DOUBLE PHASE PROBLEMS WITH VARIABLE GROWTH AND CONVECTION FOR THE BAOUENDI-GRUSHIN OPERATOR

ANOUAR BAHROUNI, VICENŢIU D. RĂDULESCU, AND PATRICK WINKERT

Abstract. In this paper we study a class of quasilinear elliptic equations with double phase energy and reaction term depending on the gradient. The main feature is that the associated functional is driven by the Baouendi-Grushin operator with variable coefficient. This partial differential equation is of mixed type and possesses both elliptic and hyperbolic regions. We first establish some new qualitative properties of a differential operator introduced recently by Bahrouni, Rădulescu and Repovš [6]. Next, under quite general assumptions on the convection term, we prove the existence of stationary waves by applying the theory of pseudomonotone operators. The analysis carried out in this paper is motivated by patterns arising in the theory of transonic flows.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with smooth boundary $\partial\Omega$ and let n, m be nonnegative integers such that $N = n+m$. This means that $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and so $z \in \Omega$ can be written as $z = (x, y)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

We consider the following double phase problem with convection term

$$\begin{aligned} -\Delta_{G(x,y)} u + A(x,y)(|u|^{G(x,y)-1} + |u|^{G(x,y)-3})u &= f\left((x,y), u, \nabla u\right) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

with

$$A(x,y) = |\nabla_x G(x,y)| + |x|^\gamma |\nabla_y G(x,y)| \quad \text{for all } (x,y) \in \Omega.$$

Here, $G : \overline{\Omega} \to (1, \infty)$ is supposed to be a continuous function and $\Delta_{G(x,y)}$ stands for the Baouendi-Grushin operator with variable coefficient, which is defined by

$$\begin{aligned} \Delta_{G(x,y)} u &= \operatorname{div}\left(\nabla_{G(x,y)} u\right)\\ &= \sum_{i=1}^n \left(|\nabla_x|^{G(x,y)-2} u_{x_i}\right)_{x_i} + |x|^\gamma \sum_{i=1}^m \left(|\nabla_y|^{G(x,y)-2} u_{y_i}\right)_{y_i}, \end{aligned}$$

where

$$\nabla_{G(x,y)} u = \mathcal{A}(x) \begin{bmatrix} |\nabla_x|^{G(x,y)-2} & \nabla_x u \\ |x|^\gamma |\nabla_y|^{G(x,y)-2} & \nabla_y u \end{bmatrix}$$

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and

$$\mathcal{A}(x) = \begin{bmatrix} I_n & 0_{n,m} \\ 0_{m,n} & |x|^\gamma I_m \end{bmatrix} \in \mathcal{M}_{N\times N}(\mathbb{R}),$$

with I_n being the identity matrix of size $n \times n$, $O_{n,m}$ is the zero matrix of size $n \times m$ and $\mathcal{M}_{N\times N}$ stands for the class of $N \times N$-matrices with real-valued entries. From the representation above it is clear that $\Delta_{G(x,y)}$ is degenerate along the m-dimensional subspace $M := \{0\} \times \mathbb{R}^m$ of $\mathbb{R}^N$.

The differential operator $\Delta_{G(x,y)}$ generalizes the degenerate operator

$$\frac{\partial^2}{\partial x^2} + x^{2r}\frac{\partial^2}{\partial y^2} \quad (r \in \mathbb{N})$$

introduced by Baouendi [7] and Grushin [17]. The Baouendi–Grushin operator can be viewed as the Tricomi operator for transonic flow restricted to subsonic regions. On the other hand, a second-order differential operator T in divergence form on the plane, can be written as an operator whose principal part is a Baouendi-Grushin-type operator, provided that the principal part of T is nonnegative and its quadratic form does not vanish at any point, see Franchi & Tesi [15].

In the right-hand side of problem (1.1) we have a nonlinearity $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ which is a Carathéodory function, that is, $f(\cdot, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f((x, y), \cdot, \cdot)$ is continuous for a.a. $(x, y) \in \Omega$.

Problem (1.1) is strictly connected with the analysis of nonlinear patterns and stationary waves for transonic flow models. We refer to the pioneering work of Morawetz [20, 21, 22] on the theory of transonic fluid flow —referring to partial differential equations that possess both elliptic and hyperbolic regions— and this remains the most fundamental mathematical work on this subject. The flow is supersonic in the elliptic region, while a shock wave is created at the boundary between the elliptic and hyperbolic regions. In the 1950s, Morawetz used functional–analytic methods to study boundary value problems for such transonic problems.

The variable coefficient $G(x, y)$ describes the geometry of a composite realized by using two materials with corresponding behaviour described by $|\nabla_x u|^{G(x,y)}$ and $|\nabla_y u|^{G(x,y)}$. Then in the region $\{z \in \Omega : \ x \neq 0\}$ the material described by the second integrand is present. In the opposite case, the material described by the first integrand is the only one that creates the composite.

The main goal of our paper is to prove the existence of at least one weak solution of problem (1.1) under very general conditions on the nonlinearity $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. The novelty of our paper is the fact that we combine a double phase operator driven by the Baouendi-Grushin operator with variable growth and a right-hand side which depends on the gradient of the solution. Such function is called convection term.

It is well known that the Caffarelli-Kohn-Nirenberg inequality is a powerful inequality and it is needed in several ways in the study of partial differential equations. We refer to the works of Adimurthi, Chaudhuri & Ramaswamy [2], Baroni, Colombo & Mingione [8], Colasuonno & Pucci [12], Colombo & Mingione [13] for relevant applications of the Caffarelli-Kohn-Nirenberg inequality. For recent contributions to the study of double-phase problems we refer to Beck & Mingione [9], Papageorgiou, Rădulescu & Repovš [24, 25], and Zhang & Rădulescu [31].

The following Caffarelli-Kohn-Nirenberg inequality [10] establishes that for given $p \in (1, N)$ and real numbers a, b and q such that

$$-\infty < a < \frac{N-p}{p}, \quad a \leq b \leq a+1, \quad q = \frac{Np}{N-p(1+a-b)},$$

there exists a positive constant $C_{a,b}$ such that for all $u \in C_c^1(\Omega)$

$$\left(\int_\Omega |x|^{-bq}|u|^q\,dx\right)^{p/q} \leq C_{a,b}\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\,.$$

This inequality was extensively studied, see for example Abdellaoui & Peral [1], Adimurthi, Chaudhuri & Ramaswamy [2], Bahrouni, Rădulescu & Repovš [5], Bahrouni, Rădulescu & Repovš [6], Catrina & Wang [11], and the references therein. In particular, Bahrouni, Rădulescu and Repovš [6] proved a new version of a Caffarelli-Kohn-Nirenberg inequality with variable exponent for the Baouendi-Grushin operator Δ_G. More precisely, the following weighted inequality has been proved.

Theorem 1.1. *Assume that G is a function of class C^1 and that $G(x,y) \in (2, N)$ for all $(x,y) \in \Omega$. Then there exists a positive constant β such that for all $u \in C_c^1(\Omega)$*

$$\begin{aligned}
&\int_\Omega (1+|x|^\gamma)\,|u|^{G(x,y)}\,dx\,dy \\
&\leq \beta\int_\Omega \left(|\nabla_x u|^{G(x,y)} + |x|^\gamma|\nabla_y u|^{G(x,y)}\right)\,dx\,dy \\
&\quad + \beta\int_\Omega |u|^{G(x,y)-1}\left(1+u^2\right)\left(|\nabla_x G(x,y)| + |x|^\gamma|\nabla_y G(x,y)|\right)\,dx\,dy.
\end{aligned}$$

The paper is organized as follows. In Section 2 we present the basic properties of variable Lebesgue and Sobolev spaces and state the main tools which will be used later; see Rădulescu and Repovš [29] for more details. New properties concerning the Baouendi-Grushin operator will be discussed in Section 3 and in the last section we state and prove our main result concerning the existence of a weak solution to problem (1.1).

2. Terminology and the abstract setting

In this section we recall some basic definitions and properties of the needed function spaces. We refer to the works of Bahrouni & Repovš [4], Hájek, Montesinos Santalucía, Vanderwerff & Zizler [18], Musielak [23], Rădulescu [27, 28], Rădulescu & Repovš [29] and the references therein. Consider the set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) \;\middle|\; p(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}$$

and define for any $p \in C_+(\overline{\Omega})$

$$p^+ := \sup_{x\in\overline{\Omega}} p(x) \qquad \text{and} \qquad p^- := \inf_{x\in\overline{\Omega}} p(x).$$

Then $1 < p^- \leq p^+ < \infty$ for each $p \in C_+(\overline{\Omega})$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} \;\middle|\; u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\mu > 0 \;\middle|\; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

It is known that $L^{p(\cdot)}(\Omega)$ is a reflexive Banach space. Moreover, continuous functions with compact support are dense in $L^{p(\cdot)}(\Omega)$.

Denote by $q(\cdot)$ the conjugate of $p(\cdot)$, that is, $1/p(x) + 1/q(x) = 1$ for all $x \in \overline{\Omega}$. If $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, then we have the following Hölder-type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|u\|_{p(\cdot)}\|v\|_{q(\cdot)}.$$

More generally, if $p_j \in C_+(\overline{\Omega})$ for $j = 1, 2, 3$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1 \quad \text{for all } x \in \overline{\Omega},$$

then we obtain for all $u \in L^{p_1(\cdot)}(\Omega)$, $v \in L^{p_2(\cdot)}(\Omega)$ and $w \in L^{p_3(\cdot)}(\Omega)$ that

$$\left|\int_\Omega uvw\,dx\right| \le \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\right)\|u\|_{p_1(\cdot)}\|v\|_{p_2(\cdot)}\|w\|_{p_3(\cdot)}.$$

Moreover, for $p_1 \le p_2$ in Ω, then there exists the continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$.

The following two propositions will be useful in the sequel.

Proposition 2.1. *Let*

$$\rho_1(u) = \int_\Omega |u|^{p(x)}\,dx \quad \textit{for all } u \in L^{p(\cdot)}(\Omega).$$

Then the following holds:

(i) $\|u\|_{p(\cdot)} < 1$ *(resp.,* $= 1; > 1$*) if and only if* $\rho_1(u) < 1$ *(resp.,* $= 1; > 1$*);*
(ii) $\|u\|_{p(\cdot)} > 1$ *implies* $\|u\|_{p(\cdot)}^{p^-} \le \rho_1(u) \le \|u\|_{p(\cdot)}^{p^+}$*;*
(iii) $\|u\|_{p(\cdot)} < 1$ *implies* $\|u\|_{p(\cdot)}^{p^+} \le \rho_1(u) \le \|u\|_{p(\cdot)}^{p^-}$.

Proposition 2.2. *Let*

$$\rho_1(u) = \int_\Omega |u|^{p(x)}\,dx \quad \textit{for all } u \in L^{p(\cdot)}(\Omega).$$

If $u, u_n \in L^{p(\cdot)}(\Omega)$ *and* $n \in \mathbb{N}$*, then the following statements are equivalent:*

(i) $\lim\limits_{n\to+\infty} \|u_n - u\|_{p(\cdot)} = 0$*;*
(ii) $\lim\limits_{n\to+\infty} \rho_1(u_n - u) = 0$*;*
(iii) $u_n(x) \to u(x)$ *in* Ω *and* $\lim\limits_{n\to+\infty} \rho_1(u_n) = \rho_1(u)$.

By $W^{1,p(\cdot)}(\Omega)$ we denote the variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \;\middle|\; |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)} + \|u\|_{p(\cdot)}.$$

Then $W^{1,p(\cdot)}(\Omega)$ is a reflexive and separable Banach space.

Our main existence result will be based on the following surjectivity result, see Gasinski & Papageorgiou [16]. First, we give the definition of pseudomonotonicity.

Definition 2.3. *Let X be a reflexive Banach space, X^* its dual space and denote by $\langle \cdot,\cdot\rangle$ its duality pairing. Let $A\colon X\to X^*$, then A is called pseudomonotone if $u_n \overset{\mathrm{w}}{\to} u$ in X and $\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle \leq 0$ imply $Au_n \overset{\mathrm{w}}{\to} Au$ and $\langle Au_n,u_n\rangle \to \langle Au,u\rangle$.*

Theorem 2.4. *Let X be a real, reflexive Banach space, and let $A\colon X\to X^*$ be a pseudomonotone, bounded, and coercive operator, and $b\in X^*$. Then the problem $Au=b$ has at least one solution.*

3. Properties of the double phase operator and the corresponding function space

In this section we recall and prove new results concerning the Baouendi-Grushin operator introduced in Section 1.

Based on Theorem 1.1, we denote by $\mathcal{W}$ the closure of $C^1_c(\Omega)$ with respect to the norm

$$\begin{aligned}\|u\| = \|\nabla_x u\|_{G(\cdot,\cdot)} &+ \left\||x|^{\frac{\gamma}{G(\cdot,\cdot)}}\nabla_y u\right\|_{G(\cdot,\cdot)}\\ &+ \left\|u\left(|\nabla_x G(x,y)|+|x|^\gamma|\nabla_y G(x,y)|\right)^{\frac{1}{G(x,y)+1}}\right\|_{G(\cdot,\cdot)+1}\\ &+ \left\|u\left(|\nabla_x G(x,y)|+|x|^\gamma|\nabla_y G(x,y)|\right)^{\frac{1}{G(x,y)-1}}\right\|_{G(\cdot,\cdot)-1}.\end{aligned}$$

Note that the norm $\|\cdot\|$ on $\mathcal{W}$ is equivalent to

$$\begin{aligned}&\|u\|_{\mathcal{W}}\\ &=\inf\left\{\mu\geq 0 \;\middle|\; \rho\left(\frac{u}{\mu}\right)\leq 1\right\}\\ &=\inf\left\{\mu\geq 0 \;\middle|\; \int_\Omega \frac{1}{G(x,y)}\left[\left|\nabla_x\left(\frac{u}{\mu}\right)\right|^{G(x,y)}+|x|^\gamma\left|\nabla_y\left(\frac{u}{\mu}\right)\right|^{G(x,y)}\right]dx\,dy\right.\\ &\qquad\left.+\int_\Omega A(x,y)\left[\frac{\left|\frac{u}{\mu}\right|^{G(x,y)+1}}{G(x,y)+1}+\frac{\left|\frac{u}{\mu}\right|^{G(x,y)-1}}{G(x,y)-1}\right]dx\,dy\leq 1\right\}.\end{aligned} \tag{3.1}$$

From now on we denote the duality pairing between $\mathcal{W}$ and its dual space $\mathcal{W}^*$ by $\langle\cdot,\cdot\rangle_{\mathcal{W}}$. Furthermore, we set

$$G^+ := \sup_{(x,y)\in\overline{\Omega}} G(x,y) \quad\text{and}\quad G^- := \inf_{(x,y)\in\overline{\Omega}} G(x,y).$$

The following compactness property was proved by Bahrouni, Rădulescu and Repovš [6].

Lemma 3.1. *Assume that G is a function of class C^1 and that $G(x,y)\in(2,N)$ for all $(x,y)\in\overline{\Omega}$. Furthermore, suppose that $s\in(1,G^-)$ and $0<\gamma<\frac{N(G^--s)}{s}$. Then $\mathcal{W}$ is compactly embedded in $L^s(\Omega)$.*

Now, we define $\rho\colon \mathcal{W} \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega \frac{1}{G(x,y)} \left[|\nabla_x(u)|^{G(x,y)} + |x|^\gamma |\nabla_y(u)|^{G(x,y)} \right] dx\,dy + \int_\Omega A(x,y) \left[\frac{|u|^{G(x,y)+1}}{G(x,y)+1} + \frac{|u|^{G(x,y)-1}}{G(x,y)-1} \right] dx\,dy.$$

The following lemma will be helpful in later treatments.

Lemma 3.2. *Let $u \in \mathcal{W}$, then the following holds:*

(i) For $u \neq 0$ we have: $\|u\|_{\mathcal{W}} = a$ if and only if $\rho(\frac{u}{a}) = 1$;
(ii) $\|u\|_{\mathcal{W}} < 1$ implies $\|u\|_{\mathcal{W}}^{G^+ +1} \leq \rho(u) \leq \|u\|_{\mathcal{W}}^{G^- -1}$;
(iii) $\|u\|_{\mathcal{W}} > 1$ implies $\|u\|_{\mathcal{W}}^{G^- -1} \leq \rho(u) \leq \|u\|_{\mathcal{W}}^{G^+ +1}$.

Proof. (i) For every fixed $u \in \mathcal{W}$, the mapping $\lambda \mapsto \rho(\lambda u)$ is a continuous, convex, even function, which is strictly increasing in $[0, +\infty)$. Thus, by the definition of ρ and the equivalent norm given in (3.1), we have

$$\|u\|_{\mathcal{W}} = a \quad \Longleftrightarrow \quad \rho\left(\frac{u}{a}\right) = 1.$$

(ii) Let $u \in \mathcal{W}$ be such that $\|u\|_{\mathcal{W}} < 1$, then

$$\begin{aligned}
&\|\nabla_x u\|_{G(\cdot,\cdot)} < 1,\\
&\left\| |x|^{\frac{\gamma}{G(x,y)}} \nabla_y u \right\|_{G(\cdot,\cdot)} < 1,\\
&\left\| u(|\nabla_x G(x,y)| + |x|^\gamma |\nabla_y G(x,y)|)^{\frac{1}{G(x,y)+1}} \right\|_{G(\cdot,\cdot)+1} < 1,\\
&\left\| u(|\nabla_x G(x,y)| + |x|^\gamma |\nabla_y G(x,y)|)^{\frac{1}{G(x,y)-1}} \right\|_{G(\cdot,\cdot)-1} < 1.
\end{aligned}$$

So, by Proposition 2.1, we get the desired result.

(iii) Let $u \in \mathcal{W}$ be such that $\|u\|_{\mathcal{W}} > 1$. By (i), we obtain

$$\rho\left(\frac{u}{\|u\|_{\mathcal{W}}}\right) = \int_\Omega \frac{1}{G(x,y)} \left[\left|\nabla_x \left(\frac{u}{\|u\|_{\mathcal{W}}}\right)\right|^{G(x,y)} + |x|^\gamma \left|\nabla_y \left(\frac{u}{\|u\|_{\mathcal{W}}}\right)\right|^{G(x,y)} \right] dx\,dy + \int_\Omega A(x,y) \left[\frac{\left|\frac{u}{\|u\|_{\mathcal{W}}}\right|^{G(x,y)+1}}{G(x,y)+1} + \frac{\left|\frac{u}{\|u\|_{\mathcal{W}}}\right|^{G(x,y)-1}}{G(x,y)-1} \right] dx\,dy = 1.$$

Then, by the mean value theorem, there exist $(x_1,y_1), (x_2,y_2), (x_3,y_3) \in \overline{\Omega}$ depending on u, G and Ω such that

$$\begin{aligned}
1 = &\frac{1}{\|u\|_{\mathcal{W}}^{G(x_1,y_1)}} \int_\Omega \frac{1}{G(x,y)} \left[|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)} \right] dx\,dy\\
&+ \frac{1}{\|u\|_{\mathcal{W}}^{G(x_2,y_2)+1}} \int_\Omega A(x,y) \frac{|u|^{G(x,y)+1}}{G(x,y)+1}\, dx\,dy\\
&+ \frac{1}{\|u\|_{\mathcal{W}}^{G(x_3,y_3)-1}} \int_\Omega A(x,y) \frac{|u|^{G(x,y)-1}}{G(x,y)-1}\, dx\,dy.
\end{aligned}$$

Since $\|u\|_{\mathcal{W}} > 1$, it follows that

$$1 \leq \frac{1}{\|u\|_{\mathcal{W}}^{G_{-}-1}} \left[\int_{\Omega} \frac{1}{G(x,y)} \left[|\nabla_x u|^{G(x,y)} + |x|^{\gamma} |\nabla_y u|^{G(x,y)} \right] dx\, dy \right] + \frac{1}{\|u\|_{\mathcal{W}}^{G_{-}-1}} \left[\int_{\Omega} A(x,y) \left[\frac{|u|^{G(x,y)+1}}{G(x,y)+1} + \frac{|u|^{G(x,y)-1}}{G(x,y)-1} \right] dx\, dy \right].$$

This finishes the proof. □

Lemma 3.3. *Assume that the assumptions of Lemma 3.1 are fulfilled. Then the following properties hold.*

(i) The functional ρ is of class C^1 and for all $u, v \in \mathcal{W}$ we have

$$\langle \rho'(u), v \rangle_{\mathcal{W}} = \int_{\Omega} \left[|\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x v + |x|^{\gamma} |\nabla_y u|^{G(x,y)-2} \nabla_y u \nabla_y v \right] dx\, dy + \int_{\Omega} A(x,y) |u|^{G(x,y)-3} (u^2+1) uv\, dx\, dy.$$

(ii) The function $\rho' \colon \mathcal{W} \to \mathcal{W}^*$ is coercive, that is, $\frac{\langle \rho'(u), u \rangle_{\mathcal{W}}}{\|u\|_{\mathcal{W}}} \to +\infty$ as $\|u\|_{\mathcal{W}} \to +\infty$.

Proof. (i) This follows directly from the definition of $\rho \colon \mathcal{W} \to \mathbb{R}$.

(ii) By Lemma 3.2, for $\|u\|_{\mathcal{W}} > 1$, we obtain

$$\langle \rho'(u), u \rangle_{\mathcal{W}} \geq \rho(u) \geq \|u\|_{\mathcal{W}}^{G^- -1}.$$

Then

$$\frac{\langle \rho'(u), u \rangle_{\mathcal{W}}}{\|u\|_{\mathcal{W}}} \geq \|u\|_{\mathcal{W}}^{G^- -2} \to +\infty$$

as $\|u\|_{\mathcal{W}} \to +\infty$ since $G \in (2, N)$ and so $G^- > 2$. □

Lemma 3.4. *Let the conditions of Lemma 3.1 be satisfied. Then there exists $\lambda_1 > 0$ such that*

$$\lambda_1 = \inf_{\substack{u \in \mathcal{W} \\ \|u\|_{\mathcal{W}} > 1}} \frac{\rho(u)}{\|u\|_{G^- -1}^{G^- -1}}.$$

Proof. By Lemma 3.1 there exists $C > 0$ such that

$$\|u\|_{\mathcal{W}} \geq C \|u\|_{G^- -1} \quad \text{for all } u \in \mathcal{W}.$$

On the other hand, by Lemma 3.2, for $\|u\|_{\mathcal{W}} > 1$ we have

$$\rho(u) \geq \|u\|_{\mathcal{W}}^{G^- -1}.$$

Combining the above inequalities we obtain

$$\rho(u) \geq C^{G^- -1} \|u\|_{G^- -1}^{G^- -1} \quad \text{for all } u \in \mathcal{W} \text{ with } \|u\|_{\mathcal{W}} > 1.$$

The proof is now complete. □

Lemma 3.5. *Assume that the conditions of Lemma 3.1 hold. Then the double phase operator $\rho' \colon \mathcal{W} \to \mathcal{W}^*$ has the following properties:*

(i) ρ' is a continuous, bounded (that is, it maps bounded sets to bounded sets), and strictly monotone operator.

(ii) ρ' *is a mapping of type* (S_+)*, that is, if* $u_n \rightharpoonup u$ *in* $\mathcal{W}$ *and*

$$\limsup_{n\to+\infty} \langle \rho'(u_n), u_n - u\rangle_{\mathcal{W}} \leq 0,$$

then $u_n \to u$ *in* $\mathcal{W}$.

(iii) ρ' *is a homeomorphism.*

Proof. (i) From Lemma 3.3 it is clear that ρ' is continuous. Next, we are going to prove that ρ' maps bounded sets to bounded sets. By Young's inequality, we obtain

$$\begin{aligned}
&\langle \rho'(u), v\rangle_{\mathcal{W}}\\
&= \int_\Omega \left[|\nabla_x u|^{G(x,y)-2}\nabla_x u \nabla_x v + |x|^\gamma |\nabla_y u|^{G(x,y)-2}\nabla_y u\nabla_y v\right] dx\,dy\\
&\quad + \int_\Omega A(x,y)\,|u|^{G(x,y)-3}(u^2+1)uv\,dx\,dy\\
&\leq (G^+-1)\int_\Omega \frac{1}{G(x,y)}\left[|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)}\right] dx\,dy\\
&\quad + \int_\Omega \frac{1}{G(x,y)}\left[|\nabla_x v|^{G(x,y)} + |x|^\gamma |\nabla_y v|^{G(x,y)}\right] dx\,dy\\
&\quad + G^+\int_\Omega A(x,y)\frac{|u|^{G(x,y)+1}}{G(x,y)+1}\,dx\,dy + \int_\Omega A(x,y)\frac{|v|^{G(x,y)+1}}{G(x,y)+1}\,dx\,dy\\
&\quad + (G^+-2)\int_\Omega A(x,y)\frac{|u|^{G(x,y)-1}}{G(x,y)-1}\,dx\,dy + \int_\Omega A(x,y)\frac{|v|^{G(x,y)-1}}{G(x,y)-1}\,dx\,dy\\
&\leq G^+\rho(u) + \rho(v).
\end{aligned}$$

Hence, from Lemma 3.2, we get

$$\|\rho'(u)\| = \sup_{\|v\|\leq 1} |\langle \rho'(u), v\rangle_{\mathcal{W}}| \leq G^+\rho(u) + 3,$$

which implies that ρ' maps bounded sets to bounded sets.

The strict monotonicity of ρ' is a direct consequence of the well-known Simon inequalities [30, formula (2.2)]

$$|x-y|^p \leq c_p\left(|x|^{p-2}x - |y|^{p-2}y\right)\cdot(x-y) \quad \text{if } p\geq 2, \tag{3.2}$$

$$\begin{aligned}
|x-y|^p \leq C_p&\left[\left(|x|^{p-2}x - |y|^{p-2}y\right)\cdot(x-y)\right]^{\frac{p}{2}}\\
&\times\left(|x|^p+|y|^p\right)^{\frac{2-p}{2}} \quad \text{if } p\in(1,2),
\end{aligned} \tag{3.3}$$

for all $x,y\in\mathbb{R}^N$, where c_p and C_p are positive constants depending only on p, see Lindqvist [19, p. 71], Filippucci, Pucci & Rădulescu [14, p. 713], and Pucci, Xiang & Zhang [26, p. 14].

(ii) Let $\{u_n\}_{n\geq 1}\subseteq \mathcal{W}$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{in } \mathcal{W} \quad \text{and} \quad \limsup_{n\to+\infty}\langle \rho'(u_n), u_n-u\rangle_{\mathcal{W}} \leq 0.$$

Then, from (i), we deduce that

$$\lim_{n\to+\infty}\langle \rho'(u_n), u_n-u\rangle_{\mathcal{W}} = 0. \tag{3.4}$$

In view of (3.2) and (3.3), the sequence $\{\nabla u_n\}_{n\geq 1}$ converges in measure to ∇u in Ω. Then there is a subsequence, still denoted by $\{\nabla u_n\}_{n\geq 1}$, that converges to ∇u a.e. in Ω.

First, we have

$$\begin{aligned}
&\langle \rho'(u_n), u_n - u\rangle_{\mathcal{W}} \\
&= \int_\Omega \Big[|\nabla_x u_n|^{G(x,y)-2} \nabla_x u_n \nabla_x (u_n - u) \\
&\qquad\qquad + |x|^\gamma |\nabla_y u_n|^{G(x,y)-2} \nabla_y u_n \nabla_y (u_n - u) \Big] \, dx\, dy \\
&\quad + \int_\Omega A(x,y) |u_n|^{G(x,y)-3} (u_n^2 + 1) u_n (u_n - u) \, dx\, dy \\
&= \int_\Omega \Big[|\nabla_x u_n|^{G(x,y)} + |x|^\gamma |\nabla_y u_n|^{G(x,y)} \Big] \, dx\, dy \\
&\quad + \int_\Omega A(x,y) |u_n|^{G(x,y)-1} (u_n^2 + 1) \, dx\, dy \\
&\quad - \int_\Omega \Big[|\nabla_x u_n|^{G(x,y)-2} \nabla_x u_n \nabla_x u + |x|^\gamma |\nabla_y u_n|^{G(x,y)-2} \nabla_y u_n \nabla_y u \Big] \, dx\, dy \\
&\quad - \int_\Omega A(x,y) |u_n|^{G(x,y)-3} (u_n^2 + 1) u_n u \, dx\, dy.
\end{aligned}$$

Then, by applying Young's inequality to the right-hand side of the last equation, we obtain

$$\begin{aligned}
&\langle \rho'(u_n), u_n - u\rangle_{\mathcal{W}} \\
&\geq \int_\Omega \Big[|\nabla_x u_n|^{G(x,y)} + |x|^\gamma |\nabla_y u_n|^{G(x,y)} \Big] \, dx\, dy \\
&\quad + \int_\Omega A(x,y) |u_n|^{G(x,y)-1} (u_n^2 + 1) \, dx\, dy \\
&\quad - \int_\Omega \frac{G(x,y)-1}{G(x,y)} \Big[|\nabla_x u_n|^{G(x,y)} + |x|^\gamma |\nabla_y u_n|^{G(x,y)} \Big] \, dx\, dy \\
&\quad - \int_\Omega \frac{1}{G(x,y)} \Big[|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)} \Big] \, dx\, dy \\
&\quad - \int_\Omega A(x,y) \frac{G(x,y)}{G(x,y)+1} |u_n|^{G(x,y)+1} \, dx\, dy \\
&\quad - \int_\Omega A(x,y) \frac{1}{G(x,y)+1} |u|^{G(x,y)+1} \, dx\, dy \\
&\quad - \int_\Omega A(x,y) \frac{G(x,y)-2}{G(x,y)-1} |u_n|^{G(x,y)-1} \, dx\, dy \\
&\quad - \int_\Omega A(x,y) \frac{1}{G(x,y)-1} |u|^{G(x,y)-1} \, dx\, dy \\
&\geq \rho(u_n) - \rho(u).
\end{aligned}$$

This finally gives

$$\langle \rho'(u_n), u_n - u\rangle_{\mathcal{W}} \geq \rho(u_n) - \rho(u). \tag{3.5}$$

Combining (3.4) and (3.5) leads to

$$\lim_{n\to+\infty}\rho(u_n)\leq\rho(u).$$

On the other hand, it follows from Fatou's lemma that

$$\liminf_{n\to+\infty}\rho(u_n)\geq\rho(u).$$

Thus, we have that

$$\lim_{n\to+\infty}\rho(u_n)=\rho(u),$$

which implies that the family of continuous functions

$$\begin{aligned}\Bigg\{&\frac{1}{G(x,y)}\left[|\nabla_x u_n|^{G(x,y)}+|x|^\gamma|\nabla_y u_n|^{G(x,y)}\right]\\&+A(x,y)\left[\frac{|u_n|^{G(x,y)+1}}{G(x,y)+1}+\frac{|u_n|^{G(x,y)-1}}{G(x,y)-1}\right]\Bigg\}_{n\geq1}\end{aligned}$$

turns out to be equicontinuous on Ω. Since

$$\begin{aligned}&\frac{1}{G(x,y)}|\nabla_x(u_n-u)|^{G(x,y)}+\frac{1}{G(x,y)}|x|^\gamma|\nabla_y(u_n-u)|^{G(x,y)}\\&\quad+A(x,y)\left[\frac{|u_n-u|^{G(x,y)+1}}{G(x,y)+1}+\frac{|u_n-u|^{G(x,y)-1}}{G(x,y)-1}\right]\\&\leq\frac{C}{G(x,y)}\left(|\nabla_x u_n|^{G(x,y)}+|\nabla_x u|^{G(x,y)}\right)\\&\quad+\frac{C}{G(x,y)}\left(|\nabla_y u_n|^{G(x,y)}+|\nabla_y u|^{G(x,y)}\right)\\&\quad+CA(x,y)\left(\frac{|u_n|^{G(x,y)+1}}{G(x,y)+1}+\frac{|u|^{G(x,y)+1}}{G(x,y)+1}\right)\\&\quad+CA(x,y)\left(\frac{|u_n|^{G(x,y)-1}}{G(x,y)-1}+\frac{|u|^{G(x,y)-1}}{G(x,y)-1}\right),\end{aligned}$$

with positive C, the integrals of the family

$$\begin{aligned}\Bigg\{&\frac{1}{G(x,y)}|\nabla_x(u_n-u)|^{G(x,y)}+\frac{1}{G(x,y)}|x|^\gamma|\nabla_y(u_n-u)|^{G(x,y)}\\&+A(x,y)\left[\frac{|u_n-u|^{G(x,y)+1}}{G(x,y)+1}+\frac{|u_n-u|^{G(x,y)-1}}{G(x,y)-1}\right]\Bigg\}_{n\geq1}\end{aligned}$$

are also equicontinuous on Ω and therefore

$$\lim_{n\to+\infty}\rho(u_n-u)=0.$$

It follows, by Proposition 2.2, that

$$u_n\to u\quad\text{in}\quad\mathcal{W}.$$

(iii) By strict monotonicity, ρ' is an injection. On the other hand, using Lemma 3.3 and the Minty-Browder theorem, ρ' is a surjection. Hence ρ' has an inverse mapping $(\rho')^{-1}\colon\mathcal{W}^*\to\mathcal{W}$. Therefore, in order to complete the proof of (iii), it suffices to prove that $(\rho')^{-1}$ is continuous. If $f_n, f\in\mathcal{W}^*, f_n\to f$, letting $u_n=$

$(\rho')^{-1}(f_n), u = (\rho')^{-1}(f)$, then $\rho'(u_n) = f_n, \rho'(u) = f$. Note that $\{u_n\}_{n\geq 1}$ is bounded in $\mathcal{W}$. Without loss of generality, we can assume that $u_n \rightharpoonup u_0$ in $\mathcal{W}$. We conclude from $f_n \to f$ that

$$\lim_{n\to+\infty} \langle \rho'(u_n) - \rho'(u_0), u_n - u_0\rangle_{\mathcal{W}} = \lim_{n\to+\infty} \langle f_n - f, u_n - u_0\rangle_{\mathcal{W}} = 0.$$

Since ρ' is of type (S_+), we know that $u_n \to u_0$ in $\mathcal{W}$ and so $u_n \to u$ in $\mathcal{W}$. □

4. Existence of a solution

We suppose the following hypotheses on the reaction term in (1.1).

(H) $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function with $f((x,y),0,0) \neq 0$ for a. a. $(x,y) \in \Omega$ such that the following holds:

(i) there exists $a \in L^\infty(\Omega \times \mathbb{R}^N)$ such that

$$|f((x,y),s,\xi)| \leq a((x,y),\xi)\left(1 + |s|^{G^- -1}\right)$$

for a. a. all $(x,y) \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$;

(ii) there exists $\vartheta \in (0,\lambda_1)$ such that

$$\limsup_{s\to+\infty} \frac{f((x,y),s,\xi)}{s^{G^- -1}} \leq \vartheta \quad \text{uniformly for a. a. } (x,y) \in \Omega$$

and for all $\xi \in \mathbb{R}^N$ with λ_1 given in Lemma 3.4.

We say that $u \in \mathcal{W}$ is a weak solution of problem (1.1) if

$$\begin{aligned}
&\int_\Omega \left[|\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x \varphi + |x|^\gamma |\nabla_y u|^{G(x,y)-2} \nabla_y u \nabla_y \varphi\right] dx\, dy \\
&\qquad + \int_\Omega A(x,y)\, |u|^{G(x,y)-3} (u^2+1) u\varphi \, dx\, dy \\
&= \int_\Omega f((x,y),u,\nabla u) v \, dx\, dy.
\end{aligned}$$

is satisfied for all $\varphi \in \mathcal{W} \setminus \{0\}$.

Now we are in the position to state our main existence result.

Theorem 4.1. *Suppose that conditions (H)(i), (ii) are fulfilled. Moreover, assume that G is a function of class C^1 and that $G(x,y) \in (2,N)$ for all $(x,y) \in \Omega$. Furthermore, suppose that $s \in (1,G^-)$ and $0 < \gamma < \frac{N(G^- -s)}{s}$. Then problem (1.1) admits at least one nontrivial weak solution.*

Proof. Let $N_f\colon \mathcal{W} \subseteq L^{G^- -1} \to L^{(G^- -1)'} \subseteq \mathcal{W}^*$ be the Nemytskij operator corresponding to the nonlinearity $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which is compact by Lemma 3.1. Now we define the operator $I\colon \mathcal{W} \to \mathcal{W}^*$ by

$$I(u) = \rho'(u) - N_f(u).$$

Because of the growth condition H(i) and Lemma 3.5(i) we know that $I\colon \mathcal{W} \to \mathcal{W}^*$ maps bounded sets into bounded sets. Let us now prove that I is pseudomonotone in the sense of Definition 2.3. To this end, let $\{u_n\}_{n\geq 1} \subseteq \mathcal{W}$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{in } \mathcal{W} \quad \text{and} \quad \limsup_{n\to+\infty} \langle I(u_n), u_n - u\rangle_{\mathcal{W}} \leq 0. \tag{4.1}$$

Recall that

$$\langle I(u_n), u_n - u\rangle = \langle \rho'(u_n), u_n - u\rangle - \int_\Omega f((x,y), u_n, \nabla u_n)(u_n - u)\,dx\,dy. \tag{4.2}$$

By Lemma 3.1 we know that

$$u_n \to u \quad \text{in } L^{G^- -1}(\Omega)$$

since $G^- - 1 < G^-$. Moreover, hypothesis H(i) implies that

$$\{N_f(u_n)\}_{n\geq 1} \subseteq L^{(G^- -1)'}(\Omega) \quad \text{is bounded.}$$

From these facts it is clear that

$$\int_\Omega f((x,y), u_n, \nabla u_n)(u_n - u)\,dx\,dy \to 0 \quad \text{as } n \to +\infty. \tag{4.3}$$

Therefore, passing to the limit in (4.2) and using (4.1) as well as (4.3) leads to

$$\limsup_{n\to+\infty} \langle \rho'(u_n), u_n - u\rangle = \limsup_{n\to+\infty} \langle I(u_n), u_n - u\rangle \leq 0. \tag{4.4}$$

From Lemma 3.5 we know that ρ' fulfills the (S_+)-property and so we conclude, in view of (4.1) and (4.4), that

$$u_n \to u \quad \text{in } \mathcal{W}.$$

Thus, because of the continuity of $I\colon \mathcal{W} \to \mathcal{W}^*$, we have $I(u_n) \to I(u)$ in $\mathcal{W}^*$ which proves that I is pseudomonotone.

Next, we have to show that the operator $I\colon \mathcal{W} \to \mathcal{W}^*$ is coercive, that is,

$$\lim_{\|u\|_{\mathcal{W}}\to\infty} \frac{\langle I(u), u\rangle_{\mathcal{W}}}{\|u\|_{\mathcal{W}}} = \infty.$$

Note that hypothesis (H)(ii) implies that for a given $\varepsilon > 0$ there exists $M = M(\varepsilon) > 1$ such that

$$f(x,s,\xi)s \leq (\vartheta + \varepsilon)s^{G^- -1} \tag{4.5}$$

for a. a. $x \in \Omega$, for all $s \geq M$ and for all $\xi \in \mathbb{R}^N$.

Let $u \in \mathcal{W}$ be such that $\|u\| > M > 1$. Applying Lemma 3.3, (4.5), Lemma 3.4 and Lemma 3.2(iii) we get

$$\begin{aligned}
\langle I(u), u\rangle &= \langle \rho'(u), u\rangle - \int_\Omega f((x,y), u, \nabla u)u\,dx\,dy\\
&\geq \rho(u) - (\vartheta + \epsilon)\int_\Omega |u|^{G^- -1}\,dx\,dy\\
&= \rho(u) - (\vartheta + \epsilon)\|u\|_{G^- -1}^{G^- -1}\\
&\geq \left(1 - \frac{\vartheta + \epsilon}{\lambda_1}\right)\rho(u)\\
&\geq \left(1 - \frac{\vartheta + \epsilon}{\lambda_1}\right)\|u\|_{\mathcal{W}}^{G^- -1}.
\end{aligned}$$

Choosing $\varepsilon \in (0, \lambda_1 - \vartheta)$ proves that $I\colon \mathcal{W} \to \mathcal{W}^*$ is coercive.

To sum up, we have shown that the operator $I\colon \mathcal{W} \to \mathcal{W}^*$ is bounded, pseudomonotone and coercive. Therefore, the main theorem on pseudomonotone operators, see Theorem 2.4, provides $u \in \mathcal{W}$, $u \neq 0$ (since $f(x,0,0) \neq 0$), such that

$I(u) = 0$. By the definition of I, the function u turns out to be a nontrivial weak solution of problem (1.1) which completes the proof. □

Concluding remarks, perspectives, and open problems

(i) The mathematical analysis carried out in this paper considers the unbalanced energy

$$\mathcal{W} \ni u \mapsto \int_\Omega \frac{1}{G(x,y)} \left[|\nabla_x(u)|^{G(x,y)} + |x|^\gamma |\nabla_y(u)|^{G(x,y)} \right] dx\, dy$$

with the associated differential operator

$$\Delta_{G(x,y)} u = \sum_{i=1}^{n} \left(|\nabla_x|^{G(x,y)-2} u_{x_i} \right)_{x_i} + |x|^\gamma \sum_{i=1}^{m} \left(|\nabla_y|^{G(x,y)-2} u_{y_i} \right)_{y_i} .$$

It appears to be worth to further investigate patterns described by the variational integral

$$\int_\Omega \left(|\nabla_x u|^{G(x,y)} + |x|^\gamma |\nabla_y u|^{G(x,y)} \right) dz \tag{4.6}$$

with corresponding anisotropic Baouendi-Grushin operator

$$\operatorname{div}_x \left(G(x,y) |\nabla_x|^{G(x,y)-2} \nabla_x \right) + \operatorname{div}_y \left(G(x,y) |x|^\gamma |\nabla_y|^{G(x,y)-2} \nabla_y \right).$$

(ii) We remark that since both ρ and the energy functional defined in (4.6) have a degenerate action on the set where the gradient vanishes, it is a natural question to study what happens if the integrand is modified in such a way that, if $|\nabla u|$ is also small, there exists an imbalance between the two terms of every integrand.

(iii) The compactness property established in Lemma 3.1 plays a key role in the proof of several crucial properties such as: coercivity of ρ' (Lemma 3.3), existence of a principal eigenvalue associated to the Rayleigh quotient (Lemma 3.4), as well as in the proof of the main result established in Theorem 4.1. This compactness property is established in a *subcritical* setting, which corresponds to the hypothesis $s < G^-$, where s describes the growth of the right-hand side of problem (1.1). In fact, Theorem 4.1 remains true if s is replaced with a variable coefficient $s(x)$, provided that $s^+ < G^-$. We do not have any knowledge about the behaviour in the *almost critical* case that arises in the following situation: there exists $x_0 \in \Omega$ such that $s(x_0) = G^-$ and $s(x) < G^-$ for all $x \in \Omega \setminus \{x_0\}$.

(iv) It is worth noting that the study of nonlinear boundary value problems involving the magnetic Baouendi-Grushin operator [3] are of real interest for mathematical physics patterns. This operator is

$$G_\mathcal{A} := -(\nabla_G + i\beta\mathcal{A}_0)^2 \quad \text{for } -\frac{1}{2} \le \beta \le \frac{1}{2},$$

where

$$\mathcal{A}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) = \left(-\frac{\partial_y d}{d}, \frac{\partial_x d}{d}, -2y\frac{\partial_t d}{d}, 2x\frac{\partial_t d}{d} \right),$$

$$\nabla_G = (\partial_x, \partial_y, 2x\partial_t, 2y\partial_t),$$

with $z = (x,y)$, $|z| = \sqrt{x^2+y^2}$, and $d(z,t) = (|z|^4 + t^2)^{1/4}$ is the Kaplan distance.

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(A. Bahrouni) Mathematics Department, University of Monastir, Faculty of Sciences, 5019 Monastir, Tunisia
Email address: `bahrounianouar@yahoo.fr`

(V.D. Rădulescu) Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland & Department of Mathematics, University of Craiova, 200585 Craiova, Romania
Email address: `radulescu@inf.ucv.ro`

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
Email address: `winkert@math.tu-berlin.de`