\theta_1}{C_\eta}\int_Q |D_s u^+|^{p(x,y)}\log(e+\alpha|D_s u^+|)\,\mathrm{d}\nu\\
&\quad - \int_\Omega \frac{f(x,\gamma u^+)}{(\gamma u^+)^{p_+ +\eta-1}}(u^+)^{p_+ +\eta}\,\mathrm{d}x,
\end{aligned}$$

which combining it with hypotheses (f3) yields $\frac{\partial G_u(\gamma,\beta_*)}{\partial\gamma}>0$ for $\gamma>0$ small enough. This means that G_u is increasing for $\gamma\in[0,\varepsilon]$ with $\varepsilon>0$ small enough, which is a contradiction to $(0,\beta_*)$ being a maximum point of G_u. Moreover, if $\theta_1=0$, we calculate that

$$\begin{aligned}
\frac{\partial G_u(\gamma,\beta_*)}{\partial\gamma} &\geq \frac{\theta_2\gamma^{\varsigma p_+ +\eta'-1}}{C_{\eta'}\varsigma}[\tilde{I}_{s,\mathcal{H}}(u^+)]^{\varsigma-1}\int_Q |D_s u^+|^{p(x,y)}\log(e+\alpha|D_s u^+|)\,\mathrm{d}\nu\\
&\quad - \int_\Omega f(x,\gamma u^+)u^+\,\mathrm{d}x,
\end{aligned}$$

where $\eta' > 0$. With the same argument of the proof for the case $\theta_1 > 0$, we can deduce a contradiction.

Similarly, $(\gamma_*, 0)$ with $\gamma_* > 0$ is also not a maximum point of G_u. Thus, the global maximum (γ_0, β_0) must be in $(0, M) \times (0, M)$ with $M > 0$. From Step 1, we infer that the unique maximum point of G_u is (γ_u, β_u). ∎

The following result will be useful later.

PROPOSITION 4.8. *Let hypotheses* (H1), (H5), *and* (f1)–(f5) *be satisfied, and let* $u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ *with* $u^\pm \neq 0$ *and* $\langle E'(u), u^+\rangle \le 0$ *as well as* $\langle E'(u), -u^-\rangle \le 0$. *Then the unique pair* (γ_u, β_u) *given by Proposition* 4.7 *fulfills* $0 < \gamma_u, \beta_u \le 1$.

Proof. For the case that $0 < \beta_u \le \gamma_u$, we suppose on the contrary that $\gamma_u > 1$. From Proposition 4.7, we see that $\gamma_u u^+ - \beta_u u^- \in \mathcal{N}$; combining this with (4.36), it follows that

(4.37)
$$\begin{aligned}
0 &= \langle E'(\gamma_u u^+ - \beta_u u^-), \gamma_u u^+\rangle \\
&= \left[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(\gamma_u u^+ - \beta_u u^-))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e+\alpha|D_s(\gamma_u u^+ - \beta_u u^-)|) + \frac{\alpha|D_s(\gamma_u u^+ - \beta_u u^-)|}{p(x,y)(e+\alpha|D_s(\gamma_u u^+ - \beta_u u^-)|)}\right] \\
&\quad \times |D_s(\gamma_u u^+ - \beta_u u^-)|^{p(x,y)-2} \left(D_s(\gamma_u u^+ - \beta_u u^-)\right)\left(D_s(\gamma_u u^+)\right) \\
&\quad + \mu(x,y)\left[\log(e+\alpha|D_s(\gamma_u u^+ - \beta_u u^-)|) + \frac{\alpha|D_s(\gamma_u u^+ - \beta_u u^-)|}{q(x,y)(e+\alpha|D_s(\gamma_u u^+ - \beta_u u^-)|)}\right] \\
&\quad \times |D_s(\gamma_u u^+ - \beta_u u^-)|^{q(x,y)-2} \left(D_s(\gamma_u u^+ - \beta_u u^-)\right)\left(D_s(\gamma_u u^+)\right) \Bigg) \mathrm{d}\nu \\
&\quad - \int_\Omega f(x, \gamma_u u^+)\gamma_u u^+ \,\mathrm{d}x \\
&\le \left[\theta_1 \gamma_u^{q_+ + 1} + \theta_2 \gamma_u^{(q_+ + 1)\varsigma}(\tilde{I}_{s,\mathcal{H}}(u))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e+\alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{p(x,y)(e+\alpha|D_s(u)|)}\right] |D_s(u)|^{p(x,y)-2} D_s u D_s(u^+) \\
&\quad + \mu(x,y)\left[\log(e+\alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right] |D_s(u)|^{q(x,y)-2} D_s u D_s(u^+) \Bigg) \mathrm{d}\nu \\
&\quad - \int_\Omega f(x, \gamma_u u^+)\gamma_u u^+ \,\mathrm{d}x.
\end{aligned}$$

Due to $\langle E'(u), u^+\rangle \le 0$, we have

(4.38)
$$\begin{aligned}
&\left[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(u))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e+\alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{p(x,y)(e+\alpha|D_s(u)|)}\right] |D_s(u)|^{p(x,y)-2} D_s u D_s(u^+) \\
&\quad + \mu(x,y)\left[\log(e+\alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right] |D_s(u)|^{q(x,y)-2} D_s u D_s(u^+) \Bigg) \mathrm{d}\nu \\
&\quad - \int_\Omega f(x, u^+)u^+ \,\mathrm{d}x \le 0.
\end{aligned}$$

Then we divide (4.37) by $(\gamma_u)^{\varsigma(q_++1)}$ and utilize (4.38) to get

(4.39)
$$\begin{aligned}
&\int_\Omega \left(\frac{f(x,\gamma_u u^+)}{(\gamma_u u^+)^{\varsigma(q_++1)-1}} - \frac{f(x,u^+)}{(u^+)^{\varsigma(q_++1)-1}}\right)(u^+)^{\varsigma(q_++1)-1}\,\mathrm{d}x \\
&\quad \le \theta_1 \left(\frac{1}{\gamma_u^{(\varsigma-1)(q_++1)}} - 1\right) \\
&\qquad \times \int_Q \left(\left[\log(e+\alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{p(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{p(x,y)-2}D_su D_s(u^+)\right. \\
&\qquad \left. + \mu(x,y)\left[\log(e+\alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{q(x,y)-2}D_su D_s(u^+)\right)\mathrm{d}\nu.
\end{aligned}$$

Recalling hypotheses (f5) and $\gamma_u > 1$, we know that the left-hand side of (4.39) is positive and the right-hand side is negative, which is a contradiction. So, it holds that $0 < \beta_u \le \gamma_u < 1$.

For the case $0 < \gamma_u \le \beta_u$, we can suppose that $\beta_u > 1$. Then

$$0 = \left\langle E'(\gamma_u u^+ - \beta_u u^-), -\beta_u u^-\right\rangle \quad \text{and} \quad \left\langle E'(u), -u^-\right\rangle \le 0$$

yield

$$\begin{aligned}
&-\int_\Omega \left(\frac{f(x,-\beta_u u^-)}{(\beta_u u^-)^{\varsigma(q_++1)-1}} - \frac{f(x,-u^-)}{(u^-)^{\varsigma(q_++1)-1}}\right)(u^-)^{\varsigma(q_++1)-1}\,\mathrm{d}x \\
&\quad \le \theta_1 \left(\frac{1}{\beta_u^{(\varsigma-1)(q_++1)}} - 1\right) \\
&\qquad \times \int_Q \left(\left[\log(e+\alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{p(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{p(x,y)-2}D_su D_s(-u^-)\right. \\
&\qquad \left. + \mu(x,y)\left[\log(e+\alpha|D_s(u)|) + \frac{\alpha|D_s(u)|}{q(x,y)(e+\alpha|D_s(u)|)}\right]|D_s(u)|^{q(x,y)-2}D_su D_s(-u^-)\right)\mathrm{d}\nu,
\end{aligned}$$

which is also a contradiction. So, $0 < \gamma_u \le \beta_u \le 1$. ◻

Let $c_c := \inf_{\mathcal{N}} E$.

PROPOSITION 4.9. *Let hypotheses* (H1), (H5), *and* (f1)–(f5) *be satisfied. Then* $c_c > 0$.

Proof. According to Proposition 4.4, there exists $\delta > 0$ sufficiently small satisfying

(4.40)
$$E(u) > 0 \quad \text{for all } u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \text{ with } 0 < [u]_{s,\mathcal{H},Q} < \delta.$$

Then, for any $u \in \mathcal{N}$, we choose $\hat{\gamma}, \hat{\beta} > 0$ satisfying $[\hat{\gamma}u^+ - \hat{\beta}u^-]_{s,\mathcal{H},Q} = \tilde{\delta} < \delta$, and it follows from (4.40) and Proposition 4.7 that

$$0 < E(\hat{\gamma}u^+ - \hat{\beta}u^-) \le E(u),$$

which indicates $c_c > 0$. ◻

In order to prove that the infimum c_c is achieved, we first show the following two propositions.

PROPOSITION 4.10. *Let hypotheses* (H1), (H5), *and* (f1)–(f5) *be satisfied. Then for any sequence* $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$ *satisfying* $[u_n]_{s,\mathcal{H},Q} \to +\infty$ *we have* $E(u_n) \to \infty$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$ be a sequence fulfilling $[u_n]_{s,\mathcal{H},Q} \to +\infty$. Let $w_n = \frac{u_n}{[u_n]_{s,\mathcal{H},Q}}$; then there exists $w \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ satisfying

$$\begin{aligned} w_n \rightharpoonup w \quad &\text{in } \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{r(\cdot)}(\Omega) \text{ and a.e. in } \Omega, \\ w_n^{\pm} \rightharpoonup w^{\pm} \quad &\text{in } \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \quad \text{and} \quad w_n^{\pm} \to w^{\pm} \quad \text{in } L^{r(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{4.41}$$

Suppose first that $w \neq 0$. By Proposition 2.2 and letting $[u_n]_{s,\mathcal{H},Q} > 1$, we obtain

$$\begin{aligned} E(u) &= \Psi(\tilde{I}_{s,\mathcal{H}}(u_n)) - \int_\Omega F(x,u_n)\,\mathrm{d}x \\ &\le \theta_1 C_\sigma [u_n]_{s,\mathcal{H},Q}^{q_+ + \sigma} + \frac{\theta_2 C_\sigma^\varsigma}{\varsigma}[u_n]_{s,\mathcal{H},Q}^{(q_+ + \sigma)\varsigma} - \int_\Omega F(x,u_n)\,\mathrm{d}x. \end{aligned} \tag{4.42}$$

Dividing (4.42) by $[u_n]_{s,\mathcal{H},Q}^{\varsigma q_+ + \eta}$ with $0 < \sigma\varsigma < \eta$ and using (f2), we arrive at $\lim_{n\to\infty} \frac{E(u_n)}{[u_n]_{s,\mathcal{H},Q}^{\varsigma q_+ + \eta}} \to -\infty$. However, according to Proposition 4.9 we know that $E(u_n) > 0$, so it must hold that $w = 0$. Thus, $w^+ = w^- = 0$. Due to $u_n \in \mathcal{N}$, note that $\tilde{I}_{s,\mathcal{H}}(w_n) = 1$. Then, by applying Proposition 4.7 and (4.41) we see that for any $(t_1,t_2) \in (0,\infty)\times(0,\infty)$ such that $0 < t_1 \le t_2$ there holds that

$$\begin{aligned} E(u_n) &\ge E(t_1 w_n^+ - t_2 w_n^-) \\ &= \left[\theta_1 + \frac{\theta_2}{\varsigma}\tilde{I}_{s,\mathcal{H}}(t_1 w_n^+ - t_2 w_n^-)^{\varsigma-1}\right] \\ &\quad \times \int_Q \Big(|D_s(t_1 w_n^+ - t_2 w_n^-)|^{p(x,y)} + \mu(x,u)|D_s(t_1 w_n^+ - t_2 w_n^-)|^{p(x,y)}\Big) \\ &\quad \times \log(e + \alpha|D_s(t_1 w_n^+ - t_2 w_n^-)|)\,\mathrm{d}\nu \\ &\quad - \int_\Omega F(x, t_1 w_n^+ - t_2 w_n^-)\,\mathrm{d}x \\ &\ge \frac{\theta_2}{\varsigma}\min\left\{t_1^{\varsigma p_-}, t_1^{\varsigma(q_+ + 1)}\right\}(\tilde{I}_{s,\mathcal{H}}(w_n))^\varsigma - \int_\Omega F(x,t_1 w_n^+) - F(x,-t_2 w_n^-)\,\mathrm{d}x \\ &\to \frac{\theta_2}{\varsigma}\min\left\{t_1^{\varsigma p_-}, t_1^{\varsigma(q_+ + 1)}\right\} \end{aligned}$$

as $n \to \infty$. This implies that if we take $t_1 > 0$ sufficiently large, then for any $K > 0$ it holds that $E(u_n) > K$ for $n \ge n_1$, where $n_1 > 0$ depends on t_1. ∎

Next, we show that the constraint set $\mathcal{N}$ is weakly closed.

PROPOSITION 4.11. *Let hypotheses* (H1), (H5), *and* (f1)–(f5) *be satisfied. Then* $\mathcal{N}$ *is weakly closed.*

Proof. We first prove that for any $M > 0$ we have

$$[u]_{s,\mathcal{H},Q} \ge M \quad \text{for any } u \in \mathcal{N}. \tag{4.43}$$

We may suppose that $[u]_{s,\mathcal{H},Q} < 1$. We first consider the case that $\theta_1 > 0$. Then, by Proposition 2.2, we obtain

$$\begin{aligned}
&\Big[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(u^+ - u^-))^{\varsigma-1}\Big]\\
&\quad\times \int_Q \Bigg(\left[\log(e+\alpha|D_s(u^+ - u^-)|) + \frac{\alpha|D_s(u^+ - u^-)|}{p(x,y)(e+\alpha|D_s u^+ - u^-)|)}\right]\\
&\quad\times |D_s(u^+ - u^-)|^{p(x,y)-2}\left(D_s(u^+ - u^-)\right)\left(D_s(u^+)\right)\\
&\quad+ \mu(x,y)\left[\log(e+\alpha|D_s(u^+ - u^-)|) + \frac{\alpha|D_s(u^+ - u^-)|}{q(x,y)(e+\alpha|D_s(u^+ - u^-)|)}\right]\\
&\quad\times |D_s(u^+ - u^-)|^{q(x,y)-2}\left(D_s(u^+ - u^-)\right)\left(D_s(u^+)\right)\Bigg)\,\mathrm{d}\nu \qquad (4.44)\\
&\geq \theta_1 \int_Q \left(|D_s(u^+)|^{p(x,y)} + \mu(x,u)|D_s(u^+)|^{p(x,y)}\right)\\
&\quad\times \log(e+\alpha|D_s(u^+)|)\,\mathrm{d}\nu = \theta_1 \tilde{I}_{s,\mathcal{H}}(u^+)\\
&\geq \frac{\theta_1}{2}\tilde{I}_{s,\mathcal{H}}(u^+) + \frac{\theta_1}{2}\int_Q |D_s u^+|^{p(x,y)}\,\mathrm{d}\nu\\
&\geq \frac{\theta_1}{2C_\sigma}[u^+]^{q_+ + \sigma}_{s,\mathcal{H},Q} + \frac{\theta_1}{2C_{\sigma'}}[u^+]^{p_+ + \sigma'}_{s,p(\cdot,\cdot)},
\end{aligned}$$

where $\sigma, \sigma', C_\sigma, C_{\sigma'} > 0$. Furthermore, since $u \in \mathcal{N}$, then $\langle E'(u^+ - u^-), u^-\rangle = 0$, and by (4.2) we calculate that

$$\begin{aligned}
&\Big[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(u^+ - u^-))^{\varsigma-1}\Big]\\
&\quad\times \int_Q \Bigg(\left[\log(e+\alpha|D_s(u^+ - u^-)|) + \frac{\alpha|D_s(u^+ - u^-)|}{p(x,y)(e+\alpha|D_s u^+ - u^-)|)}\right]\\
&\quad\times |D_s(u^+ - u^-)|^{p(x,y)-2}\left(D_s(u^+ - u^-)\right)\left(D_s(u^+)\right)\\
&\quad+ \mu(x,y)\left[\log(e+\alpha|D_s(u^+ - u^-)|) + \frac{\alpha|D_s(u^+ - u^-)|}{q(x,y)(e+\alpha|D_s(u^+ - u^-)|)}\right]\\
&\quad\times |D_s(u^+ - u^-)|^{q(x,y)-2}\left(D_s(u^+ - u^-)\right)\left(D_s(u^+)\right)\Bigg)\,\mathrm{d}\nu \qquad (4.45)\\
&= \int_\Omega f(x,u^+)u^+\,\mathrm{d}x\\
&\leq \frac{\varepsilon}{p_-}\int_\Omega |u^+|^{p_+ + \sigma'}\,\mathrm{d}x + C_\varepsilon \rho_{r(\cdot)}(u^+)\\
&\leq \frac{\varepsilon}{p_-}\|u^+\|^{p_+ + \sigma'}_{p_+ + \sigma'} + C_\varepsilon \rho_{r(\cdot)}(u^+)\\
&\leq \frac{C_{e3}\varepsilon}{p_-}[u^+]^{p_+ + \sigma'}_{s,p(\cdot,\cdot)} + C_\varepsilon \max_{k\in\{r_+, r_-\}}\left\{C^k_{e1}[u^+]^k_{s,\mathcal{H},Q}\right\},
\end{aligned}$$

where C_{e1} is the embedding constant of $\widetilde{W}^{s,\mathcal{H}}_0(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ and C_{e3} is the embedding constant of $\widetilde{W}^{s,p(\cdot,\cdot)}_0(\Omega) \hookrightarrow L^{p_+ + \sigma'}(\Omega)$ with $\sigma' \leq (p_-)^*_s - p_+$. Combining (4.44) and (4.45), we see that if choose $0 < \varepsilon < \frac{\theta_1 p_-}{2C_{e3}C_{\sigma'}}$, it holds that

$$\frac{\theta_1}{2C_\sigma}[u^+]^{q_+ + \sigma}_{s,\mathcal{H},Q} \leq C_\varepsilon C^k_{e1}[u^+]^k_{s,\mathcal{H},Q},$$

with $k \in \{r_+, r_-\}$. Since $q_+ < r_-$, we can choose $0 < \sigma < r_- - q_+$ to get

$$[u]_{s,\mathcal{H},Q} \geq \min_{k\in\{r_+,r_-\}} \left(\frac{\theta_1}{2C_\sigma C_\varepsilon C_{e1}^k} \right)^{\frac{1}{k-q_+ +\sigma}} =: M.$$

As done above, applying (4.3) we can verify the results for the case that $\theta_1 = 0$ since $\varsigma q_+ < r_-$.

Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$ be such that $u_n \rightharpoonup u$, which implies $u_n^+ \rightharpoonup u^+$ and $u_n^- \rightharpoonup u^-$ in $\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ with $u^+, u^- \geq 0$. It follows that

$$u_n^+ \to u^+ \quad \text{and} \quad u_n^- \to u^- \quad \text{in } L^{r(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \tag{4.46}$$

Then we verify that $u^+ \neq 0 \neq u^-$. In fact, if $u^+ = 0$, recalling that $u_n \in \mathcal{N}$, we get

$$\begin{aligned}
0 &= \left\langle E'(u_n), u_n^+ \right\rangle \\
&= \left[\theta_1 + \theta_2 (\tilde{I}_{s,\mathcal{H}}(u_n^+ - u_n^-))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(u_n^+ - u_n^-)|) + \frac{\alpha|D_s(u_n^+ - u_n^-)|}{p(x,y)(e+\alpha|D_s u_n^+ - u_n^-)|)}\right] \\
&\quad \times |D_s(u_n^+ - u_n^-)|^{p(x,y)-2} \left(D_s(u_n^+ - u_n^-)\right) \left(D_s(u^+)\right) \\
&\quad + \mu(x,y) \left[\log(e + \alpha|D_s(u_n^+ - u_n^-)|) + \frac{\alpha|D_s(u_n^+ - u_n^-)|}{q(x,y)(e+\alpha|D_s(u_n^+ - u_n^-)|)}\right] \\
&\quad \times |D_s(u_n^+ - u_n^-)|^{q(x,y)-2} \left(D_s(u_n^+ - u_n^-)\right) \left(D_s(u_n^+)\right) \Bigg) \mathrm{d}\nu \\
&\quad - \int_\Omega f(x, u_n^+) u_n^+ \,\mathrm{d}x \\
&\geq \theta_2 \tilde{I}_{s,\mathcal{H}}(u_n^+)^\varsigma - \int_\Omega f(x, u_n^+) u_n^+ \,\mathrm{d}x.
\end{aligned}$$

According to (4.46) and hypotheses (f1), we get

$$\theta_2 \tilde{I}_{s,\mathcal{H}}(u_n^+)^\varsigma \leq \int_\Omega f(x, u_n^+) u_n^+ \,\mathrm{d}x \to \int_\Omega f(x, u^+) u^+ \,\mathrm{d}x \to 0$$

as $n \to \infty$. Therefore, $\tilde{I}_{s,\mathcal{H}}(u_n^+) \to 0$ and hence $u_n^+ \to 0$ in $\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$, which is a contradiction to (4.43). So, we get $u^+ \neq 0$, and similarly $u^- \neq 0$. By Proposition 4.7, we can find a pair $(\gamma_u, \beta_u) \in (0,\infty) \times (0,\infty)$ such that $\gamma_u u^+ - \beta_u u^- \in \mathcal{N}$. Note that

$$\begin{aligned}
&\left[\theta_1 + \theta_2 (\tilde{I}_{s,\mathcal{H}}(u^+ - u^-))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(u^+ - u^-)|) + \frac{\alpha|D_s(u^+ - u^-)|}{p(x,y)(e+\alpha|D_s u^+ - u^-)|)}\right] \\
&\quad \times |D_s(u^+ - u^-)|^{p(x,y)-2} \left(D_s(u^+ - u^-)\right) \left(D_s(u^+)\right) \\
&\quad + \mu(x,y) \left[\log(e + \alpha|D_s(u^+ - u^-)|) + \frac{\alpha|D_s(u^+ - u^-)|}{q(x,y)(e+\alpha|D_s(u^+ - u^-)|)}\right] \\
&\quad \times |D_s(u^+ - u^-)|^{q(x,y)-2} \left(D_s(u^+ - u^-)\right) \left(D_s(u^+)\right) \Bigg) \mathrm{d}\nu
\end{aligned} \tag{4.47}$$

is weak lower semicontinuous since it is convex and continuous. From this along with $u_n \in \mathcal{N}$, (4.46), and (f1) we see that

$$
\begin{aligned}
&\left\langle E'(u), \pm u^{\pm}\right\rangle \\
&\quad = \left[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(u^+ - u^-))^{\varsigma - 1}\right] \\
&\qquad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(u^+ - u^-)|) + \frac{\alpha|D_s(u^+ - u^-)|}{p(x,y)(e + \alpha|D_s u^+ - u^-|)}\right] \\
&\qquad \times |D_s(u^+ - u^-)|^{p(x,y)-2} \left(D_s(u^+ - u^-)\right)\left(D_s(\pm u^{\pm})\right) \\
&\qquad + \mu(x,y)\left[\log(e + \alpha|D_s(u^+ - u^-)|) + \frac{\alpha|D_s(u^+ - u^-)|}{q(x,y)(e + \alpha|D_s(u^+ - u^-)|)}\right] \\
&\qquad \times |D_s(u^+ - u^-)|^{q(x,y)-2} \left(D_s(u^+ - u^-)\right)\left(D_s(\pm u^{\pm})\right) \Bigg) \mathrm{d}\nu \\
&\qquad - \int_\Omega f(x, \pm u^{\pm}) \pm u^{\pm} \,\mathrm{d}x \\
&\quad \le \liminf_{n\to\infty} \left[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(u_n^+ - u_n^-))^{\varsigma - 1}\right] \\
&\qquad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(u_n^+ - u_n^-)|) + \frac{\alpha|D_s(u_n^+ - u_n^-)|}{p(x,y)(e + \alpha|D_s u_n^+ - u_n^-)|)}\right] \\
&\qquad \times |D_s(u_n^+ - u_n^-)|^{p(x,y)-2} \left(D_s(u_n^+ - u_n^-)\right)\left(D_s(\pm u_n^{\pm})\right) \\
&\qquad + \mu(x,y)\left[\log(e + \alpha|D_s(u_n^+ - u_n^-)|) + \frac{\alpha|D_s(u_n^+ - u_n^-)|}{q(x,y)(e + \alpha|D_s(u_n^+ - u_n^-)|)}\right] \\
&\qquad \times |D_s(u_n^+ - u_n^-)|^{q(x,y)-2} \left(D_s(u_n^+ - u_n^-)\right)\left(D_s(\pm u_n^{\pm})\right) \Bigg) \mathrm{d}\nu \\
&\qquad - \lim_{n\to\infty} \int_\Omega f(x, u_n^+) u_n^+ \,\mathrm{d}x \\
&\quad = \liminf_{n\to\infty} \left\langle E'(u_n), \pm u_n^{\pm}\right\rangle = 0.
\end{aligned}
$$

Due to the above inequalities, by applying Proposition 4.8 we see that $(\gamma_u, \beta_u) \in (0,1] \times (0,1]$. By applying (f1), (f4), (4.46), and $u_n, \gamma_u u^+ - \beta_u u^- \in \mathcal{N}$ with $(\gamma_u, \beta_u) \in (0,1] \times (0,1]$ as well as the lower semicontinuity of (4.47) and E we obtain

$$
\begin{aligned}
c_c &= \inf_{\mathcal{N}} E \le E(\gamma_u u^+ - \beta_u u^-) \\
&= E(\gamma_u u^+ - \beta_u u^-) - \frac{1}{\varsigma q_+ (1 + \frac{\kappa}{p_-})} \left\langle E'(\gamma_u u^+ - \beta_u u^-, \gamma_u u^+ - \beta_u u^-)\right\rangle \\
&= \left[\theta_1 + \frac{\theta_2}{\varsigma}\left(\tilde{I}_{s,\mathcal{H}}(\gamma_u u^+ - \beta_u u^-)\right)^{\varsigma-1}\right] \\
&\quad \times \Bigg[\int_Q \left(|D_s(\gamma u^+ - \beta u^-)|^{p(x,y)} + \mu(x,y)|D_s(\gamma u^+ - \beta u^-)|^{q(x,y)}\right) \\
&\qquad \log(e + \alpha|D_s(\gamma u^+ - \beta u^-)|)\,\mathrm{d}\nu \Bigg] \\
&\quad - \frac{1}{\varsigma q_+ (1 + \frac{\kappa}{p_-})} \left[\theta_1 + \theta_2(\tilde{I}_{s,\mathcal{H}}(\gamma_u u^+ - \beta_u u^-))^{\varsigma-1}\right] \\
&\quad \times \int_Q \Bigg(\left[\log(e + \alpha|D_s(\gamma_u u^+ - \beta_u u^-)|) + \frac{\alpha|D_s(\gamma_u u^+ - \beta_u u^-)|}{p(x,y)(e + \alpha|D_s(\gamma_u u^+ - \beta_u u^-)|)}\right] \\
&\quad \times |D_s(\gamma_u u^+ - \beta_u u^-)|^{p(x,y)}
\end{aligned}
$$

$$
\begin{aligned}
&+\mu(x,y)\left[\log(e+\alpha|D_s(\gamma_u u^+-\beta_u u^-)|)+\frac{\alpha|D_s(\gamma_u u^+-\beta_u u^-)|}{q(x,y)(e+\alpha|D_s(\gamma_u u^+-\beta_u u^-)|)}\right]\\
&\times|D_s(\gamma_u u^+-\beta_u u^-)|^{q(x,y)}\Big)\,\mathrm{d}\nu\\
&+\int_\Omega\frac{1}{\varsigma q_+(1+\frac{\kappa}{p_-})}f(x,\gamma_u u^+-\beta_u u^-)(\gamma_u u^+-\beta_u u^-)\\
&-F(x,\gamma_u u^+-\beta_u u^-)(\gamma_u u^+-\beta_u u^-)\,\mathrm{d}x\\
\leq&\left[\theta_1+\frac{\theta_2}{\varsigma}\left(\tilde{I}_{s,\mathcal{H}}(u^+-u^-)\right)^{\varsigma-1}\right]\\
&\times\left[\int_Q\left(|D_s(\gamma u^+-\beta u^-)|^{p(x,y)}+\mu(x,y)|D_s(\gamma u^+-\beta u^-)|^{q(x,y)}\right)\right.\\
&\left.\times\log(e+\alpha|D_s(\gamma u^+-\beta u^-)|)\,\mathrm{d}\nu\right]\\
&-\frac{1}{\varsigma q_+(1+\frac{\kappa}{p_-})}\left[\theta_1+\theta_2(\tilde{I}_{s,\mathcal{H}}(u^+-u^-))^{\varsigma-1}\right]\\
&\times\int_Q\left(\left[\log(e+\alpha|D_s(u^+-u^-)|)+\frac{\alpha|D_s(u^+-u^-)|}{p(x,y)(e+\alpha|D_s(u^+-u^-)|)}\right]\right.\\
&\times|D_s(u^+-u^-)|^{p(x,y)}\\
&+\mu(x,y)\left[\log(e+\alpha|D_s(u^+-u^-)|)+\frac{\alpha|D_s(u^+-u^-)|}{q(x,y)(e+\alpha|D_s(u^+-u^-)|)}\right]\\
&\left.\times|D_s(u^+-u^-)|^{q(x,y)}\right)\mathrm{d}\nu\\
&+\int_\Omega\frac{1}{\varsigma q_+(1+\frac{\kappa}{p_-})}f(x,u^+-u^-)(u^+-u^-)-F(x,u^+-u^-)(u^+-u^-)\,\mathrm{d}x\\
\leq&\liminf_{n\to\infty}\left(E(u_n^+-u_n^-)-\frac{1}{\varsigma q_+(1+\frac{\kappa}{p_-})}\left\langle E'(u_n^+-u_n^-),u_n^+-u_n^-\right\rangle\right)=c_c,
\end{aligned}
$$

which indicates that $\gamma_u=\beta_u=1$. Hence, $u\in\mathcal{N}$, and therefore the constraint set $\mathcal{N}$ is weakly closed. ◻

Now we are ready to show that the infimum of E over $\mathcal{N}$ is achieved.

PROPOSITION 4.12. *Let hypotheses* (H1), (H5), *and* (f1)–(f5) *be satisfied. Then there exists $u_c\in\mathcal{N}$ satisfying $E(u_c)=c_c$.*

Proof. Suppose that $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{N}$ is a minimizing sequence, namely,

$$E(u_n)\searrow c_c.$$

Then Proposition 4.10 implies that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Hence, from the reflexivity of $\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ one can find $u\in\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ such that $u_n\rightharpoonup u_c\in\widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ with

$$E(u_c)=c_c=\inf_{u\in\mathcal{N}}E(u).$$

Applying the weak closedness of the set $\mathcal{N}$, we see that $u_c\in\mathcal{N}$. ◻

THEOREM 4.13. *Let hypotheses* (H1), (H5), *and* (f1)–(f5) *be satisfied, and let $u_c\in\mathcal{N}$ be such that $E(u_c)=c_c$. Then u_c is a critical point of E. In particular, it is a least energy sign-changing weak solution of problem* (4.1).

Proof. Suppose that $E'(u_c) \neq 0$. Then there exist $\lambda, \delta_1 > 0$ satisfying

$$\|E'(u)\|_{\widetilde{W}_0^{s,\mathcal{H}}(\Omega)^*} \geq \lambda \quad \text{for all } u \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega) \text{ with } [u - u_c]_{s,\mathcal{H},Q} < 3\delta_1.$$

Denote by C_{e4} the embedding constant of $\widetilde{W}_0^{s,\mathcal{H}}(\Omega) \hookrightarrow L^{p_-}(\Omega)$. By Proposition 4.11, we see that $u_c^+ \neq 0 \neq u_c^-$; then for any $w \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ there holds that

$$[u_c - w]_{s,\mathcal{H},Q} \geq C_{e4}^{-1} \|u_c - w\|_{p_-} \geq \begin{cases} C_{e4}^{-1} \|u_c^-\|_{p_-} & \text{if } w^- = 0, \\ C_{e4}^{-1} \|u_c^+\|_{p_-} & \text{if } w^+ = 0. \end{cases}$$

Choosing $0 < \delta_2 < \min\{C^{-1}\|u_c^-\|_{p_-}, C^{-1}\|u_c^+\|_{p_-}\}$, we get $w^+ \neq 0 \neq w^-$ for any $w \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ such that $[u_c - w]_{s,\mathcal{H},Q} < \delta_2$.

Now we take $\delta = \min\{\delta_1, \delta_2/2\}$. Due the continuity of $[0,\infty) \times [0,\infty) \ni (\gamma,\beta) \mapsto \gamma u_c^+ - \beta u_c^- \in \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$, one can find $0 < m < 1$ such that for all $\gamma, \beta \geq 0$ fulfilling $\max\{|\gamma - 1|, |\beta - 1|\} < m$, there holds that

$$[\gamma u_c^+ - \beta u_c^- - u_c]_{s,\mathcal{H},Q} < \delta.$$

Denoting $D = (1-m, 1+m) \times (1-m, 1+m)$ and applying Proposition 4.7, we get

$$E(\gamma u_c^+ - \beta u_c^-) < E(u_c^+ - u_c^-) = \inf_{u \in \mathcal{N}} E(u) \tag{4.48}$$

for all $\gamma, \beta \geq 0$ satisfying $(\gamma, \beta) \neq (1,1)$. Then

$$C_m := \max_{(\gamma,\beta) \in \partial D} E(\gamma u_c^+ - \beta u_c^-) < \inf_{u \in \mathcal{N}} E(u).$$

Thus, we see that the assumptions of Lemma 2.7 (quantitative deformation lemma) are satisfied with

$$S = B(u_c, \delta), \quad c = \inf_{u \in \mathcal{N}} E(u), \quad \varepsilon = \min\left\{\frac{c - C_m}{4}, \frac{m\delta}{8}\right\},$$

where δ is given above, and note that $S_{2\delta} = B(u_c, 3\delta)$. Thus, we can find a mapping η fulfilling the properties of the quantitative deformation lemma. By the definition of ε, for all $(\gamma, \beta) \in \partial D$ we get

$$E(\gamma u_c^+ - \beta u_c^-) \leq C_m + c - c < c - \left(\frac{c - C_m}{2}\right) \leq c - 2\varepsilon. \tag{4.49}$$

Next, we define $\mathcal{P}\colon [0,\infty) \times [0,\infty) \to \widetilde{W}_0^{s,\mathcal{H}}(\Omega)$ and $\mathcal{T}\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ as

$$\begin{aligned} &\mathcal{P}(\gamma,\beta) = \eta(1, \gamma u_c^+ - \beta u_c^-), \\ &\mathcal{T}(\gamma,\beta) = \left[\left\langle E'(\mathcal{P}(\gamma,\beta), \mathcal{P}^+(\gamma,\beta))\right\rangle, \left\langle E'(\mathcal{P}(\gamma,\beta), -\mathcal{P}^-(\gamma,\beta))\right\rangle\right]. \end{aligned}$$

Due to the continuity of η and the differentiability of E, we know that $\mathcal{P}$ and $\mathcal{T}$ are continuous. According to Lemma 2.7 and (4.49), we infer that $\mathcal{P}(\gamma,\beta) = \gamma u_c^+ - \beta u_c^-$ and

$$\mathcal{T}(\gamma,\beta) = \left[\left\langle E'(\gamma u_c^+ - \beta u_c^-, \gamma u_c^+)\right\rangle, \left\langle E'(\gamma u_c^+ - \beta u_c^-, -\beta u_c^-)\right\rangle\right]$$

for all $(\gamma,\beta) \in \partial D$. Let $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$; then from (4.25)–(4.28) we arrive at

$$\mathcal{T}_1(1-m, s) > 0 > \mathcal{T}_1(1+m, s) \quad \text{and} \quad \mathcal{T}_2(s, 1-m) > 0 > \mathcal{T}_2(s, 1+m)$$

for all $s \in [1-m, 1+m]$. Then utilizing Theorem 2.8 one can find $(\gamma_*, \beta_*) \in D$ such that $\mathcal{T}(\gamma_*, \beta_*) = 0$, that is,

$$\left\langle E'(\mathcal{P}(\gamma_*, \beta_*), \mathcal{P}^+(\gamma_*, \beta_*)) \right\rangle = 0 \tag{4.50}$$

and

$$\left\langle E'(\mathcal{P}(\gamma_*, \beta_*), -\mathcal{P}^-(\gamma_*, \beta_*)) \right\rangle = 0. \tag{4.51}$$

By the choice of m, we deduce from Lemma 2.7(iv) that

$$[\mathcal{P}(\gamma_*, \beta_*) - u_c]_{s,\mathcal{H},Q} \leq 2\delta \leq \delta_2.$$

Due to the definition of δ_2, we infer from the above inequalities that $\mathcal{P}^+(\gamma_*, \beta_*) \neq 0 \neq -\mathcal{P}^-(\gamma_*, \beta_*)$, which by (4.50) and (4.51) implies that $\mathcal{P}(\gamma_*, \beta_*) \in \mathcal{N}$. However, by the choice of m and (4.48), it follows from Lemma 2.7(ii) that $E(\mathcal{P}(\gamma_*, \beta_*)) \leq c - \varepsilon$, which is a contradiction. So, u_c is indeed a critical point of E and therefore a least energy sign-changing weak solution of problem (4.1). □

Finally, we give an example of function f satisfying hypotheses (H6).

Example 4.14. Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x,t) = |t|^{\varsigma q_+\left(1+\frac{\kappa}{p_-}\right)+\eta-2} t,$$

with $\varsigma q_+(1+\frac{\kappa}{p_-}) < \varsigma(q_+ + 1) < (p_-)^*_s$ and $0 < \eta < 1 - \frac{q_+\kappa}{p_-}$. Then the function f given above fulfills hypotheses (H6) with $\varsigma q_+(1+\frac{\kappa}{p_-}) + \eta < r_- \leq r_+ < (p_-)^*_s$ in (f2).

5. Summary and discussion. This paper presents a systematic study of elliptic inclusions and Kirchhoff-type problems driven by a fractional double phase operator with variable exponents and a logarithmic perturbation. The main contribution lies in providing the first unified treatment of three challenging mathematical concepts: variable exponent growth, double phase behavior, and logarithmic perturbation, all considered within a fractional framework. We develop the corresponding variational setting and establish several existence results in this generalized context.

As observed in previous studies, single-valued double phase problems with variable exponents and logarithmic perturbations were investigated in [70] and [49]. These works established fundamental properties of double phase operators and the corresponding Musielak–Orlicz spaces generated by the N-functions (1.6) and (1.7), respectively, and proved existence and uniqueness results of weak solutions by means of surjectivity theorems for operators. In contrast, the present paper concentrates on fractional double phase problems and provides a deeper qualitative analysis of their weak solutions:

- For the elliptic inclusion, we employ the sub- and supersolution methods to establish, for the first time, the existence, extremality, and compactness properties of the solution set for a class of multivalued variational inequalities.
- For the Kirchhoff problem, by combining variational methods, the quantitative deformation lemma, and the Poincaré–Miranda theorem, we establish the existence of multiple solutions: specifically, at least one positive solution, one negative solution, and one sign-changing solution, despite the intrinsic difficulties arising from the interaction between the nonlocal operator and the strongly nonlinear structure.

While this paper establishes the existence of solutions to the fractional double phase problem (3.1) and the existence of sign-changing solutions for the Kirchhoff-type problem (4.1), the uniqueness of solutions is not addressed. In the case of the inclusion problem (3.1), the presence of the multivalued lower order operator $\mathcal{F}$ generally prevents uniqueness. To recover it, one could impose strong monotonicity on f in the following sense: assume there exists $m>0$ such that for any $\eta_1 \in f(x,t_1), \eta_2 \in f(x,t_2)$, we have

$$(\eta_1-\eta_2)(t_1-t_2) \geq m|t_1-t_2|^2,$$

which in turn would imply strong monotonicity of the composite operator $(-\Delta)^s_{\mathcal{H}} + \partial I_K + \mathcal{F}$. For the nonlocal Kirchhoff problem, uniqueness is particularly difficult to obtain, even in much simpler situations, and typically requires highly restrictive assumptions on ψ and f, such as global monotonicity conditions on the nonlinearity f.

We identify several promising directions for future research:

- **Regularity theory:** Investigating higher regularity properties, such as Hölder continuity or differentiability of solutions to these problems, remains a major challenge due to the combined effects of nonlocality, variable exponents, double phase behavior, and logarithmic nonlinearities. Progress in this direction would not only improve the physical relevance of the solutions (for instance, by excluding nonphysical singularities) but also provide a solid foundation for the development of reliable numerical methods.
- **Multiphase extensions:** A natural continuation of this work is to study multiphase problems involving more than two growth modes. This requires establishing essential functional analytic tools, including embedding theorems, compactness results, and convergence principles, under appropriately adapted assumptions on the variable exponents and weight functions.

Appendix A. Basic notations and results. In the following, we recall properties of variable exponent spaces, Musielak–Orlicz spaces, and fractional Musielak–Sobolev spaces. Most of the results are taken from Diening et al. [28], Fan and Zhao [31], Harjulehto and Hästö [38], Kováčik and Rákosník [41], Lu, Vetro, and Zeng [48], and de Albuquerque [26].

First, we define $C_+(\overline{\Omega})$ by

$$C_+(\overline{\Omega}) := \left\{ g \in C(\overline{\Omega}) \colon 1 < \inf_{x\in\overline{\Omega}} g(x) \text{ for all } x \in \overline{\Omega} \right\}.$$

For any $\iota \in C_+(\overline{\Omega})$, we denote

$$\iota_- := \inf_{x\in\overline{\Omega}} \iota(x) \quad \text{and} \quad \iota_+ := \sup_{x\in\overline{\Omega}} \iota(x).$$

By $\iota' \in C_+(\overline{\Omega})$ we mean the conjugate variable exponent of ι, that is,

$$\frac{1}{\iota(x)} + \frac{1}{\iota'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Let $M(\Omega)$ be the set of measurable functions from Ω to $\mathbb{R}$. For any fixed $\iota \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space $L^{\iota(\cdot)}(\Omega)$ is given by

$$L^{\iota(\cdot)}(\Omega) = \left\{ u \in M(\Omega) \colon \varrho_{\iota(\cdot)}(u) < \infty \right\},$$

where $\varrho_{\iota(\cdot)}$ is the related modular defined by

$$\varrho_{\iota(\cdot)}(u) = \int_\Omega |u|^{\iota(x)}\,\mathrm{d}x.$$

Note that $L^{\iota(\cdot)}(\Omega)$ endowed with the Luxemburg norm

$$\|u\|_{\iota(\cdot)} = \inf\left\{\lambda > 0\colon \int_\Omega \left(\frac{|u|}{\lambda}\right)^{\iota(x)} \mathrm{d}x \leq 1\right\}$$

forms a separable and reflexive Banach space. The dual space of $L^{\iota(\cdot)}(\Omega)$ is $L^{\iota'(\cdot)}(\Omega)$, and for all $u \in L^{\iota(\cdot)}(\Omega)$, $\omega \in L^{\iota'(\cdot)}(\Omega)$ the Hölder-type inequality of the form

$$\int_\Omega |u\omega|\,\mathrm{d}x \leq \left[\frac{1}{\iota_-} + \frac{1}{\iota'_-}\right] \|u\|_{\iota(\cdot)} \|\omega\|_{\iota'(\cdot)} \leq 2\|u\|_{r(\cdot)}\|\omega\|_{\iota'(\cdot)}$$

holds. Moreover, if $\iota_1, \iota_2 \in C_+(\overline{\Omega})$ fulfilling $\iota_1(x) \leq \iota_2(x)$ for all $x \in \overline{\Omega}$, we have the continuous embedding

$$L^{\iota_2(\cdot)}(\Omega) \hookrightarrow L^{\iota_1(\cdot)}(\Omega).$$

Next, we consider the definitions of N- and generalized N-functions.

Definition A.1.

(i) *A function $\varphi\colon [0,\infty) \to [0,\infty)$ is said to be an N-function if it is continuous and convex, and $\varphi(t) = 0$ if and only if $t = 0$. Also, it holds that*

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0 \quad \textit{and} \quad \lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty.$$

(ii) *A function $\varphi\colon \Omega\times\Omega\times[0,\infty) \to [0,\infty)$ is said to be a generalized N-function (denoted by $\varphi \in N(\Omega\times\Omega)$) if it is measurable for all $t \geq 0$ $\varphi(\cdot,\cdot,t)$ and $\varphi(x,x,\cdot)$ is an N-function for a.a. $(x,x) \in \Omega\times\Omega$. Similarly, one can define functions $\varphi \in N(\Omega)$.*

Next, we recall some definitions related to N-functions and generalized N-functions.

Definition A.2.

(i) *A function $\varphi\colon \Omega\times[0,\infty) \to [0,\infty)$ is said to be locally integrable if $\varphi(\cdot,t)$ belongs to $L^1(\Omega)$ for all $t > 0$.*

(ii) *For $\varphi, \psi \in \mathrm{N}(\Omega)$, then φ is weaker than ψ ($\varphi \prec \psi$) if*

$$\varphi(x,t) \leq c_1\psi(x, c_2 t) + g(x) \quad \textit{for a.a. } x \in \Omega \textit{ and for all } t \geq 0,$$

with $c_1, c_2 > 0$ and $0 \leq g(\cdot) \in L^1(\Omega)$. We say that φ, ψ are equivalent, denoted by $\varphi \sim \psi$, if $\varphi \prec \psi$ and $\psi \prec \varphi$.

(iii) *For $\varphi, \psi \in N(\Omega)$, we say that φ increases essentially slower than ψ near infinity (we write $\varphi \ll \psi$) if for every $k > 0$*

$$\lim_{t\to\infty} \frac{\varphi(x,kt)}{\psi(x,t)} = 0 \textit{ uniformly for a.a. } x \in \Omega.$$

For a fixed $\varphi \in N(\Omega)$, the associated modular function is defined as

$$\rho_\varphi(u) = \int_\Omega \varphi(x, |u|)\,\mathrm{d}x,$$

while the corresponding Musielak–Orlicz space $L^\varphi(\Omega)$ is defined by

$$L^\varphi(\Omega) := \{u \in M(\Omega) \colon \text{there exists } \lambda > 0 \text{ such that } \rho_\varphi(\lambda u) < +\infty\},$$

endowed with the Luxemburg norm

$$\|u\|_{\varphi,\Omega} := \inf\left\{\lambda > 0 \colon \rho_\varphi\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

In what follows, we may denote the above norm by $\|u\|_\varphi$ instead of $\|u\|_{\varphi,\Omega}$.

According to Musielak [55, Theorem 8.5], the following useful embedding result holds.

PROPOSITION A.3. *Let $\varphi \in N(\Omega)$ and $\psi \in N(\Omega)$ such that $\varphi \prec \psi$; then $L^\psi(\Omega) \hookrightarrow L^\varphi(\Omega)$.*

Now let us recall some basic definitions and notations of fractional Musielak–Orlicz Sobolev spaces; see de Albuquerque et al. [26]. For this purpose, we define

$$\mathcal{H}(x,y,t) = \int_0^t h(x,y,\tau)\,\mathrm{d}\tau,$$

where $h \colon \Omega \times \Omega \times [0,\infty) \to [0,\infty)$. We suppose the following assumptions:

(φ_1) $\lim_{t\to 0} h(x,y,t) = 0$ and $\lim_{t\to\infty} h(x,y,t) = +\infty$ with $t \mapsto h(x,y,t)$ being continuous on the interval $(0,\infty)$ for a.a. $(x,y) \in \Omega \times \Omega$;

(φ_2) $t \mapsto h(\cdot,\cdot,t)$ is increasing on $(0,\infty)$;

(φ_3) it holds that

$$\ell \le \frac{h(x,y,t)}{\mathcal{H}(x,y,t)} \le m$$

for $1 < \ell \le m < +\infty$, for a.a. $(x,y) \in \Omega \times \Omega$, and for all $t \in (0,\infty)$.

If h fulfills (φ_1)–(φ_3) and $h(\cdot,\cdot,t)$ is measurable for all $t \ge 0$, then we deduce that $\mathcal{H}$ is a generalized N-function.

In what follows, we present some useful results concerning $\mathcal{H}$ and the associated fractional Musielak–Sobolev space $W_0^{s,\mathcal{H}}(\Omega)$.

DEFINITION A.4. *Let $\mathcal{H} \in N(\Omega \times \Omega)$. We say that $\mathcal{H}$ fulfills the fractional boundedness condition if*

$$(B_f) \qquad 0 < C_1 \le \mathcal{H}(x,y,1) \le C_2 \quad \text{for a.a.}\,(x,y) \in \Omega \times \Omega,$$

with $C_1, C_2 > 0$.

Under hypotheses (H1), it is easy to see that $\mathcal{H}$ fulfills hypotheses (B_f), with $C_1 = 1$ and $C_2 = (1 + \|\mu\|_\infty)\log(e + \alpha)$.

The next proposition can be found in the paper by Azroul et al. [7, Theorem 2.3].

PROPOSITION A.5. *Let hypotheses* (H1) *be satisfied, let $s \in (0,1)$, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Then there holds that*

$$\|u\|_{\widehat{\mathcal{H}}} \le C[u]_{s,\mathcal{H}}$$

for all $u \in W_0^{s,\mathcal{H}}(\Omega)$ with $C > 0$.

For all $u \in W_0^{s,\mathcal{H}}(\Omega)$, it follows from Proposition A.5 that

$$\int_\Omega \widehat{\mathcal{H}}(x,|u(x)|)\,\mathrm{d}x \le \lambda_1 \int_\Omega \int_\Omega \mathcal{H}(x,y,|D_s u(x,y)|)\,\mathrm{d}\nu, \tag{A.1}$$

where $\lambda_1 > 0$. Moreover, $[\,\cdot\,]_{s,\mathcal{H}}$ turns out to be an equivalent norm of $\|\cdot\|_{s,\mathcal{H}}$ on $W_0^{s,\mathcal{H}}(\Omega)$; namely, there exist constants $C', C'' > 0$ such that for all $u \in W_0^{s,\mathcal{H}}(\Omega)$ we have

$$C'[u]_{s,\mathcal{H}} \le \|u\|_{s,\mathcal{H}} \le C''[u]_{s,\mathcal{H}}.$$

The next proposition describes the relation of the norm for the space $L^{\widehat{\mathcal{H}}}(\Omega)$ and its modular; see Lu, Vetro, and Zeng [49, Theorem 2.21] for a detailed proof.

Proposition A.6. *Let hypotheses* (H1) *be satisfied, let $u \in L^{\widehat{\mathcal{H}}}(\Omega)$, and let*

$$\rho_{\widehat{\mathcal{H}}}(u) = \int_\Omega \left[|u|^{p(x)} + \mu(x)|u|^{q(x)}\right] \log(e+\alpha|u|)\,\mathrm{d}x \quad \textit{for all } u \in L^{\widehat{\mathcal{H}}}(\Omega).$$

Then, for $\sigma > 0$, the following hold:
- (i) $\|u\|_{\widehat{\mathcal{H}}} = \lambda \Leftrightarrow \rho_{\widehat{\mathcal{H}}}(\frac{u}{\lambda}) = 1$ *with* $u \neq 0$;
- (ii) $\|u\|_{\widehat{\mathcal{H}}} < 1$ *(resp.,* $=1, >1$*)* $\Leftrightarrow \rho_{\widehat{\mathcal{H}}}(u) < 1$ *(resp.,* $=1, >1$*)*;
- (iii) *if* $\|u\|_{\widehat{\mathcal{H}}} < 1$, *then* $C_\sigma^{-1}\|u\|_{\widehat{\mathcal{H}}}^{q_+ + \sigma} \le \rho_{\widehat{\mathcal{H}}}(u) \le \|u\|_{\widehat{\mathcal{H}}}^{p_-}$;
- (iv) *if* $\|u\|_{\widehat{\mathcal{H}}} > 1$, *then* $\|u\|_{\widehat{\mathcal{H}}}^{p_-} \le \rho_{\widehat{\mathcal{H}}}(u) \le C_\sigma\|u\|_{\widehat{\mathcal{H}}}^{q_+ + \sigma}$;
- (v) $\|u\|_{\widehat{\mathcal{H}}} \to 0 \Leftrightarrow \rho_{\widehat{\mathcal{H}}}(u) \to 0$;
- (vi) $\|u\|_{\widehat{\mathcal{H}}} \to \infty \Leftrightarrow \rho_{\widehat{\mathcal{H}}}(u) \to \infty$;
- (vii) $\|u\|_{\widehat{\mathcal{H}}} \to 1 \Leftrightarrow \rho_{\widehat{\mathcal{H}}}(u) \to 1$;
- (viii) *if* $u_n \to u$ *in* $L^{\widehat{\mathcal{H}}}(\Omega)$, *then* $\rho_{\widehat{\mathcal{H}}}(u_n) \to \rho_{\widehat{\mathcal{H}}}(u)$.

Similar to Proposition A.6, we get some results concerning $\rho_{s,\mathcal{H}}(\cdot)$ and the $(s,\mathcal{H})$-Gagliardo seminorm $[\cdot]_{s,\mathcal{H}}$.

Proposition A.7. *Let hypotheses* (H1) *be satisfied, let $u \in W^{s,\mathcal{H}}(\Omega)$, and let $\sigma > 0$. Then the following hold:*
- (i) $[u]_{s,\mathcal{H}} < 1 \Rightarrow C_\sigma^{-1}[u]_{s,\mathcal{H}}^{q_+ + \sigma} \le \rho_{s,\mathcal{H}}(u) \le [u]_{s,\mathcal{H}}^{p_-}$;
- (ii) $[u]_{s,\mathcal{H}} > 1 \Rightarrow [u]_{s,\mathcal{H}}^{p_-} \le \rho_{s,\mathcal{H}}(u) \le C_\sigma[u]_{s,\mathcal{H}}^{q_+ + \sigma}$.

Due to conditions (φ_1)–(φ_3), we infer that $\widehat{\mathcal{H}}\colon [0,+\infty) \to [0,+\infty)$ is an increasing homeomorphism. Next, we denote by $\widehat{\mathcal{H}}^{-1}$ the inverse function of $\widehat{\mathcal{H}}$ such that

$$\int_0^1 \frac{\widehat{\mathcal{H}}^{-1}(x,\tau)}{\tau^{\frac{N+s}{N}}}\,\mathrm{d}\tau < \infty \quad \text{and} \quad \int_1^\infty \frac{\widehat{\mathcal{H}}^{-1}(x,\tau)}{\tau^{\frac{N+s}{N}}}\,\mathrm{d}\tau = \infty \quad \text{for a.a. } x \in \Omega.$$

Denoting the Musielak–Orlicz Sobolev conjugate function of $\widehat{\mathcal{H}}$ by $\widehat{\mathcal{H}}_s^*$, we can give the definition for the inverse of $\widehat{\mathcal{H}}_s^*$ as

$$(\widehat{\mathcal{H}}_s^*)^{-1}(x,\phi) = \int_0^\phi \frac{\widehat{\mathcal{H}}^{-1}(x,\tau)}{\tau^{\frac{N+s}{N}}}\,\mathrm{d}\tau \quad \text{for a.a. } x \in \Omega \text{ and for all } \phi \ge 0.$$

The next embedding result is taken from Azroul et al. [7, Lemma 2.3].

Lemma A.8. *Let $0 < s' < s < 1$, Ω be a bounded domain in $\mathbb{R}^N$, and suppose hypotheses* (H1). *Then there exists the continuous embedding $W^{s,\mathcal{H}}(\Omega) \hookrightarrow W^{s',r}(\Omega)$ with $r \in [1, p_-)$.*

For more sharp embedding results, we introduce the following definition of a Young function.

DEFINITION A.9. *A function* $\varphi\colon [0,\infty) \to [0,\infty]$ *is called a Young function if it is convex, continuous, and nonconstant,* $\varphi(0) = 0$*, and* $\varphi(t) = \int_0^t a(\tau)\,\mathrm{d}\tau$*, where* $a\colon [0,\infty) \to [0,\infty]$ *is a nondecreasing function. Moreover, we denote the left-continuous inverse of* φ *by* $\varphi^{-1}\colon [0,\infty) \to [0,\infty)$*, which is given as*

$$\varphi^{-1}(t) = \inf\{\tau \geq 0\colon \varphi(\tau) \geq t\}$$

for $t \geq 0$*.*

Let Y be a Young function such that

$$\int_1^\infty \left(\frac{t}{Y(t)}\right)^{\frac{s}{N-s}} \mathrm{d}t = \infty \quad \text{and} \quad \int_0^1 \left(\frac{t}{Y(t)}\right)^{\frac{s}{N-s}} \mathrm{d}t < \infty. \tag{A.2}$$

Then the corresponding Orlicz target is defined as

$$Y_{\frac{N}{s}}(t) = Y(T^{-1}(t)) \tag{A.3}$$

for all $t \geq 0$, with

$$T(t) = \left(\int_0^t \left(\frac{\tau}{Y(\tau)}\right)^{\frac{s}{N-s}} \mathrm{d}\tau\right)^{\frac{N-s}{N}}$$

for all $t \geq 0$.

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