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On Polyharmonic Kirchhoff Problems with Double Phase Structure and Subcritical Nonlinearities

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Abstract

This article studies subcritical elliptic problems driven by a polyharmonic double phase operator and establishes the existence of an unbounded sequence of weak solutions. Our approach relies on the symmetric mountain pass theorem of Ambrosetti and Rabinowitz and successfully treats the delicate degenerate regime of the operator. The results appear to be the first in the literature to address polyharmonic double phase problems within this framework.

Keywords Multiple solutions · Musielak-Orlicz Sobolev spaces · Polyharmonic double phase operator · Subcritical growth problem · Variational methods · Symmetric mountain pass theorem

Mathematics Subject Classification 35A01 · 35G20 · 35J30 · 35J91

1 Introduction

In this work we investigate the existence of a sequence of weak solutions to the Kirchhoff-type polyharmonic double phase problem

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$$\begin{cases} \mathcal{M}\left(\int_\Omega \left(\frac{|\nabla^m u|^p}{p} + a(x)\frac{|\nabla^m u|^q}{q}\right) \mathrm{d}x\right) \mathcal{L}^m_{p,q}(u) = g(x,u), & \text{in } \Omega, \\ u = \nabla u = \cdots = \nabla^{m-1} u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz boundary $\partial\Omega$. For $m \in \mathbb{N}$, we define the m-order differential operator associated with a function u by

$$\nabla^m u = \begin{cases} \nabla \Delta^{\frac{(m-1)}{2}} u & \text{if } m \text{ is odd}, \\ \Delta^{\frac{m}{2}} u & \text{if } m \text{ is even}, \end{cases}$$

where ∇ and Δ denote the classical gradient and Laplacian operators. The operator $\mathcal{L}^m_{p,q}(u)$ denotes the polyharmonic double phase operator associated with problem (1.1) and is given by

$$-\nabla \cdot (\Delta^{\frac{m-1}{2}} \{|\nabla \Delta^{\frac{m-1}{2}} u|^{p-2} \nabla \Delta^{\frac{m-1}{2}} u + a(x) |\nabla \Delta^{\frac{m-1}{2}} u|^{q-2} \nabla \Delta^{\frac{m-1}{2}} u\}), \tag{1.2}$$

when m is odd, and by

$$\Delta^{\frac{m}{2}} \left(|\Delta^{\frac{m}{2}} u|^{p-2} \Delta^{\frac{m}{2}} u + a(x) |\Delta^{\frac{m}{2}} u|^{q-2} \Delta^{\frac{m}{2}} u\right), \tag{1.3}$$

when m is even. We impose the following assumptions:

(H_1) Assume that $1 < p < q < \frac{N}{m}$ with $m \in \mathbb{N}$, and that $a(\cdot) \in L^\infty(\Omega)$ satisfies $a(x) \geq 0$ for a.a. $x \in \Omega$.

In the following, we define the critical Sobolev exponents corresponding to p and q by

$$p^* = \frac{Np}{N - mp} \quad \text{and} \quad q^* = \frac{Nq}{N - mq},$$

respectively.

(H_2) The Kirchhoff function $\mathcal{M} \colon [0, \infty) \to [0, \infty)$ is continuous and there exists a constant $\gamma \in \left[1, \frac{p^*}{q}\right)$ such that

$$\tau\, \mathcal{M}(\tau) \leq \gamma\, M(\tau), \quad \text{for all } \tau \geq 0, \tag{1.4}$$

where $M(\tau) = \int_0^\tau \mathcal{M}(s)\, \mathrm{d}s$. Moreover, for each $\sigma > 0$ there exists a constant $k = k(\sigma) > 0$ such that

$$\mathcal{M}(\tau) \geq k \quad \text{for all } \tau \geq \sigma.$$

(H_3) The function $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is of Carathéodory type, that is, $g(\cdot,t)$ is measurable for all $t\in\mathbb{R}$ and $g(x,\cdot)$ is continuous for a.a. $x\in\Omega$. Moreover, g satisfy the following conditions:

(i) $g(x,\cdot)$ is odd, i.e., $g(x,-t)=-g(x,t)$ for a.a. $x\in\Omega$ and for all $t\in\mathbb{R}$.

(ii) There exist constants $C>0$ and $r\in(\gamma q,p^*)$ such that

$$|g(x,t)|\leq C\big(1+|t|^{r-1}\big)$$

for a.a. $x\in\Omega$ and for all $t\in\mathbb{R}$, where γ is from (H_2).

(iii) There exist $\mu>\gamma q$, $c>0$ and $U\geq 0$ such that

$$c\leq \mu G(x,t)\leq tg(x,t) \tag{1.5}$$

for a.a. $x\in\Omega$ and for all $|t|\geq U$, where $G(x,t)=\int_0^t g(x,s)\,\mathrm{d}s$.

(iv) $\displaystyle\lim_{t\to 0^+}\frac{G(x,t)}{t^{\gamma q}}=0$ uniformly for a.a. $x\in\Omega$.

Remark 1.1 Hypothesis (H_2) ensures, in particular, that $M(\tau)>0$ for every $\tau>0$. Moreover, (H_2) implies that

$$M(\tau)\geq M(1)\tau^{\gamma}\quad\text{for all } \tau\in[0,1]. \tag{1.6}$$

In addition, for any $\varepsilon>0$, we have

$$M(\tau)\leq \delta\tau^{\gamma}\quad\text{for all } \tau\geq\varepsilon, \tag{1.7}$$

where $\delta=\delta(\varepsilon)=M(\varepsilon)/\varepsilon^{\gamma}>0$. Consequently, inequality (1.7) leads to

$$\lim_{\tau\to\infty}\tau^{-\mu/q}M(\tau)=0, \tag{1.8}$$

where $\mu>\gamma q$.

Remark 1.2 We also observe that assumptions (H_3) (ii) and (H_3) (iii) necessarily imply the relation $\mu\leq r$. Condition (H_3) (iii), commonly referred to as the Ambrosetti-Rabinowitz superquadraticity condition, plays a central role in our analysis. In particular, it ensures the boundedness of Palais-Smale sequences associated with the corresponding energy functional, which is essential for verifying the Palais-Smale condition within the variational framework.

Example 1.3 A concrete example of a Kirchhoff function satisfying hypothesis (H_2) is

$$\mathcal{M}(\tau)=a+b\gamma\tau^{\gamma-1},\quad a,b\geq 0,\quad a+b>0,\quad \gamma\in\begin{cases}\left(1,\frac{p^*}{q}\right), & \text{if } b>0,\\ 1, & \text{if } b=0.\end{cases} \tag{1.9}$$

If $\mathcal{M}$ is given by (1.9), the corresponding problem (1.1) is called nondegenerate when $a > 0$ and $b \geq 0$, while it is called degenerate when $a = 0$ and $b > 0$. A typical example of a function $g\colon \mathbb{R} \to \mathbb{R}$ satisfying (H$_3$) is

$$g(t) = \begin{cases} |t|^{r-2}t - |t|^{\wp-2}t, & \text{if } |t| \leq 1, \\ |t|^{r-2}t - |t|^{q-2}t, & \text{if } |t| \geq 1, \end{cases}$$

where the exponents $\wp, p, q, r$ satisfy $1 < \gamma q < \wp$ and $\gamma q < r < p^*$, with $\gamma \in \big[1, \frac{p^*}{q}\big)$. In this situation one easily verifies that (H$_3$) (i), (ii), and (iv) hold with $C = 2$, since $\gamma q < \max\{\wp, r\}$. Furthermore, one can show that for every $\mu \in (\gamma q, r]$ there exist

$$U = \max\left\{(r/q)^{1/(r-q)}, \left[\mu(\wp - q)\,/\,\wp(\mu - q)\right]^{1/q}\right\} > 1$$

and

$$c = \frac{\mu(\wp - q)}{\wp q},$$

such that condition (H$_3$) (iii) is also satisfied.

Let $W_0^{m,\mathcal{A}}(\Omega)$ denote the homogeneous Musielak-Orlicz Sobolev space, which will be introduced in Sect. 2, and define

$$\Phi_{\mathcal{A}}(u) = \int_\Omega \left(\frac{|u|^p}{p} + a(x)\frac{|u|^q}{q}\right) \mathrm{d}x.$$

The energy functional $J\colon W_0^{m,\mathcal{A}}(\Omega) \to \mathbb{R}$ associated with problem (1.1) belongs to $C^1\big(W_0^{m,\mathcal{A}}(\Omega)\big)$ and is given by

$$J(u) = M(\Phi_{\mathcal{A}}(\nabla^m u)) - \int_\Omega G(x, u)\,\mathrm{d}x. \tag{1.10}$$

A weak solution of problem (1.1) is a function $u \in W_0^{m,\mathcal{A}}(\Omega)$ that is a critical point of the energy functional J defined in (1.10). Equivalently, u satisfies

$$\begin{aligned} &\mathcal{M}(\Phi_{\mathcal{A}}(\nabla^m u)) \int_\Omega (|\nabla^m u|^{p-2} + a(x)|\nabla^m u|^{q-2})\nabla^m u \nabla^m \varphi\,\mathrm{d}x \\ &= \int_\Omega g(x, u)\varphi\,\mathrm{d}x \end{aligned} \tag{1.11}$$

for all $\varphi \in W_0^{m,\mathcal{A}}(\Omega)$.

The main result in this paper reads as follows.

Theorem 1.4 *Let hypotheses* (H$_1$)–(H$_3$) *be satisfied. Then problem* (1.1) *admits an unbounded sequence of weak solutions.*

Problem (1.1) involves a nonlocal Kirchhoff term coupled with a nonhomogeneous operator of double phase type. When $m = 1$, the operators in (1.2)–(1.3) reduce to the variational operator associated with the functional

$$\omega \mapsto \int_\Omega \left(|\nabla \omega|^p + a(x)|\nabla \omega|^q\right) \mathrm{d}x, \tag{1.12}$$

introduced by Zhikov [33]. Functionals of this kind arise naturally in the study of strongly anisotropic materials within homogenization and elasticity theory. In such models, the hardening response of the material varies spatially, and the coefficient $a(\cdot)$ governs the interaction between two components exhibiting different growth behaviors corresponding to the exponents p and q. Beyond applications in material science, functionals of type (1.12) play a fundamental role in several areas of nonlinear analysis, such as duality theory and the investigation of the Lavrentiev gap phenomenon, see Zhikov [34, 35]. Moreover, (1.12) is part of the more general class of variational integrals with nonstandard growth, extensively studied in the pioneering works of Marcellini [28, 29]. We also refer to recent contributions by Cupini–Marcellini–Mascolo [12] and Marcellini [27], where operators with explicit dependence on the solution u are analyzed.

Another noteworthy aspect of problem (1.1) is the appearance of the nonlocal Kirchhoff term, which traces back to the classical model proposed by Kirchhoff [24],

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, ρ_0, h, E and L are physical constants. This equation represents a nonlocal modification of the classical D'Alembert wave equation and has become a central model in various applications in physics and engineering. Since the seminal work of Lions [25], which established the analytical foundation for Kirchhoff-type equations, a rich and diverse literature has developed around these problems, addressing them under different structural conditions and employing a wide range of techniques. For further background and representative contributions, see for example Autuori–Pucci–Salvatori [5], D'Ancona–Spagnolo [13], Figueiredo [15], Fiscella [16], and Fiscella–Valdinoci [19].

Despite the extensive literature on Kirchhoff-type equations, problems involving double phase operators have received comparatively little attention. A first contribution in this direction is due to Fiscella–Pinamonti [18], who studied the Dirichlet problem

$$-\mathcal{M}\left[\int_\Omega \left(\frac{|\nabla u|^p}{p} + \mu(x)\frac{|\nabla u|^q}{q}\right) \mathrm{d}x\right]\mathcal{G}(u) = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

under subcritical assumptions on the reaction term satisfying the Ambrosetti-Rabinowitz condition. Using the mountain pass theorem, they proved the existence of a nontrivial weak solution. More recently, Arora–Fiscella–Mukherjee–Winkert [4] considered the singular double phase Kirchhoff problem

$$-\mathcal{M}\left[\int_\Omega\left(\frac{|\nabla u|^p}{p}+\mu(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x\right]\mathcal{G}(u)=\lambda u^{-\gamma}+u^{r-1}\quad\text{in }\Omega,$$

with $u = 0$ on $\partial\Omega$, and established, via a suitable splitting of the Nehari manifold, the existence of two distinct positive solutions at different energy levels. Its critical analogue was later analyzed in Arora–Fiscella–Mukherjee–Winkert [3]. Further developments include degenerate double phase Kirchhoff problems by Cen–Vetro–Zeng [7], Neumann-type double phase problems by Fiscella–Marino–Pinamonti–Verzellesi [17], and mountain pass-type results by Gupta–Dwivedi [20]. Ho–Winkert [22] obtained infinitely many solutions via abstract critical point theory, while Crespo-Blanco–Gasiński–Winkert [10] established the existence of a sign-changing solution with least energy. In the context of polyharmonic Kirchhoff equations, Colasuonno–Pucci [8] proved the existence of an unbounded sequence of solutions for $p(\cdot)$-polyharmonic problems. Mishra–Goyal–Sreenadh [30] obtained nonnegative solutions combining the mountain pass theorem with concentration-compactness principle. More recently, Bousgheiri–Ourraoui [6] studied Dirichlet $(p_1,\ldots,p_N)$-polyharmonic Kirchhoff problems, while Harrabi–Hamdani–Fiscella [21] established multiplicity for higher-order Kirchhoff equations on unbounded domains. Deng–Huang [14] addressed polyharmonic Kirchhoff-type problems with singular exponential nonlinearities without assuming the Ambrosetti-Rabinowitz condition. Despite these advances in Kirchhoff double phase problems and polyharmonic equations, the polyharmonic double phase problem has remained essentially unexplored. This gap in the literature provides the main motivation for the present work.

It is worth emphasizing that the double phase operator treated in this work can be viewed as a particular case of the broader class of Musielak-Orlicz operators. A notable recent advancement in this area concerns the analysis of the logarithmic double phase operator, driven by the Musielak-Orlicz function

$$\Phi(x,t)=t^p+a(x)t^q\log(e+t),\quad 1<p<q<N, \tag{1.13}$$

for all $x\in\Omega$ and for every $t\geq 0$, where e denotes Euler's number. Compared with the classical double phase energy density, this function displays a more delicate growth pattern and naturally fits within the Musielak-Orlicz framework. The functional analytic structure of the associated Musielak-Orlicz Sobolev spaces, together with the corresponding variational setting in the logarithmic double phase case, has recently been examined by Arora–Crespo-Blanco–Winkert [2]. Later, Vetro [32] investigated Kirchhoff-type problems driven by the logarithmic double phase operator, obtaining existence and multiplicity results. Problems involving generalized critical growth for this operator have been further analyzed in [1]. Although the present paper establishes existence and multiplicity results for polyharmonic double phase operators, extending these conclusions to the polyharmonic logarithmic double phase framework (1.13) constitutes a compelling and technically demanding open direction. The logarithmic contribution generates a genuinely nonstandard growth behavior and creates additional analytical challenges, especially with respect to compactness arguments and the derivation of appropriate higher-order embedding theorems in the related Musielak-Orlicz Sobolev spaces.

This paper is organized as follows. In Sect. 2, we recall the fundamental properties of Musielak-Orlicz Sobolev spaces $W_0^{m,\mathcal{A}}(\Omega)$ and present the main embedding results required in our analysis. Section 3 contains the proof of Theorem 1.4, where we establish the existence of an unbounded sequence of weak solutions to problem (1.1).

2 Preliminaries

In this section, we briefly recall the fundamental properties and embedding results related to Musielak-Orlicz Sobolev spaces. Throughout the paper, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$. For any $s \in [1, \infty)$, we denote by $L^s(\Omega)$ the classical Lebesgue spaces equipped with the norm $\|\cdot\|_s$. Moreover, for $1 < s < \infty$, the Sobolev space $W_0^{m,s}(\Omega)$ is taken with the equivalent norm $\|\nabla^m \cdot\|_s$.

Let hypothesis (H_1) hold and consider the nonlinear function $\mathcal{A}\colon \Omega \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{A}(x,t) = t^p + a(x)t^q.$$

Let $M(\Omega)$ denote the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. We define the associated Musielak-Orlicz Lebesgue space by

$$L^{\mathcal{A}}(\Omega) = \{u \in M(\Omega)\colon \varrho_{\mathcal{A}}(u) < \infty\},$$

where the modular $\varrho_{\mathcal{A}}$ is given by

$$\varrho_{\mathcal{A}}(u) = \int_\Omega \mathcal{A}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + a(x)|u|^q\right)\mathrm{d}x.$$

The space $L^{\mathcal{A}}(\Omega)$ is equipped with the Luxemburg norm

$$\|u\|_{\mathcal{A}} = \inf\left\{t > 0\colon \varrho_{\mathcal{A}}\left(\frac{u}{t}\right) \le 1\right\}.$$

We also introduce the seminormed space

$$L_a^q(\Omega) = \left\{u \in M(\Omega)\colon \int_\Omega a(x)|u|^q\,\mathrm{d}x < \infty\right\}.$$

endowed with the seminorm

$$\|u\|_{q,a} = \left(\int_\Omega a(x)|u|^q\,\mathrm{d}x\right)^{1/q}.$$

The Luxemburg norm $\|\cdot\|_{\mathcal{A}}$ and the modular function $\varrho_{\mathcal{A}}$ satisfy the following relations, see Liu–Dai [26, Proposition 2.1].

Proposition 2.1 *Let hypothesis* (H_1) *be satisfied,* $u \in L^{\mathcal{A}}(\Omega)$ *and* $c > 0$*. Then the following hold:*

(i) *If* $u \neq 0$*, then* $\|u\|_{\mathcal{A}} = c$ *if and only if* $\varrho_{\mathcal{A}}\left(\frac{u}{c}\right) = 1$*,*
(ii) $\|u\|_{\mathcal{A}} < 1$ *(resp.* > 1*,* $= 1$*) if and only if* $\varrho_{\mathcal{A}}(u) < 1$ *(resp.* > 1*,* $= 1$*),*
(iii) *If* $\|u\|_{\mathcal{A}} < 1$*, then* $\|u\|_{\mathcal{A}}^q \leq \varrho_{\mathcal{A}}(u) \leq \|u\|_{\mathcal{A}}^p$*,*
(iv) *If* $\|u\|_{\mathcal{A}} > 1$*, then* $\|u\|_{\mathcal{A}}^p \leq \varrho_{\mathcal{A}}(u) \leq \|u\|_{\mathcal{A}}^q$*,*
(v) $\|u\|_{\mathcal{A}} \to 0$ *if and only if* $\varrho_{\mathcal{A}}(u) \to 0$*,*
(vi) $\|u\|_{\mathcal{A}} \to \infty$ *if and only if* $\varrho_{\mathcal{A}}(u) \to \infty$*.*

As a direct consequence of Proposition 2.1 (iii) and (iv), we obtain

$$\min\{\|u\|_{\mathcal{A}}^p, \|u\|_{\mathcal{A}}^q\} \leq \int_\Omega (|u|^p + a(x)|u|^q)\,\mathrm{d}x \leq \max\{\|u\|_{\mathcal{A}}^p, \|u\|_{\mathcal{A}}^q\}, \tag{2.1}$$

for all $u \in L^{\mathcal{A}}(\Omega)$.

The Musielak-Orlicz Sobolev space $W^{m,\mathcal{A}}(\Omega)$ consists of all functions $u \in L^{\mathcal{A}}(\Omega)$ such that $|\nabla^k u| \in L^{\mathcal{A}}(\Omega)$ for all $k \in \{0, 1, 2, \cdots, m\}$. It is endowed with the norm

$$\|u\|_{m,\mathcal{A}} = \sum_{k=0}^m \|\nabla^k u\|_{\mathcal{A}},$$

where we use the notation $\|\nabla^k u\|_{\mathcal{A}} = \|\,|\nabla^k u|\,\|_{\mathcal{A}}$ for all $k \in \{0, 1, 2, \cdots, m\}$. With this norm, $W^{m,\mathcal{A}}(\Omega)$ is a reflexive Banach space. Moreover, the space $W_0^{m,\mathcal{A}}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{m,\mathcal{A}}(\Omega)$. It is also a reflexive Banach space.

The following Poincaré-type inequality can be found in Colasuonno–Squassina [9] and Crespo-Blanco–Gasiński–Harjulehto–Winkert [11].

Theorem 2.2 *There exists a constant* $k > 0$ *such that*

$$\|\phi\|_{\mathcal{A}} \leq k\|\nabla\phi\|_{\mathcal{A}}, \quad \textit{for all } \phi \in W_0^{1,\mathcal{A}}(\Omega). \tag{2.2}$$

In fact, estimate (2.2) can be established for $W_0^{m,\mathcal{A}}(\Omega)$ by following an argument similar to that in [11, Proposition 2.18]. Consequently, we obtain the following result.

Corollary 2.3 $\|\nabla^m\phi\|_{\mathcal{A}}$ *defines a norm on* $W_0^{m,\mathcal{A}}(\Omega)$ *that is equivalent to the standard norm* $\|\phi\|_{m,\mathcal{A}}$*.*

By Corollary 2.3, we may introduce the equivalent norm $\|u\| = \|\nabla^m u\|_{\mathcal{A}}$ on $W_0^{m,\mathcal{A}}(\Omega)$. Since reflexivity is preserved under equivalent norms, it follows that $(W_0^{m,\mathcal{A}}(\Omega), \|\cdot\|)$ is a reflexive Banach space. Applying inequality (2.1), we further obtain

$$\min\{\|u\|^p, \|u\|^q\} \leq \int_\Omega \left(|\nabla^m u|^p + a(x)|\nabla^m u|^q\right)\mathrm{d}x \leq \max\{\|u\|^p, \|u\|^q\}, \tag{2.3}$$

for all $u \in W_0^{m,\mathcal{A}}(\Omega)$.

Next, we recall some basic definitions from the theory of Musielak-Orlicz spaces.

Definition 2.4 A function $\varphi\colon [0,\infty) \to [0,\infty)$ is called a Φ-function, if the following conditions are satisfied:

(i) φ is continuous and convex,
(ii) $\varphi(0) = 0$,
(iii) $\varphi(t) > 0$ for all $t > 0$.

Definition 2.5 A function $\varphi\colon \Omega\times[0,\infty) \to [0,\infty)$ is called a generalized Φ-function, denoted by $\varphi \in \Phi(\Omega)$, if it satisfies:

(i) for each fixed $t \geq 0$, the mapping $x \mapsto \varphi(x,t)$ is measurable,
(ii) for a.a. $x \in \Omega$, the mapping $t \mapsto \varphi(x,t)$ is a Φ-function.

Definition 2.6 A function $\varphi\colon [0,\infty) \to [0,\infty)$ is called an $\mathcal{N}$-function, if it is a Φ function and satisfies

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

Definition 2.7 A function $\varphi\colon \Omega \times [0,\infty) \to [0,\infty)$ is called a generalized $\mathcal{N}$-function, written $\varphi \in N(\Omega)$, if the following hold:

(i) for each fixed $t \geq 0$, the mapping $x \mapsto \varphi(x,t)$ is measurable,
(ii) for a.a. $x \in \Omega$, the mapping $t \mapsto \varphi(x,t)$ is an $\mathcal{N}$-function.

Definition 2.8 Let $\varphi, \psi \in \Phi(\Omega)$. We say that φ is dominated by ψ, and write $\varphi \preceq \psi$, if there exist constants $C_1, C_2 > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\varphi(x,t) \leq C_1\psi(x, C_2 t) + h(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } t \geq 0.$$

The following result can be found in Musielak [31, Theorem 8.5].

Proposition 2.9 *Let $\varphi, \psi \in N(\Omega)$ satisfy $\varphi \preceq \psi$. Then the space $L^\psi(\Omega)$ is continuously embedded in $L^\varphi(\Omega)$, that is, $L^\psi(\Omega) \hookrightarrow L^\varphi(\Omega)$.*

Proposition 2.10 *Let hypothesis* (H_1) *be satisfied. Then the following statements hold:*

(i) *the embeddings $L^{\mathcal{A}}(\Omega) \hookrightarrow L^s(\Omega)$ and $W_0^{m,\mathcal{A}}(\Omega) \hookrightarrow W_0^{m,s}(\Omega)$ are continuous for all $s \in [1,p]$,*
(ii) *the embedding $W_0^{m,\mathcal{A}}(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $s \in [1,p^*]$ and compact for $s \in [1,p^*)$,*
(iii) *the embeddings $L^q(\Omega) \hookrightarrow L^{\mathcal{A}}(\Omega) \hookrightarrow L^q_a(\Omega)$ are continuous.*

Proof Define $\mathcal{A}_p(x,t) = t^p$ for a.a. $x \in \Omega$ and for all $t \geq 0$. Since $\mathcal{A}_p \preceq \mathcal{A}$, Proposition 2.9 yields the continuous embeddings $L^{\mathcal{A}}(\Omega) \hookrightarrow L^p(\Omega)$ and $W_0^{m,\mathcal{A}}(\Omega) \hookrightarrow W_0^{m,p}(\Omega)$. This proves part (i). Assertion (ii) follows directly from the classical

Sobolev embedding theorem combined with part (i), and is therefore omitted. For (iii), observe that

$$\int_\Omega a(x)|u|^q \,\mathrm{d}x \le \int_\Omega (|u|^p + a(x)|u|^q)\,\mathrm{d}x = \varrho_{\mathcal{A}}(u)$$

for all $u \in L^{\mathcal{A}}(\Omega)$. Moreover, for $u \neq 0$,

$$\int_\Omega a(x)\left|\frac{u}{\|u\|_{\mathcal{A}}}\right|^q \mathrm{d}x \le 1 \quad \text{if and only if} \quad \|u\|_{q,a} \le \|u\|_{\mathcal{A}}.$$

By hypothesis (H_1), we have

$$\mathcal{A}(x,t) \le 1 + (1 + \|a\|_\infty)t^q, \quad \text{for a.a. } x \in \Omega \text{ and for all } t \ge 0.$$

Together with Proposition 2.9, this implies the desired embeddings in (iii). □

Remark 2.11 By Proposition 2.10 (ii), the embedding $W_0^{m,\mathcal{A}}(\Omega) \hookrightarrow L^s(\Omega)$ is compact for $1 \le s < p^*$. In particular it is continuous, and hence there exists a constant $\mathcal{S}_s > 0$ such that

$$\|u\|_s \le \mathcal{S}_s \|u\| \quad \text{for all } u \in W_0^{m,\mathcal{A}}(\Omega) \tag{2.4}$$

Let X be a real Banach space and let $J\colon X \to \mathbb{R}$ be a C^1-functional. A sequence $\{u_j\}_{j\in\mathbb{N}} \subset X$ is called a Palais-Smale sequence for J ((PS)-sequence for short) if

$$\{J(u_j)\}_{j\in\mathbb{N}} \text{ is bounded} \quad \text{and} \quad J'(u_j) \to 0 \quad \text{in } X^*.$$

We say that J satisfies the Palais-Smale condition ((PS)-condition for short) if every (PS)-sequence for J admits a convergent subsequence.

The following result is taken from Jabri [23, Theorem 11.5], known as the symmetric mountain pass theorem.

Theorem 2.12 *Let X be a real, infinite-dimensional Banach space and let $J \in C^1(X,\mathbb{R})$ satisfy the* (PS)*-condition. Assume in addition that:*

(i) $J(0) = 0$*, and there exist constants $\nu > 0$ and $\alpha > 0$ such that $J(u) \ge \alpha$ for all $u \in \partial B_\nu$, where $\partial B_\nu = \{u \in X\colon \|u\| = \nu\}$,*
(ii) *J is even, that is, $J(-u) = J(u)$ for all $u \in X$,*
(ii) *for every finite-dimensional subspace $\widetilde{X} \subset X$, there exists $K = K(\widetilde{X}) > 0$ such that*

$$J(u) \le 0 \quad \textit{for all } u \in X \setminus B_K(\widetilde{X}),$$

where

$$B_K(\widetilde{X}) = \{u \in \widetilde{X}\colon \|u\| < K\}.$$

Then J admits an unbounded sequence of critical values obtained via a minimax argument.

3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Our approach relies on the application of the symmetric mountain pass theorem stated in Theorem 2.12.

Lemma 3.1 *The energy functional J defined in* (1.10) *satisfies the* (PS)*-condition.*

Proof Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{m,\mathcal{A}}(\Omega)$ be a (PS)-sequence for J, that is,

$$\sup_{j\in\mathbb{N}} |J(u_j)| = A < \infty \quad \text{and} \quad J'(u_j) \to 0 \quad \text{in } W_0^{m,\mathcal{A}}(\Omega)^*. \tag{3.1}$$

We distinguish two cases.

Case 1. $\inf_{j\in\mathbb{N}} \|u_j\| = d > 0$. We first show that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{m,\mathcal{A}}(\Omega)$. Let $\kappa = \kappa(d)$ be the constant corresponding to $\sigma \geq \min\{\frac{d^p}{q}, \frac{d^q}{q}\}$ in hypothesis (H_2). Then $\mathcal{M}(\Phi_{\mathcal{A}}(\nabla^m u_j)) \geq \kappa$ for all $j \in \mathbb{N}$. Using (1.10), (1.11), and assumptions (1.4) and (1.5) we compute

$$\begin{aligned}
&J(u_j) - \frac{1}{\mu}\langle J'(u_j), u_j\rangle \\
\geq &M(\Phi_{\mathcal{A}}(\nabla^m u_j)) - \int_\Omega G(x, u_j)\,\mathrm{d}x \\
&- \frac{q}{\mu}\left(\mathcal{M}(\Phi_{\mathcal{A}}(\nabla^m u_j))\Phi_{\mathcal{A}}(\nabla^m u_j) - \int_\Omega u_j g(x, u_j)\,\mathrm{d}x\right) \\
\geq &\left(\frac{1}{\gamma} - \frac{q}{\mu}\right)\mathcal{M}(\Phi_{\mathcal{A}}(\nabla^m u_j))\Phi_{\mathcal{A}}(\nabla^m u_j) - \int_\Omega \left[G(x, u_j) - \frac{1}{\mu}u_j g(x, u_j)\right]\mathrm{d}x, \\
\geq &C_d \Phi_{\mathcal{A}}(\nabla^m u_j) - \int_{\Omega_j} \left[G(x, u_j) - \frac{1}{\mu}u_j g(x, u_j)\right]^+ \mathrm{d}x,
\end{aligned}$$

where $\Omega_j = \{x \in \Omega\colon |u_j(x)| \leq U\}$ and

$$C_d = \left(\frac{1}{\gamma} - \frac{q}{\mu}\right)\kappa > 0$$

By assumption (1.5), the quantity

$$d_U := \sup_{x\in\Omega, |t|\leq U} [G(x,t) - tg(x,t)/\mu]^+ < \infty$$

is finite. Hence,

$$J(u_j) - \frac{1}{\mu}\langle J'(u_j), u_j\rangle \geq C_d \Phi_{\mathcal{A}}(\nabla^m u) - D, \tag{3.2}$$

where $D = d_U|\Omega| < \infty$. On the other hand, since $J'(u_j) \in W_0^{m,\mathcal{A}}(\Omega)^*$, by (3.1), we obtain

$$J(u_j) - \frac{1}{\mu}\langle J'(u_j), u_j\rangle \le A + \frac{1}{\mu}\langle J'(u_j), u_j\rangle \le A + B\|u_j\|, \tag{3.3}$$

where $\mu B = \sup_j \|J'(u_j)\|_*$ with $\|\cdot\|_*$ being the norm of $W_0^{m,\mathcal{A}}(\Omega)^*$. Combining (3.2), (3.3), and (2.3), we deduce

$$A + B\|u_j\| \ge C_d \Phi_{\mathcal{A}}(\nabla^m u_j) - D \ge \frac{C_d}{q}\min\left\{\|u_j\|^p, \|u_j\|^q\right\} - D,$$

which implies that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{m,\mathcal{A}}(\Omega)$. Since $W_0^{m,\mathcal{A}}(\Omega)$ is reflexive, there exists $u \in W_0^{m,\mathcal{A}}(\Omega)$ such that, up to a subsequence, $u_j \rightharpoonup u$ weakly in $W_0^{m,\mathcal{A}}(\Omega)$.

We now show that the convergence is strong. For every $j \in \mathbb{N}$

$$\begin{aligned}&\langle J'(u_j), u_j - u\rangle\\ &= \mathcal{M}(\Phi_{\mathcal{A}}(\nabla^m u_j))\int_\Omega \left(|\nabla^m u_j|^{p-2} + a(x)|\nabla^m u_j|^{q-2}\right)\nabla^m u_j \nabla^m(u_j - u)\,\mathrm{d}x\\ &\quad - \int_\Omega g(x, u_j)(u_j - u)\,\mathrm{d}x.\end{aligned} \tag{3.4}$$

By (3.1) and the boundedness of $\{u_j\}_{j\in\mathbb{N}}$ in $W_0^{m,\mathcal{A}}(\Omega)$

$$|\langle J'(u_j), u_j - u\rangle| \le \|J'(u_j)\|_*\|u_j - u\| \to 0, \tag{3.5}$$

as $j \to \infty$. Using Proposition 2.10 (ii), the embedding $W_0^{m,\mathcal{A}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact. Hence, $\{u_j\}_{j\in\mathbb{N}} \to u$ strongly in $L^r(\Omega)$. By hypotheses (H_3) (ii), (iii) and Hölder's inequality, we have

$$\begin{aligned}&\int_\Omega g(x, u_j)(u_j - u)\,\mathrm{d}x\\ &\le C\int_\Omega (1 + |u_j|^{r-1})|u_j - u|\,\mathrm{d}x\\ &= C\left(\|u_j - u\|_1 + \int_\Omega |u_j|^{r-1}|u_j - u|\,\mathrm{d}x\right) \le C_1\|u_j - u\|_r \to 0,\end{aligned} \tag{3.6}$$

as $j \to \infty$, where $C_1 = 2C\max\{|\Omega|^{1/r'}, \sup_j \|u_j\|_r^{r-1}\} < \infty$ and r' is the conjugate exponent of r. Combining (3.4), (3.5), and (3.6) yields

$$\mathcal{M}(\Phi_{\mathcal{A}}(\nabla^m u_j))\int_\Omega \left(|\nabla^m u_j|^{p-2} + a(x)|\nabla^m u_j|^{q-2}\right)\nabla^m u_j \nabla^m(u_j - u)\,\mathrm{d}x \to 0$$

as $j \to \infty$. Since $\mathcal{M}(\Phi_{\mathcal{A}}(\nabla^m u_j)) \geq \kappa > 0$ for all $j \in \mathbb{N}$, where $\kappa = \kappa(d)$ is given by hypothesis (H_2), this implies

$$\int_\Omega \left(|\nabla^m u_j|^{p-2} + a(x)|\nabla^m u_j|^{q-2}\right) \nabla^m u_j \nabla^m (u_j - u)\,\mathrm{d}x \to 0 \tag{3.7}$$

as $j \to \infty$. By the weak lower semicontinuity of the norm, we have $\|\nabla^m u_j\|_p^p \leq \liminf_{j\to\infty} \|\nabla^m u_j\|_p^p$. On the other hand, by convexity,

$$\|\nabla^m u\|_p^p + p \int_\Omega |\nabla^m u_j|^{p-2} \nabla^m u_j \nabla^m (u_j - u)\,\mathrm{d}x \geq \|\nabla^m u_j\|_p^p,$$

Passing to the limit and using (3.7), we obtain

$$\|\nabla^m u\|_p^p \geq \limsup_{j\to\infty} \|\nabla^m u_j\|_p^p.$$

Consequently,

$$\lim_{j\to\infty} \|\nabla^m u_j\|_p = \|\nabla^m u\|_p.$$

By an analogous argument, it also follows that

$$\lim_{j\to\infty} \|\nabla^m u_j\|_{q,a} = \|\nabla^m u\|_{q,a}.$$

Therefore, $\Phi_{\mathcal{A}}(\nabla^m u_j) \to \Phi_{\mathcal{A}}(\nabla^m u)$ as $j \to \infty$. Using the uniform convexity of the integrand associated with the modular function, we deduce that

$$\Phi_{\mathcal{A}}\left(\frac{\nabla^m u_j - \nabla^m u}{2}\right) \to 0.$$

Finally, Proposition 2.1(v) yields $u_j \to u$ strongly in $W_0^{m,\mathcal{A}}(\Omega)$. This completes the proof of Case 1.

Case 2. $\inf_{j\in\mathbb{N}} \|u_j\| = 0$. If 0 is a limit point of the sequence $\{\|u_j\|\}_{j\in\mathbb{N}}$, then there exists a subsequence of $\{u_j\}_{j\in\mathbb{N}}$ that converges strongly to $u = 0$ in $W_0^{m,\mathcal{A}}(\Omega)$, and the conclusion follows immediately. Otherwise, 0 is an isolated point of the set $\{\|u_j\|\}_{j\in\mathbb{N}}$. Hence, we may extract a subsequence, not relabeled, such that $\inf_{j\in\mathbb{N}} \|u_j\| = d > 0$. In this situation, the arguments developed in Case 1 apply verbatim to this subsequence, and therefore the desired conclusion follows. □

Lemma 3.2 *There exist constants $\nu, \alpha > 0$ such that $J(u) \geq \alpha$ for every $u \in W_0^{m,\mathcal{A}}(\Omega)$ with $\|u\| = \nu$.*

Proof By assumption (H_3) (iv), for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|G(x,u)| \leq \varepsilon\, |u|^{\gamma q} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq u < \delta.$$

Moreover, applying (H_3) (ii) for $u \geq \delta$, we obtain

$$|G(x,u)| \leq \int_0^u |g(x,t)|\,\mathrm{d}t \leq C\left(u + \frac{u^r}{r}\right) \leq c_\varepsilon |u|^r,$$

where $c_\varepsilon = 2C\max\{\delta^{1-r}, \frac{1}{r}\} > 0$. Since $G(x,\cdot)$ is an even function by (H_3) (i), it follows that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|G(x,u)| \leq \varepsilon |u|^{\gamma q} + c_\varepsilon |u|^r \quad \text{for a.a. } x \in \Omega \text{ and for all } u \in \mathbb{R}. \tag{3.8}$$

Choose $\varepsilon \in \left(0, \frac{M(1)}{q^\gamma \mathcal{S}_{\gamma q}^{\gamma q}}\right)$, where $\mathcal{S}_{\gamma q}$ denotes the Sobolev embedding constant in (2.4) corresponding to $s = \gamma q$. Define

$$\kappa_\varepsilon = \frac{1}{q^\gamma} M(1) - \varepsilon \mathcal{S}_{\gamma q}^{\gamma q} > 0 \quad \text{and} \quad C_\varepsilon = c_\varepsilon\, \mathcal{S}_r^r,$$

where $\mathcal{S}_r$ is the embedding constant in (2.4) for $s = r$. Next, choose $\nu \in (0,1]$ sufficiently small such that $\nu^{r-\gamma q} < \kappa_\varepsilon / C_\varepsilon$. Then, for every $u \in W_0^{m,\mathcal{A}}(\Omega)$ with $\|u\| = \nu$, using (1.10), (2.3), (1.6), and (3.8), we derive

$$J(u) \geq \frac{1}{q^\gamma} M(1)\|u\|^{\gamma q} - \varepsilon \mathcal{S}_{\gamma q}^{\gamma q} \|u\|^{\gamma q} - C_\varepsilon \|u\|^r = \nu^{\gamma q}\left(\kappa_\varepsilon - C_\varepsilon \nu^{r-\gamma q}\right).$$

Setting

$$\alpha = \nu^{\gamma q}\left(\kappa_\varepsilon - C_\varepsilon \nu^{r-\gamma q}\right) > 0,$$

we conclude that $J(u) \geq \alpha$ for all u with $\|u\| = \nu$. This completes the proof. □

Lemma 3.3 *Let W be a finite-dimensional subspace of $W_0^{m,\mathcal{A}}(\Omega)$. Then there exists a constant $K_0 = K_0(W) > 0$ such that*

$$J(u) \leq 0 \quad \text{for all } u \in W \setminus B_{K_0}(W),$$

where $B_{K_0}(W) = \{u \in W \colon \|u\| < K_0\}$.

Proof For all $u > U$, assumption (H_3) (iii) yields

$$\frac{g(x,u)}{G(x,u)} \geq \frac{\mu}{u}.$$

Integrating on the interval $[U, u]$, we deduce that

$$G(x,u) \geq d_\mu u^\mu,$$

where $d_\mu = c/(\mu U^\mu) > 0$. Moreover, by assumption (H$_3$) (i), the function $g(x,\cdot)$ is odd, which implies that $G(x,\cdot)$ is even. Consequently,

$$G(x,u) \geq d_\mu |u|^\mu \quad \text{for a.a. } x \in \Omega \text{ and for all } |u| \geq U.$$

Using again assumption (H$_3$) (iii), we obtain

$$d_0 = \sup_{x\in\Omega,\ |u|\leq U} \left[G(x,u) - d_\mu |u|^\mu\right]^- < \infty,$$

which implies that, for a.a. $x \in \Omega$ and for all $u \in \mathbb{R}$,

$$G(x,u) \geq d_\mu |u|^\mu - d_0.$$

Let W be a finite-dimensional subspace of $W_0^{m,\mathcal{A}}(\Omega)$. For any $u \in W$ with $\|u\| = 1$ and for every $t > 0$, we estimate

$$J(tu) \leq \left[M(\Phi_{\mathcal{A}}(\nabla^m(tu)))t^{-\mu} - D_\mu\right] t^\mu + d_0|\Omega|,$$

where $D_\mu = d_\mu c_W^\mu > 0$, and c_W is a constant such that $\|w\|_\mu \geq c_W\|w\|$ for all $w \in W$. Fix $\varepsilon \in (0, D_\mu)$. By (2.3), (1.7), and (1.8), there exists $t_0 > 0$ such that

$$M(\Phi_{\mathcal{A}}(\nabla^m(tu)))t^{-\mu} < \varepsilon \quad \text{whenever} \quad \Phi_{\mathcal{A}}(\nabla^m tu) \geq t_0.$$

Therefore,

$$J(tu) \leq -\beta_\varepsilon t^\mu + d_0|\Omega| \to -\infty \quad \text{as } t \to \infty,$$

where $\beta_\varepsilon = D_\mu - \varepsilon > 0$. It follows that

$$\sup_{\|u\|=K,\, u\in W} J(u) = \sup_{\|u\|=K,\, u\in W} J(Ku) \to -\infty \quad \text{as } K \to \infty.$$

Hence, there exists some $K_0 > 0$ sufficiently large such that $J(u) \leq 0$ for all $u \in W$ with $\|u\| = K$ and $K \geq K_0$. This completes the proof. □

Proof of Theorem 1.4 By assumption (H$_3$) (i), the function $G(x,\cdot)$ is even on $\mathbb{R}$, which implies that the associated energy functional J is also even. Moreover, $J(0) = 0$ holds trivially. By Lemmas 3.1, 3.2, and 3.3, together with Theorem 2.12, all the hypotheses of the symmetric mountain pass theorem are satisfied. Therefore, the functional J admits an unbounded sequence of critical values. Consequently, problem (1.1) admits an unbounded sequence of weak solutions. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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