SEQUENCES OF NODAL SOLUTIONS FOR CRITICAL DOUBLE PHASE PROBLEMS WITH VARIABLE EXPONENTS

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Abstract. In this paper, we study a double phase problem with both variable exponents. Such problem has a reaction consisting of a Carathéodory perturbation defined only locally and of a critical term. The presence of the critical term does not permit to use results of the critical point theory for the corresponding energy functional. Consequently, using suitable cut-off functions and truncation techniques we focus on an auxiliary coercive problem on which, differently from our main problem, we can act with variational tools. In this way, we are able to produce a sequence of sign-changing solutions to our main problem converging to 0 in L^∞ and in the Musielak-Orlicz Sobolev space.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Given $r \in C(\overline{\Omega})$, we define

$$r^- = \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r^+ = \max_{x\in\overline{\Omega}} r(x).$$

In this paper, we focus on the following critical double phase Dirichlet problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) &= f(x,u) + |u|^{p^*(x)-2}u \quad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here, we suppose the following hypotheses on the exponents, the weight function and the perturbation term:

(H1) $p, q \in C(\overline{\Omega})$ are such that $1 < p(x) < N$, $p(x) < q(x) < p^*(x) := \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega) \setminus \{0\}$.

(H2) $f\colon \Omega \times [-\eta_0, \eta_0] \to \mathbb{R}$, with $\eta_0 > 0$, is a Carathéodory function such that $f(x,\cdot)$ is odd for a.a. $x \in \Omega$ and

(i) there exists $a_0 \in L^\infty(\Omega)$ such that

$$|f(x,s)| \leq a_0(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \leq \eta_0;$$

(ii) there exist $\delta \in \left(0, \min\{\frac{\eta_0}{2}, 1\}\right)$ and $\tau \in C(\overline{\Omega})$ with $1 \leq \tau(x) \leq \tau^+ < p^-$ such that

$$c_0\,|s|^{\tau(x)} \leq f(x,s)\,s \leq \tilde{c}_0\,|s|^{\tau(x)}$$

for some $c_0 > \frac{2\tau^+}{p^-}$ and $\tilde{c}_0 > c_0$, for a.a. $x \in \Omega$ and for all $|s| \leq \delta$.

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So, in the right-hand side of problem (1.1) we find the combined effects of a Carathéodory perturbation $f(x,\cdot)$ which is defined only locally and of a critical term $u \to |u|^{p^*(x)-2}u$ with $p^*(\cdot)$ being the critical exponent corresponding to $p(\cdot)$.

A function $u \in W_0^{1,\mathcal{H}}(\Omega)$ (the Musielak-Orlicz Sobolev space, see Section 2) is said to be a weak solution of problem (1.1) if

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla h \,\mathrm{d}x = \int_\Omega \left(f(x,u) + |u|^{p^*(x)-2}u\right) h \,\mathrm{d}x$$

is satisfied for all $h \in W_0^{1,\mathcal{H}}(\Omega)$.

We point out that the main goal of this paper is to produce nodal (that is, sign-changing) solutions for problem (1.1). Precisely, we here establish the following result.

Theorem 1.1. *Let hypotheses* (H1) *and* (H2) *be satisfied. Then, problem* (1.1) *has a sequence*

$$\{z_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$$

of nodal (that is, sign-changing) solutions such that

$$\|z_n\| \to 0 \quad \textit{and} \quad \|z_n\|_\infty \to 0 \quad \textit{as } n \to +\infty.$$

The proof of Theorem 1.1 is based on the use of cut-off functions, truncation techniques and also a generalized version of the symmetric mountain pass theorem due to Kajikiya [19, Theorem 1]. Hence, using suitable cut-off functions and truncation techniques we introduce an auxiliary coercive problem (see problem (3.4), Section 3) on which differently from problem (1.1) we can act with variational tools. In fact, we point out that the presence of the critical term $u \to |u|^{p^*(x)-2}u$ in the right-hand side of problem (1.1) does not permit us to apply results of the critical point theory to the corresponding energy functional. Therefore, we focus on the auxiliary coercive problem (3.4). We show the existence of extremal constant sign solutions for such problem (see again Section 3). After that, using these extremal solutions and Kajikiya's theorem we are able to produce a sequence of sign-changing solutions for problem (1.1) (see Section 4).

Also, we remark that our work was inspired by a recent paper of Liu-Papageorgiou [23]. Similar to our finding here, the authors in [23] consider a double phase Dirichlet problem exhibiting the combined effects of a Carathéodory perturbation defined only locally and of a critical term, but they work with constant exponents and under more restrictive conditions. Thus, we extend the results of Liu-Papageorgiou [23] to the case of a double phase operator with both variable exponents and under weaker conditions. Indeed, we are able to skip condition H1 (iii) in [23]. In addition, we point out that our main result extends the one of Papageorgiou-Vetro-Winkert [31] to the case of two variable exponents. In [31] $p \equiv p(x)$ has to be a constant.

Finally, we mention that functionals of type

$$\mathcal{F}(\omega) = \int_\Omega \left(|\nabla \omega|^p + \mu(x)|\nabla \omega|^q\right)\mathrm{d}x, \quad 1 < p < q < N,$$

were studied by Marcellini [26] and Zhikov [40] in order to describe strongly anisotropic materials in the context of homogenization and elasticity. Such functionals also find application in the study of duality theory and of the Lavrentiev gap phenomenon, see Zhikov [41, 42], and in the context of problems of the calculus of variations, see Marcellini [25, 26].

A first mathematical framework for such functionals has been provided by Baroni-Colombo-Mingione [4], see also the related works by the same authors in [5, 6] and of De Filippis-Mingione [10] about nonautonomous integrals.

So far, there are only few results involving the variable exponent double phase operator. We refer to the recent results of Aberqi-Bennouna-Benslimane-Ragusa [1] for existence results in complete manifolds, Albalawi-Alharthi-Vetro [2] for convection problems with $(p(\cdot), q(\cdot))$-Laplace type operator, Bahrouni-Rădulescu-Winkert [3] for problems with Baouendi-Grushin type operator, Crespo-Blanco-Gasiński-Harjulehto-Winkert [8] for double phase convection problems, Kim-Kim-Oh-Zeng [20] for concave-convex-type double phase problems, Leonardi-Papageorgiou [21] for concave-convex problems, Liu-Pucci [24] for problems without supposing the Ambrosetti-Rabinowitz condition, Vetro-Winkert [35] for parametric problems involving superlinear nonlinearities and Zeng-Rădulescu-Winkert [39] for multivalued problems, see also the references therein. We also recall the papers of Colasuonno-Squassina [7] for eigenvalue problems of double phase type, Farkas-Winkert [12] for Finsler double phase problems, Gasiński-Papageorgiou [13] for locally Lipschitz right-hand sides, Gasiński-Winkert [14, 15] for convection problems and constant sign-solutions, Liu-Dai [22] for a Nehari manifold approach, Papageorgiou-Vetro [29] for superlinear problems, Papageorgiou-Vetro-Vetro [30] for parametric Robin problems, Perera-Squassina [33] for Morse theoretical approach, Vetro-Winkert [36] for parametric convective problems, Vetro-Winkert [37] for critical Robin double phase problems with one variable exponent and Zeng-Bai-Gasiński-Winkert [38] for implicit obstacle problems with multivalued operators.

2. Mathematical background

Given a bounded domain $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, we denote by $M(\Omega)$ the set of all measurable functions $u\colon \Omega \to \mathbb{R}$. Let $r \in C(\overline{\Omega})$ be such that $r(x) > 1$ for all $x \in \overline{\Omega}$. Then, the usual variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ is defined by

$$L^{r(\cdot)}(\Omega) = \left\{ u \in M(\Omega) \,:\, \rho_{r(\cdot)}(u) := \int_\Omega |u|^{r(x)} \,\mathrm{d}x < +\infty \right\}.$$

We furnish this space with the Luxemburg norm

$$\|u\|_{r(\cdot)} = \inf \left\{ \alpha > 0 \,:\, \rho_{r(\cdot)}\left(\frac{u}{\alpha}\right) \leq 1 \right\}.$$

We write $W^{1,r(\cdot)}(\Omega)$ and $W_0^{1,r(\cdot)}(\Omega)$ to denote the corresponding Sobolev spaces equipped with the norms $\|\cdot\|_{1,r(\cdot)}$ and $\|\nabla \cdot\|_{r(\cdot)}$, respectively, where $\|\cdot\|_{1,r(\cdot)}$ is given by

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)},$$

see also Diening-Harjulehto-Hästö-Růžička [11] and Harjulehto-Hästö [17]. Below, we recall the relations between $\|\cdot\|_{r(\cdot)}$ and $\rho_{r(\cdot)}$, see again [11].

Proposition 2.1. *Let $r \in C(\overline{\Omega})$ be such that $r(x) > 1$ for all $x \in \overline{\Omega}$. Then the following hold:*

(i) $\|u\|_{r(\cdot)} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{r(\cdot)}(u) < 1$ *(resp.* $> 1, = 1$*);*
(ii) *if* $\|u\|_{r(\cdot)} < 1$ *then* $\|u\|_{r(\cdot)}^{r^+} \leq \rho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r^-}$*;*
(iii) *if* $\|u\|_{r(\cdot)} > 1$ *then* $\|u\|_{r(\cdot)}^{r^-} \leq \rho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r^+}$*;*
(iv) $\|u\|_{r(\cdot)} \to 0$ *if and only if* $\rho_{r(\cdot)}(u) \to 0$*;*
(v) $\|u\|_{r(\cdot)} \to +\infty$ *if and only if* $\rho_{r(\cdot)}(u) \to +\infty$*.*

Assume that hypothesis (H1) is satisfied, then we can consider the nonlinear function $\mathcal{H}\colon \Omega \times [0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{H}(x,t) = t^{p(x)} + \mu(x)t^{q(x)} \quad \text{for all } x \in \Omega \text{ and for all } t \geq 0.$$

Here, we denote by $\rho_{\mathcal{H}}(\cdot)$ the corresponding modular function, that is,

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right)\mathrm{d}x.$$

Using $\rho_{\mathcal{H}}(\cdot)$ we introduce the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ by

$$L^{\mathcal{H}}(\Omega) = \left\{u \in M(\Omega) \,:\, \rho_{\mathcal{H}}(u) < +\infty\right\}.$$

As usual, we equip this space with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\alpha > 0 \,:\, \rho_{\mathcal{H}}\left(\frac{u}{\alpha}\right) \leq 1\right\}.$$

We underline that also the modular $\rho_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$ are related by a close relation, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [8, Proposition 2.13].

Proposition 2.2. *Let hypotheses (H1) be satisfied. Then the following hold:*
(i) $\|u\|_{\mathcal{H}} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{\mathcal{H}}(u) < 1$ *(resp.* $> 1, = 1$*);*
(ii) *if* $\|u\|_{\mathcal{H}} < 1$ *then* $\|u\|_{\mathcal{H}}^{q^+} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{p^-}$*;*
(iii) *if* $\|u\|_{\mathcal{H}} > 1$ *then* $\|u\|_{\mathcal{H}}^{p^-} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{q^+}$*;*
(iv) $\|u\|_{\mathcal{H}} \to 0$ *if and only if* $\rho_{\mathcal{H}}(u) \to 0$*;*
(v) $\|u\|_{\mathcal{H}} \to +\infty$ *if and only if* $\rho_{\mathcal{H}}(u) \to +\infty$*.*

Now, starting from the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$, we can define the corresponding Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}.$$

We consider on this space the norm

$$\|u\|_{1,\mathcal{H}} = \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} := \|\,|\nabla u|\,\|_{\mathcal{H}}$. Also, by $W^{1,\mathcal{H}}_0(\Omega)$ we mean the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$.

From Crespo-Blanco-Gasiński-Harjulehto-Winkert [8, Proposition 2.12] we know that the spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ are reflexive Banach spaces. Further, on account of Proposition 2.18 in [8] we can endow the space $W^{1,\mathcal{H}}_0(\Omega)$ with the equivalent norm

$$\|u\| = \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W^{1,\mathcal{H}}_0(\Omega).$$

We recall that the classical Sobolev embedding theorem extends to the space $W^{1,\mathcal{H}}_0(\Omega)$ as follows, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [8, Proposition 2.16].

Proposition 2.3. *Let hypotheses (H1) be satisfied. Then the following hold:*

(i) $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ *is continuous for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p(x)$ *for all* $x \in \overline{\Omega}$*;*
(ii) $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is compact for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$*.*

Finally, for any $s \in \mathbb{R}$ we put $s_\pm = \max\{\pm s, 0\}$ and we have $s = s_+ - s_-$ and $|s| = s_+ + s_-$. In addition, for any function $u\colon \Omega \to \mathbb{R}$ we define $u_\pm(\cdot) = [u(\cdot)]_\pm$.

Given a Banach space X and its dual space X^*, we recall that a functional $\varphi \in C^1(X)$ satisfies the Palais-Smale condition (PS-condition for short), if every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(x_n) \to 0 \quad \text{in } X^* \text{ as } n \to +\infty,$$

admits a strongly convergent subsequence. We denote by K_φ the set of all critical points of φ, that is,

$$K_\varphi = \{u \in X \, : \, \varphi'(u) = 0\}.$$

We also recall that a set $\mathcal{S} \subseteq X$ is said to be downward directed if for given $u_1, u_2 \in \mathcal{S}$ we can find $u \in \mathcal{S}$ such that $u \leq u_1$ and $u \leq u_2$. Analogously, $\mathcal{S} \subseteq X$ is said to be upward directed if for given $v_1, v_2 \in \mathcal{S}$ we can find $v \in \mathcal{S}$ such that $v_1 \leq v$ and $v_2 \leq v$.

3. Auxiliary problem

In this section, using suitable cut-off functions and truncation techniques, we introduce an auxiliary coercive problem on which we can act with variational tools. The study of this problem helps us to prove the existence of nodal (that is, sign-changing) solutions for our main problem (1.1).

Let $\theta \in C^1(\mathbb{R})$ be an even cut-off function satisfying the following conditions:

$$\operatorname{supp}\theta \subseteq [-\eta_0, \eta_0], \quad \theta\big|_{\left[\frac{-\eta_0}{2}, \frac{\eta_0}{2}\right]} \equiv 1 \quad \text{and} \quad 0 < \theta \leq 1 \quad \text{on } (-\eta_0, \eta_0), \tag{3.1}$$

where η_0 is the positive constant from hypothesis (H2). Using θ, we introduce the Carathéodory function $k\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k(x,s) = \theta(s)\left[f(x,s) + |s|^{p^*(x)-2}s\right] + (1-\theta(s))|s|^{\tau(x)-2}s \tag{3.2}$$

for all $(x,s) \in \Omega \times \mathbb{R}$, where τ is given in (H2)(ii). The assumptions on θ (see (3.1)) and hypothesis (H2) guarantee that

$$|k(x,s)| \leq c\left(1 + |s|^{\tau(x)-1}\right) \tag{3.3}$$

for some $c > 0$, for a.a. $x \in \Omega$ and for all $|s| \leq \delta$.

Then, we consider the following auxiliary double phase Dirichlet problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) &= k(x,u) \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

Here, we denote by $\mathcal{S}_+$ and $\mathcal{S}_-$ the sets of positive and negative solutions of problem (3.4), respectively. First, we prove that these sets are nonempty.

Proposition 3.1. *Let hypotheses* (H1) *and* (H2) *be satisfied. Then* $\mathcal{S}_+$ *and* $\mathcal{S}_-$ *are nonempty subsets in* $W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$*.*

Proof. We start by the set $\mathcal{S}_+$ and show that it is nonempty. To this purpose, we consider the C^1-functional $\phi_+\colon W_0^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$ defined by

$$\phi_+(u)=\int_\Omega\left[\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right]\mathrm{d}x-\int_\Omega K(x,u_+)\,\mathrm{d}x,$$

for all $u\in W_0^{1,\mathcal{H}}(\Omega)$, where $K(x,s)=\int_0^s k(x,t)\,\mathrm{d}t$. Due to (3.3) we obtain

$$\begin{aligned}
\phi_+(u)&\geq\frac{1}{p^+}\int_\Omega|\nabla u|^{p(x)}\,\mathrm{d}x+\frac{1}{q^+}\int_\Omega\mu(x)|\nabla u|^{q(x)}\,\mathrm{d}x-\int_\Omega K(x,u_+)\,\mathrm{d}x\\
&\geq\frac{1}{q^+}\,\rho_{\mathcal{H}}(|\nabla u|)-\int_\Omega K(x,u_+)\,\mathrm{d}x\\
&\geq\frac{1}{q^+}\,\rho_{\mathcal{H}}(|\nabla u|)-\int_\Omega\int_0^{u_+}c\left(1+|s|^{\tau(x)-1}\right)\,\mathrm{d}s\\
&\geq\frac{1}{q^+}\,\rho_{\mathcal{H}}(|\nabla u|)-c\int_\Omega u_+\,\mathrm{d}x-\frac{c}{\tau^-}\int_\Omega u_+^{\tau(x)}\,\mathrm{d}x\\
&\geq\frac{1}{q^+}\,\rho_{\mathcal{H}}(|\nabla u|)-c\int_\Omega|u|\,\mathrm{d}x-\frac{c}{\tau^-}\int_\Omega|u|^{\tau(x)}\,\mathrm{d}x.
\end{aligned}$$

We recall that if $\|u\|_{\tau(\cdot)}>1$, due to Proposition 2.1(iii), we have that $\rho_{\tau(\cdot)}(u)\leq\|u\|_{\tau(\cdot)}^{\tau^+}$. Similarly, if $\|u\|:=\|\nabla u\|_{\mathcal{H}}>1$ using Proposition 2.2(iii) we know that $\rho_{\mathcal{H}}(|\nabla u|)\geq\|\nabla u\|_{\mathcal{H}}^{p^-}$. According of this, for any $u\in W_0^{1,\mathcal{H}}(\Omega)$ such that $\|u\|_{\tau(x)}>1$ and $\|u\|>1$ we can further write

$$\begin{aligned}
\phi_+(u)&\geq\frac{1}{q^+}\,\|u\|^{p^-}-c\|u\|_1-\frac{c}{\tau^-}\|u\|_{\tau(\cdot)}^{\tau^+}\\
&\geq\frac{1}{q^+}\|u\|^{p^-}-c_1\|u\|-\frac{c_1}{\tau^-}\|u\|^{\tau^+}
\end{aligned}$$

for some $c_1>0$ since the embeddings $W_0^{1,\mathcal{H}}(\Omega)\hookrightarrow L^1(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)\hookrightarrow L^{\tau(\cdot)}(\Omega)$ are compact, see Proposition 2.3(ii). From this, taking into account that $\tau^+<p^-$ (see (H2)(ii)), we conclude that ϕ_+ is coercive. Moreover, using the compactness of the embedding $W_0^{1,\mathcal{H}}(\Omega)\hookrightarrow L^{r(\cdot)}(\Omega)$ for any $r\in C(\overline{\Omega})$ with $1\leq r(x)<p^*(x)$ for all $x\in\overline{\Omega}$ (see Proposition 2.3(ii)), we infer that the functional ϕ_+ is sequentially weakly lower semicontinuous. Thus, there exists $u_0\in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\phi_+(u_0)=\inf\left[\phi_+(u)\,:\,u\in W_0^{1,\mathcal{H}}(\Omega)\right].$$

We show that u_0 is nontrivial. Hence, we consider a function $\tilde{u}\in\operatorname{int}C(\overline{\Omega})_+$ and we take $t\in(0,1)$ small enough so that $t\tilde{u}(x)\in(0,\delta]$ for all $x\in\overline{\Omega}$. Then, we have

$$\begin{aligned}
&\phi_+(t\tilde{u})\\
&=\int_\Omega\left[\frac{1}{p(x)}|\nabla(t\tilde{u})|^{p(x)}+\frac{\mu(x)}{q(x)}|\nabla(t\tilde{u})|^{q(x)}\right]\mathrm{d}x-\int_\Omega K\left(x,t\tilde{u}\right)\,\mathrm{d}x\\
&\leq\frac{t^{p^-}}{p^-}\left[\int_\Omega|\nabla\tilde{u}|^{p(x)}\,\mathrm{d}x+\int_\Omega\mu(x)|\nabla\tilde{u}|^{q(x)}\,\mathrm{d}x\right]-\int_\Omega K\left(x,t\tilde{u}\right)\,\mathrm{d}x.
\end{aligned}\tag{3.5}$$

Taking into account that $t\tilde{u}(x)\in(0,\delta]$ and further $\delta\leq\frac{\eta_0}{2}$ (see (H2)(ii)), from (3.1) we have that $\theta(t\tilde{u}(x))=1$. This in addition gives

$$k(x,t\tilde{u})=f(x,t\tilde{u}(x))+(t\tilde{u}(x))^{p^*(x)-2}t\tilde{u}(x)\geq f(x,t\tilde{u}(x)).\tag{3.6}$$

Now, since $t\tilde{u}(x) \in (0,\delta]$ and $\delta \leq \frac{\eta_0}{2}$, thanks to hypothesis (H2)(ii) we know that

$$c_0 \, |t\tilde{u}|^{\tau(x)-1} \leq f(x, t\tilde{u}). \tag{3.7}$$

So, using (3.6) and (3.7) in (3.5) we have

$$\begin{aligned}
&\phi_+(t\tilde{u})\\
&\leq \frac{t^{p^-}}{p^-}\left[\int_\Omega |\nabla \tilde{u}|^{p(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla \tilde{u}|^{q(x)}\,\mathrm{d}x\right] - \frac{c_0\, t^{\tau^+}}{\tau^+}\int_\Omega |\tilde{u}|^{\tau(x)}\,\mathrm{d}x\\
&= t^{\tau^+}\left[\frac{t^{p^- - \tau^+}}{p^-}\left[\int_\Omega |\nabla \tilde{u}|^{p(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla \tilde{u}|^{q(x)}\,\mathrm{d}x\right] - \frac{c_0}{\tau^+}\int_\Omega |\tilde{u}|^{\tau(x)}\,\mathrm{d}x\right].
\end{aligned}$$

Now, taking into account that $\tau^+ < p^-$ (see (H2)(ii)), choosing $t \in (0,1)$ small enough we have that

$$\frac{t^{p^- - \tau^+}}{p^-}\left[\int_\Omega |\nabla \tilde{u}|^{p(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla \tilde{u}|^{q(x)}\,\mathrm{d}x\right] - \frac{c_0}{\tau^+}\int_\Omega |\tilde{u}|^{\tau(x)}\,\mathrm{d}x < 0.$$

Recall that $\int_\Omega |\nabla \tilde{u}|^{p(x)}\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla \tilde{u}|^{q(x)}\,\mathrm{d}x$ and $\int_\Omega |\tilde{u}|^{\tau(x)}\,\mathrm{d}x$ are fixed for a given $\tilde{u} \in \operatorname{int} C(\overline{\Omega})_+$. Consequently, we conclude that $\phi_+(t\tilde{u}) < 0 = \phi_+(0)$ for $t \in (0,1)$ sufficiently small. This guarantees that $u_0 \neq 0$.

Finally, we remark that u_0 is a global minimizer of ϕ_+, so $\phi_+'(u_0) = 0$. This implies that

$$\int_\Omega \left(|\nabla u_0|^{p(x)-2}\nabla u_0 + \mu(x)|\nabla u_0|^{q(x)-2}\nabla u_0\right)\cdot \nabla h\,\mathrm{d}x = \int_\Omega k(x,(u_0)_+)\,h\,\mathrm{d}x \tag{3.8}$$

for all $h \in W_0^{1,\mathcal{H}}(\Omega)$. Now, using Proposition 2.17 of Crespo-Blanco-Gasiński-Harjulehto-Winkert [8] which gives $\pm u_\pm \in W_0^{1,\mathcal{H}}(\Omega)$ for any $u \in W_0^{1,\mathcal{H}}(\Omega)$, we can choose $h = -(u_0)_-$ in (3.8). In this way, we get that $(u_0)_- = 0$ and thus we deduce that $u_0 \geq 0$. As $u_0 \neq 0$ we conclude that u_0 is a nontrivial positive weak solution of problem (3.4), thus $\mathcal{S}_+ \neq \emptyset$. Also, from Crespo-Blanco-Winkert [9, Theorem 3.1] we have that $u_0 \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.

In order to obtain a nontrivial negative weak solution for problem (3.4), we can use the C^1-functional $\phi_- \colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\phi_-(u) = \int_\Omega \left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right]\mathrm{d}x - \int_\Omega K(x,-u_-)\,\mathrm{d}x$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$. Then, arguing as in the case of the positive solution, we show that it has a global minimizer which turns out to be nontrivial and nonpositive. Therefore, it is a nontrivial negative weak solution of problem (3.4). □

Next, our aim is to show the existence of extremal constant sign solutions for problem (3.4). So, we consider the following auxiliary problem

$$\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) &= c_0|u|^{\tau(x)-2}u \qquad && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega
\end{aligned} \tag{3.9}$$

and establish the following result.

Proposition 3.2. *Let hypothesis (H1) be satisfied. Then, problem (3.9) has a unique positive solution $\overline{u} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$. Further, $\overline{v} = -\overline{u}$ is the unique negative solution of problem (3.9).*

Proof. We point out that in order to have the claim it is sufficient to show that problem (3.9) has a unique positive solution $\overline{u} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$. Indeed, taking into account that the problem (3.9) is odd, this guarantees that $\overline{v} = -\overline{u}$ is a negative solution of problem (3.9) and further it is the unique negative solution.

Now, in order to prove the existence of a positive solution for problem (3.9), we consider the C^1-functional $\psi_+ \colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \int_\Omega \left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right] \mathrm{d}x - \int_\Omega \frac{c_0}{\tau(x)}(u_+)^{\tau(x)}\,\mathrm{d}x,$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$. We know that

$$\begin{aligned}
\psi_+(u) &\geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^+}\int_\Omega \mu(x)|\nabla u|^{q(x)}\,\mathrm{d}x - \frac{c_0}{\tau^-}\int_\Omega (u_+)^{\tau(x)}\,\mathrm{d}x\\
&\geq \frac{1}{q^+}\,\rho_{\mathcal{H}}(|\nabla u|) - \frac{c_0}{\tau^-}\int_\Omega |u|^{\tau(x)}\,\mathrm{d}x.
\end{aligned}$$

With similar arguments as in the proof of Proposition 3.1, we can show that $\psi_+(\cdot)$ is coercive and sequentially weakly semicontinuous. This ensures, thanks to the Weierstraß-Tonelli theorem, that we can find $\overline{u} \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\psi_+(\overline{u}) = \inf\left[\psi_+(u)\,:\, u \in W_0^{1,\mathcal{H}}(\Omega)\right].$$

Reasoning again as in the proof of Proposition 3.1, we are also able to show that $\overline{u} \neq 0$ and further $\overline{u} \geq 0$. So, we conclude that $\overline{u} \in W_0^{1,\mathcal{H}}(\Omega)$ is a nontrivial positive weak solution of problem (3.9).

Next, we prove that such positive solution is unique. To this purpose, we consider the integral functional $j \colon L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u^{\frac{1}{p^-}}|^{p(x)} + \int_\Omega \frac{\mu(x)}{q(x)}|\nabla u^{\frac{1}{p^-}}|^{q(x)} & \text{if } u \geq 0 \text{ and } u^{\frac{1}{p^-}} \in W_0^{1,\mathcal{H}}(\Omega),\\ +\infty & \text{otherwise}\end{cases}$$

and let

$$\operatorname{dom} j = \left\{u \in L^1(\Omega)\,:\, j(u) < +\infty\right\}$$

be the effective domain of $j(\cdot)$. We point out that from the anisotropic Díaz-Saa inequality, see Takáč-Giacomoni [34], we have that $j(\cdot)$ is convex.

Now, let $\overline{w} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ be another nontrivial positive weak solution of problem (3.9). Given $\varepsilon > 0$, let $\overline{u}^\varepsilon = \overline{u} + \varepsilon \in \operatorname{int} L^\infty(\Omega)_+$ and $\overline{w}^\varepsilon = \overline{w} + \varepsilon \in \operatorname{int} L^\infty(\Omega)_+$. Thanks to Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [27], we know that

$$\frac{\overline{u}^\varepsilon}{\overline{w}^\varepsilon} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\overline{w}^\varepsilon}{\overline{u}^\varepsilon} \in L^\infty(\Omega). \tag{3.10}$$

We put $h = (\overline{u}^\varepsilon)^{p^-} - (\overline{w}^\varepsilon)^{p^-} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$. We note that from (3.10) and the convexity of $j(\cdot)$ we see that the directional derivatives of $j(\cdot)$ at $(\overline{u}^\varepsilon)^{p^-}$ and at

$(\overline{w}^\varepsilon)^{p^-}$ in the direction h exist and are equal to

$$\begin{aligned}
j'\left((\overline{u}^\varepsilon)^{p^-}\right)(h) &= \frac{1}{p^-}\int_\Omega \frac{-\Delta_{p(\cdot)}\overline{u}-\mu(x)\Delta_{q(\cdot)}\overline{u}}{(\overline{u}^\varepsilon)^{p^--1}}\, h\,\mathrm{d}x\\
&= \frac{1}{p^-}\int_\Omega \frac{c_0\,\overline{u}^{\tau(x)-1}}{(\overline{u}^\varepsilon)^{p^--1}}\, h\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
j'((\overline{w}^\varepsilon)^{p^-})(h) &= \frac{1}{p^-}\int_\Omega \frac{-\Delta_{p(\cdot)}\overline{w}-\mu(x)\Delta_{q(\cdot)}\overline{w}}{(\overline{w}^\varepsilon)^{p^--1}}\, h\,\mathrm{d}x\\
&= \frac{1}{p^-}\int_\Omega \frac{c_0\,\overline{w}^{\tau(x)-1}}{(\overline{w}^\varepsilon)^{p^--1}}\, h\,\mathrm{d}x,
\end{aligned}$$

respectively. Moreover, the convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. On account of this, we have that

$$0\leq c_0\int_\Omega\left(\frac{\overline{u}^{\tau(x)-1}}{(\overline{u}^\varepsilon)^{p^--1}}-\frac{\overline{w}^{\tau(x)-1}}{(\overline{w}^\varepsilon)^{p^--1}}\right)\left((\overline{u}^\varepsilon)^{p^-}-(\overline{w}^\varepsilon)^{p^-}\right)\mathrm{d}x$$

Hence, for $\varepsilon\to 0^+$, using Lebesgue's Theorem, we obtain

$$0\leq c_0\int_\Omega\left(\frac{1}{\overline{u}^{p^--\tau(x)}}-\frac{1}{\overline{w}^{p^--\tau(x)}}\right)\left(\overline{u}^{p^-}-\overline{w}^{p^-}\right)\mathrm{d}x.$$

It follows that $\overline{u}=\overline{w}$ and thus we have the claim.

Finally, we point out that the double phase maximum principle leads to $\overline{u}(x)>0$ for a.a. $x\in\Omega$. $\square$

Now, we are ready to prove the existence of a smallest positive solution $u_*\in\mathcal{S}_+$ and the existence of a largest negative solution $v_*\in\mathcal{S}_-$.

Proposition 3.3. *Let hypotheses (H1) and (H2) be satisfied. Then there exists $u_*\in\mathcal{S}_+$ such that $u_*\leq u$ for all $u\in\mathcal{S}_+$ and there exists $v_*\in\mathcal{S}_-$ such that $v_*\geq v$ for all $v\in\mathcal{S}_-$.*

Proof. We only show the existence of a smallest positive solution for problem (3.4), the case of a largest negative solution works similarly.

Arguing in a similar way to the proof of Proposition 7 in Papageorgiou-Rădulescu-Repovš [28] we can deduce that $\mathcal{S}_+$ is downward directed. On account of this, we can use Lemma 3.10 of Hu-Papageorgiou [18, p. 178] which gives the existence of a decreasing sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{S}_+$ such that

$$\inf_{n\in\mathbb{N}} u_n=\inf\mathcal{S}_+.$$

As $u_n\in\mathcal{S}_+$ it follows that

$$\int_\Omega\left(|\nabla u_n|^{p(x)-2}\nabla u_n+\mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla h\,\mathrm{d}x=\int_\Omega k(x,u_n)\,h\,\mathrm{d}x \tag{3.11}$$

for all $h\in W_0^{1,\mathcal{H}}(\Omega)$ and for all $n\in\mathbb{N}$. Hence, choosing $h=u_n$ in (3.11) and using (3.3) along with $0\leq u_n\leq u_1$, we have that

$$\rho_{\mathcal{H}}(\nabla u_n)=\int_\Omega|\nabla u_n|^{p(x)}\,\mathrm{d}x+\int_\Omega\mu(x)|\nabla u_n|^{q(x)}\,\mathrm{d}x\leq d_1$$

for some $d_1 > 0$ and for all $n \in \mathbb{N}$. This fact along with Proposition 2.2 shows that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega)$ is bounded. Hence, we can assume that

$$u_n \rightharpoonup u_* \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad u_n \to u_* \quad \text{in } L^{q(\cdot)}(\Omega).$$

Thanks to (3.1), (3.2) and hypothesis (H2)(ii) we have

$$|k(x,s)| \leq d_2|s|^{\tau(x)-1} \tag{3.12}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for some $d_2 > 0$. From (3.12) along with $0 \leq u_n \leq u_1$, we in particular deduce that

$$|k(x,u_n)| \leq d_2|u_n|^{\tau(x)-1} \leq d_2|u_1|^{\tau(x)-1}.$$

Then, taking into account that $d_2|u_1|^{\tau(\cdot)-1} \in L^\infty(\Omega)$, from (3.11) and (3.12) along with a Moser-iteration type argument as it was explained in Guedda-Veron [16], we can obtain that

$$\|u_n\|_\infty \leq O(u_n).$$

Now let us check that $u_* \neq 0$. Indeed, if $u_* = 0$ we have that $\|u_n\|_\infty \to 0$ as $n \to +\infty$. This implies that we can find $n_0 \in \mathbb{N}$ such that

$$0 < u_n(x) \leq \delta$$

for a.a. $x \in \Omega$ and for all $n \geq n_0$, where $\delta \in \left(0, \min\{\frac{\eta_0}{2}, 1\}\right)$. Then, we fix $n \geq n_0$ and consider the Carathéodory function $l_n\colon \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$l_n(x,s) = \begin{cases} c_0(s_+)^{\tau(x)-1} & \text{if } s \leq u_n(x), \\ c_0|u_n(x)|^{\tau(x)-1} & \text{if } u_n(x) < s. \end{cases}$$

Then we introduce the C^1-functional $\sigma_+\colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \int_\Omega \left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right] \mathrm{d}x - \int_\Omega L_n(x,u)\,\mathrm{d}x$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$, where $L_n(x,s) = \int_0^s l_n(x,t)\,\mathrm{d}t$. Similar arguments to the ones in the proof of Proposition 3.1 show that this functional is coercive and sequentially weakly lower semicontinuous. Hence, we can find $\hat{u} \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\sigma_+(\hat{u}) = \inf\left[\sigma_+(u) \,:\, u \in W_0^{1,\mathcal{H}}(\Omega)\right] < 0 = \sigma_+(0),$$

which gives $\hat{u} \neq 0$. Also, we can deduce that $\hat{u} \geq 0$. In addition, we point out that $K_{\sigma_+} \subseteq [0, u_n]$. In fact, let $v \in K_{\sigma_+}$ with $v \neq 0, u_n$, then we have

$$\int_\Omega \left(|\nabla v|^{p(x)-2}\nabla v + \mu(x)|\nabla v|^{q(x)-2}\nabla v\right) \cdot \nabla h\,\mathrm{d}x = \int_\Omega l_n(x,v)\,h\,\mathrm{d}x \tag{3.13}$$

for all $h \in W_0^{1,\mathcal{H}}(\Omega)$. Taking the test function $h = (v-u_n)_+$ in (3.13), using (H2)(ii) along with $0 < u_n \leq \delta$ and $u_n \in S_+$, we obtain

$$\begin{aligned}
&\int_\Omega \left(|\nabla v|^{p(x)-2}\nabla v + \mu(x)|\nabla v|^{q(x)-2}\nabla v\right) \cdot \nabla(v-u_n)_+\,\mathrm{d}x \\
&= \int_\Omega l_n(x,v)(v-u_n)_+\,\mathrm{d}x \\
&= \int_\Omega c_0(u_n)^{\tau(x)-1}(v-u_n)_+\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_\Omega f(x,u_n)(v-u_n)_+ \,\mathrm{d}x\\
&\leq \int_\Omega \left[f(x,u_n)+(u_n)^{p^*(x)-2}u_n\right](v-u_n)_+ \,\mathrm{d}x\\
&= \int_\Omega k(x,u_n)(v-u_n)_+ \,\mathrm{d}x\\
&= \int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n+\mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(v-u_n)_+ \,\mathrm{d}x
\end{aligned}
$$

as $0<u_n\leq\delta\leq\frac{\eta_0}{2}$ and (3.1) give $\theta(u_n)=1$ and so $k(x,u_n)=f(x,u_n)+(u_n)^{p^*(x)-2}u_n$. Consequently, we have

$$
\begin{aligned}
&\int_\Omega \left(|\nabla v|^{p(x)-2}\nabla v-|\nabla u_n|^{p(x)-2}\nabla u_n\right)\cdot\nabla(v-u_n)_+ \,\mathrm{d}x\\
&+\int_\Omega \mu(x)\left(|\nabla v|^{q(x)-2}\nabla v-|\nabla u_n|^{q(x)-2}\nabla u_n\right)\cdot\nabla(v-u_n)_+ \,\mathrm{d}x\leq 0,
\end{aligned}
$$

which implies that $v\leq u_n$. Furthermore, choosing the test function $h=-(v_-)$ in (3.13), we easily see that $v_-=0$ and thus we have that $v\geq 0$. We finally conclude that $K_{\sigma_+}\subseteq[0,u_n]$.

Now, we recall that $\hat u$ is a global minimizer of σ_+ and then $\hat u\in K_{\sigma_+}\subseteq[0,u_n]$. So, we know that

$$
\int_\Omega \left(|\nabla\hat u|^{p(x)-2}\nabla\hat u+\mu(x)|\nabla\hat u|^{q(x)-2}\nabla\hat u\right)\cdot\nabla h\,\mathrm{d}x=\int_\Omega c_0(\hat u_+)^{\tau(x)-1}\,h\,\mathrm{d}x
$$

for all $h\in W_0^{1,\mathcal{H}}(\Omega)$. This clearly implies that $\hat u$ is a positive solution of problem (3.9). Since such solution is unique (see Proposition 3.2), we conclude that $\hat u=\overline{u}$. From this, it follows that $\overline{u}\leq u_n$ for all $n\geq n_0$, which contradicts the hypothesis $u_*=0$. Consequently, we have that $u_*\neq 0$ and further $u_*\in S_+$. Therefore, we have that u_* is the smallest positive solution of (3.4) in S_+.

□

4. Proof of Theorem 1.1

In this section, we use the extremal constant sign solutions u_* and v_* given in Proposition 3.3 in order to produce a sequence of nodal (that is, sign-changing) solutions for problem (1.1). In addition, we show that such sequence converges to 0 in $W_0^{1,\mathcal{H}}(\Omega)$ and in $L^\infty(\Omega)$.

Considering truncations of $k(x,\cdot)$ at $v_*(x)$ and $u_*(x)$, we introduce the Carathéodory function $k_*\colon\Omega\times\mathbb{R}\to\mathbb{R}$ given by

$$
k_*(x,s):=\begin{cases} k(x,v_*(x)) & \text{if } s<v_*(x),\\ k(x,s) & \text{if } v_*(x)\leq s\leq u_*(x),\\ k(x,u_*(x)) & \text{if } u_*(x)<s.\end{cases}
$$

Let $\Phi_*\colon W_0^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$ be the C^1-functional defined by

$$
\Phi_*(u)=\int_\Omega\left[\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right]\mathrm{d}x-\int_\Omega K_*(x,u)\,\mathrm{d}x,
$$

for all $u\in W_0^{1,\mathcal{H}}(\Omega)$, where $K_*(x,s)=\int_0^s k_*(x,t)\,\mathrm{d}t$.

Note that the set K_{Φ_*} of the critical points of the functional Φ_* is contained in the order interval $[v_*, u_*]$. In fact, for a given $u \in K_{\Phi_*} \setminus \{u_*, v_*\}$ we know that

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla h \,\mathrm{d}x = \int_\Omega k_*(x,u)h \,\mathrm{d}x \tag{4.1}$$

holds for all $h \in W_0^{1,\mathcal{H}}(\Omega)$. Then, choosing $h = (u-u_*)_+$ in (4.1), we have that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla (u-u_*)_+ \,\mathrm{d}x\\
&= \int_\Omega k_*(x,u)(u-u_*)_+ \,\mathrm{d}x\\
&= \int_\Omega k(x,u_*)(u-u_*)_+ \,\mathrm{d}x\\
&= \int_\Omega \left(|\nabla u_*|^{p(x)-2}\nabla u_* + \mu(x)|\nabla u_*|^{q(x)-2}\nabla u_*\right) \cdot \nabla (u-u_*)_+ \,\mathrm{d}x,
\end{aligned}$$

since $u_* \in \mathcal{S}_+$. Hence, we deduce that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla u_*|^{p(x)-2}\nabla u_*\right) \cdot \nabla (u-u_*)_+ \,\mathrm{d}x\\
&+ \int_\Omega \mu(x) \left(|\nabla u|^{q(x)-2}\nabla u - |\nabla u_*|^{q(x)-2}\nabla u_*\right) \cdot \nabla (u-u_*)_+ \,\mathrm{d}x = 0,
\end{aligned}$$

and consequently we have that $u \leq u_*$. An analogous reasoning and the choice of $h = (v_* - u)_+$ in (4.1) gives that $v_* \leq u$.

Now, let $V \subseteq W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ be a finite dimensional subspace. Then, given $v \in V$ we put

$$\begin{aligned}
\{v < v_*\} &:= \{x \in \Omega \,:\, v(x) < v_*(x)\},\\
\{v_* \leq v \leq u_*\} &:= \{x \in \Omega \,:\, v_*(x) \leq v(x) \leq u_*(x)\},\\
\{u_* < v\} &:= \{x \in \Omega \,:\, u_*(x) < v(x)\}.
\end{aligned}$$

Since V is finite dimensional, all norms on V are equivalent, see for example Papageorgiou-Winkert [32, Proposition 3.1.17, p.183]. Hence, we know that there exists a positive constant e_V, independent of $v \in V$, such that

$$e_V \,\|v\| \leq \|v\|_{\tau(\cdot)}. \tag{4.2}$$

Using this fact, we are able to establish the following result.

Proposition 4.1. *Let hypotheses* (H1) *and* (H2) *be satisfied. Then, we can find* $r_V > 0$ *such that*

$$\sup\,[\Phi_*(v) \,:\, v \in V,\, \|v\| = r_V] < 0.$$

Proof. Since all norms on V are equivalent, we can find $r_V > 0$ such that

$$v \in V \quad \text{and} \quad \|v\| = r_V \qquad \text{imply} \qquad |v(x)| \leq \delta \quad \text{for a.a. } x \in \Omega$$

with $\delta \in \left(0, \min\{\frac{\eta_0}{2}, 1\}\right)$. Recall that from $\delta \leq \frac{\eta_0}{2}$ it follows $\theta(v(x)) = 1$ for a.a. $x \in \Omega$, see (3.1). Hence, given $v \in V$ with $\|v\| = r_V$ we know that

$$k_*(x, v(x)) = \begin{cases} f(x, v_*(x)) + |v_*(x)|^{p^*(x)-2}v_*(x) & \text{if } v(x) < v_*(x),\\ f(x, v(x)) + |v(x)|^{p^*(x)-2}v(x) & \text{if } v_*(x) \leq v(x) \leq u_*(x),\\ f(x, u_*(x)) + |u_*(x)|^{p^*(x)-2}u_*(x) & \text{if } u_*(x) < v(x).\end{cases} \tag{4.3}$$

Let $f_*\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f_*(x,v(x))=\begin{cases} f(x,v_*(x)) & \text{if } v(x)<v_*(x),\\ f(x,v(x)) & \text{if } v_*(x)\le v(x)\le u_*(x),\\ f(x,u_*(x)) & \text{if } u_*(x)<v(x),\end{cases}$$

and in addition let $F_*(x,s):=\int_0^s f_*(x,t)\,\mathrm{d}t$. Then, for $v<v_*$ we see that

$$\begin{aligned} F_*(x,v)&=\int_0^{v_*} f_*(x,s)\,\mathrm{d}s+\int_{v_*}^{v} f_*(x,s)\,\mathrm{d}s\\ &=\int_0^{v_*} f(x,s)\,\mathrm{d}s+f(x,v_*)(v-v_*).\end{aligned}$$

Note that according to (H2)(ii) we have

$$f(x,s)<0 \text{ for all } -\delta<s\le 0 \quad\text{and}\quad f(x,s)>0 \text{ for all } 0<s\le\delta. \tag{4.4}$$

Using (4.4), we see that $f(x,v_*)$ is negative and hence we infer that $f(x,v_*)(v-v_*)>0$. Therefore, we can write

$$\begin{aligned} F(x,v)-F_*(x,v)&=F(x,v)-F(x,v_*)+f(x,v_*)(v_*-v)\\ &\le F(x,v)-F(x,v_*),\end{aligned}$$

where $F(x,s):=\int_0^s f(x,t)\,\mathrm{d}t$. Moreover, for $u_*<v$ we have that

$$F_*(x,v)=F(x,u_*)+f(x,u_*)(v-u_*).$$

Taking into account that from (4.4) it follows $f(x,u_*)(u_*-v)<0$, we deduce that

$$\begin{aligned} F(x,v)-F_*(x,v)&=F(x,v)-F(x,u_*)+f(x,u_*)(u_*-v)\\ &\le F(x,v)-F(x,u_*).\end{aligned}$$

Using this facts along with the fact that the terms

$$\frac{1}{p^*(x)}|v_*|^{p^*(x)},\qquad \frac{1}{p^*(x)}|v|^{p^*(x)}\quad\text{and}\quad \frac{1}{p^*(x)}|u_*|^{p^*(x)}$$

are positive, we get that

$$\begin{aligned} \Phi_*(v)&=\int_\Omega\left[\frac{1}{p(x)}|\nabla v|^{p(x)}+\frac{\mu(x)}{q(x)}|\nabla v|^{q(x)}\right]\mathrm{d}x-\int_\Omega K_*(x,v)\,\mathrm{d}x\\ &\le\frac{1}{p^-}\int_\Omega|\nabla v|^{p(x)}\,\mathrm{d}x+\frac{1}{q^-}\int_\Omega\mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x\\ &\quad-\int_{\{v<v_*\}}\left[F_*(x,v)+\frac{1}{p^*(x)}|v_*|^{p^*(x)}\right]\mathrm{d}x\\ &\quad-\int_{\{v_*\le v\le u_*\}}\left[F(x,v)+\frac{1}{p^*(x)}|v|^{p^*(x)}\right]\mathrm{d}x\\ &\quad-\int_{\{u_*<v\}}\left[F_*(x,v)+\frac{1}{p^*(x)}|u_*|^{p^*(x)}\right]\mathrm{d}x\\ &\le\frac{1}{p^-}\int_\Omega|\nabla v|^{p(x)}\,\mathrm{d}x+\frac{1}{q^-}\int_\Omega\mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x\\ &\quad-\int_{\{v<v_*\}}F_*(x,v)\,\mathrm{d}x-\int_{\{v_*\le v\le u_*\}}F(x,v)\,\mathrm{d}x-\int_{\{u_*<v\}}F_*(x,v)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{p^-}\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x+\frac{1}{q^-}\int_\Omega \mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x-\int_\Omega F(x,v)\,\mathrm{d}x\\
&\quad+\int_{\{v<v_*\}}[F(x,v)-F_*(x,v)]\,\mathrm{d}x+\int_{\{u_*<v\}}[F(x,v)-F_*(x,v)]\,\mathrm{d}x\\
&\leq \frac{1}{p^-}\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x+\frac{1}{q^-}\int_\Omega \mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x\ -\int_\Omega F(x,v)\,\mathrm{d}x\\
&\quad+\int_{\{v<v_*\}}[F(x,v)-F(x,v_*)]\,\mathrm{d}x+\int_{\{u_*<v\}}[F(x,v)-F(x,u_*)]\,\mathrm{d}x\\
&\leq \frac{1}{p^-}\ \rho_{\mathcal{H}}(\nabla v)\ -\int_\Omega F(x,v)\,\mathrm{d}x\\
&\quad+\int_{\{v<v_*\}}[F(x,v)-F(x,v_*)]\,\mathrm{d}x+\int_{\{u_*<v\}}[F(x,v)-F(x,u_*)]\,\mathrm{d}x.
\end{aligned}
$$

Now, due to hypothesis (H2)(ii) we can find $\delta\in\left(0,\min\{\frac{\eta_0}{2},1\}\right)$ such that

$$
c_0\,|s|^{\tau(x)-1}\leq |f(x,s)|\quad\text{for all } |s|\leq\delta.
$$

Consequently, as $\tau^+<p^-$ due to hypothesis (H2)(ii), we can choose

$$
r_V<(e_V)^{\frac{\tau^+}{p^--\tau^+}}
$$

with $e_V\,r_V<1$, where e_V is the positive constant introduced in (4.2), so that

$$
\int_{\{v<v_*\}}[F(x,v)-F(x,v_*)]\,\mathrm{d}x+\int_{\{u_*<v\}}[F(x,v)-F(x,u_*)]\,\mathrm{d}x<\frac{c_0\,(e_V\,r_V)^{\tau^+}}{2\,\tau^+},
$$

and in addition

$$
\Phi_*(v)\leq\frac{1}{p^-}\ \rho_{\mathcal{H}}(\nabla v)\ -\frac{c_0}{\tau^+}\int_\Omega |v|^{\tau(x)}\,\mathrm{d}x+\frac{c_0\,(e_V\,r_V)^{\tau^+}}{2\,\tau^+}.
$$

Next, using Proposition 2.2 (ii), (iii) we have

$$
\rho_{\mathcal{H}}(\nabla v)\leq\max\{\|v\|^{p^-},\ \|v\|^{q^+}\},
$$

and using Proposition 2.1 (ii), (iii) we have

$$
\rho_{\tau(\cdot)}(v)\geq\min\{\|v\|_{\tau(x)}^{\tau^-},\ \|v\|_{\tau(x)}^{\tau^+}\}.
$$

Further, thanks to (4.2) we deduce that

$$
\rho_{\tau(\cdot)}(v)\geq\min\{(e_V\,\|v\|)^{\tau^-},\ (e_V\,\|v\|)^{\tau^+}\}.
$$

Now, as $r_V<1$ and $e_V r_V<1$, for $v\in V$ with $\|v\|=r_V$, we have

$$
\rho_{\mathcal{H}}(\nabla v)\leq r_V^{p^-}\quad\text{and}\quad\rho_{\tau(\cdot)}(v)\geq(e_V r_V)^{\tau^+}.
$$

Hence, we get

$$
\begin{aligned}
\Phi_*(v)&\leq\frac{1}{p^-}r_V^{p^-}-\frac{c_0\,(e_V\,r_V)^{\tau^+}}{\tau^+}+\frac{c_0\,(e_V\,r_V)^{\tau^+}}{2\,\tau^+}\\
&=\left(\frac{r_V^{p^--\tau^+}}{p^-}-\frac{c_0\,e_V^{\tau^+}}{2\,\tau^+}\right)r_V^{\tau^+}.
\end{aligned}
$$

Now, we recall that $c_0 > \frac{2\tau^+}{p^-}$ (see (H2)(ii)) and $r_V < (e_V)^{\frac{\tau^+}{p^- - \tau^+}}$. Therefore, we have that

$$\frac{r_V^{p^- - \tau^+}}{p^-} - \frac{c_0\, e_V^{\tau^+}}{2\,\tau^+} < \frac{r_V^{p^- - \tau^+}}{p^-} - \frac{e_V^{\tau^+}}{p^-} < 0.$$

From this, we conclude that $\Phi_*(v) < 0$ for all $v \in V$ with $\|v\| = r_V$ and $r_V < (e_V)^{\frac{\tau^+}{p^- - \tau^+}}$ such that $e_V\, r_V < 1$. This clearly gives the assertion of the proposition. $\square$

We are now ready to establish the proof of Theorem 1.1 which is based on a generalized version of the symmetric mountain pass theorem due to Kajikiya [19, Theorem 1].

Proof of Theorem 1.1. According to the definition of $k_* \colon \Omega \times \mathbb{R} \to \mathbb{R}$ given in (4.3), we can show with similar arguments as in the proof of Proposition 3.1 that the functional Φ_* is even and coercive. This permits us to deduce that Φ_* is bounded from below. Further, from Papageorgiou-Rădulescu-Repovš [27, Proposition 5.1.15], we also have that Φ_* satisfies the PS-condition. Therefore, taking into account that Proposition 4.1 holds, we can use Theorem 1 of Kajikiya [19] which guarantees the existence of a sequence $\{z_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ satisfying the following properties

$$z_n \in K_{\Phi_*} \subseteq [v_*, u_*], \quad z_n \neq 0, \quad \Phi_*(z_n) \leq 0 \quad \text{for all } n \in \mathbb{N}$$

and

$$\|z_n\| \to 0 \quad \text{as } n \to +\infty.$$

Since v_* and u_* are extremal solutions of problem (3.4), from

$$z_n \in K_{\Phi_*} \subseteq [v_*, u_*] \quad \text{and} \quad z_n \neq 0 \quad \text{for all } n \in \mathbb{N},$$

we conclude that z_n is a nodal (that is, sign-changing) solution of problem (3.4) for all $n \in \mathbb{N}$. Now, we recall that

$$\|z_n\|_\infty \leq O(u_n)$$

(see the proof of Proposition 3.3). This along with $\|z_n\| \to 0$ yields $\|z_n\|_\infty \to 0$ as $n \to +\infty$.

Finally, from $\|z_n\|_\infty \to 0$ as $n \to +\infty$ we also get that there exists $n_0 \in \mathbb{N}$ such that $|z_n(x)| \leq \frac{\eta_0}{2}$ for a.a. $x \in \Omega$ and for all $n \geq n_0$. Thus, we have that $\theta(z_n(x)) = 1$ for a.a. $x \in \Omega$ and for all $n \geq n_0$. On account of this, in view of (3.2), we conclude that z_n is a nodal solution of problem (1.1) for all $n \geq n_0$. $\square$

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