SMALL PERTURBATIONS OF ROBIN PROBLEMS DRIVEN BY THE p-LAPLACIAN PLUS A POSITIVE POTENTIAL

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Abstract. We consider a quasilinear Robin problem driven by the p-Laplacian plus a positive potential and with a small perturbation. We assume that the main term in the equation has an Ekeland structure but we do not suppose any growth condition for the perturbation term. Applying variational methods, we prove the existence of at least one nontrivial weak solution.

1. Introduction

In this paper, we study the following quasilinear Robin p-Laplace problem with small perturbation given by

$$\begin{aligned} -\Delta_p u + V(x)u &= a(x)|u|^{q-1}u + \lambda g(x,u) + f(x) \qquad && \text{in } \Omega,\\ |\nabla u|^{p-2}\nabla u\cdot\nu + \beta(x)|u|^{p-2}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N>2)$ is a bounded domain with a C^2-boundary $\partial\Omega$, $p\geq 2$, λ is a real parameter, $0<q<p-1$ and $\nu(x)$ denotes the outer unit normal of Ω at $x\in\partial\Omega$. This problem is driven by the p-Laplacian, which is defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{for } u \in W^{1,p}(\Omega).$$

Our aim is to prove the existence of at least one weak solution of problem (1.1) by applying variational methods like Ekeland's variational principle. In order to state our main result we need to give first the precise assumptions on the data of problem (1.1).

H(a): $a\in L^\infty(\Omega)$ and there exists $\alpha>0$ such that

$$a(x)\leq -\alpha \quad \text{for all } x\in\Omega;$$

H(β): $\beta\in L^\infty(\partial\Omega)$ and $\beta(x)\geq 0$ for all $x\in\partial\Omega$ with $\beta\not\equiv 0$;
H(g): $g\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is continuous;
H(V): $V\in L^\infty(\Omega)$ and

$$\inf_{x\in\Omega} V(x) > 1;$$

H(f): $f\in L^\infty(\Omega)$ and there exist $x_0\in\Omega$ and $R_0>0$ such that

$$f(x)>0 \quad \text{for all } x\in B(x_0,R_0).$$

Our main result reads as follows.

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Theorem 1.1. *Assume that conditions H(a), H(β), H(f), H(g) and H(V) hold. Moreover, suppose that $N < p$ and $0 < q < p - 1$. Then there exists a positive number λ_0 such that if $|\lambda| < \lambda_0$, problem (1.1) has at least one nontrivial weak solution.*

In order to treat problem (1.1), we first study the existence of solutions of the following auxiliary problem

$$\begin{aligned} -\Delta_p u + V(x)u &= a(x)|u|^{q-1}u + f(x) \qquad && \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu + \beta(x)|u|^{p-2}u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

For V we suppose a weaker condition in the following way.

H(V'): $V \in L^\infty(\Omega)$ and

$$\|V^-\|_{\frac{N}{2}} \leq S,$$

where $V^\pm = \max\{V^\pm, 0\}$ and S denotes the best Sobolev constant of the compact embedding $\mathcal{D}^{1,2}(\Omega)$ into $L^r(\Omega)$ for $r \in [1, 2^*)$, see Anello and Cordaro [1, Lemma 2.1], that is,

$$S = \inf\left\{ \frac{\|\nabla u\|_2^2}{\|u\|_2^2} \;\middle|\; u \in \mathcal{D}^{1,2}(\Omega) \right\},$$

where $\mathcal{D}^{1,2}(\Omega)$ is the completion of the space of continuous functions on Ω with compact support with respect to the norm

$$\left(\int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

Remark 1.2. *It is clear that condition H(V) implies assumption H(V').*

For problem (1.2) we are going to prove the following result.

Theorem 1.3. *Let H(a), H(β), H(f) and H(V') be satisfied and let $0 < q < p - 1$. Then there exists a positive number λ_0 such that if $|\lambda| < \lambda_0$, problem (1.2) admits a nontrivial weak solution.*

The main result in this paper establishes that problem (1.1) has a solution provided that a suitable perturbation of the second reaction term is sufficiently small. This perturbation is described in terms of the real parameter λ in relationship with the small values of the first reaction term with respect to a certain topology.

Such existence type results have been investigated by Papageorgiou and Rădulescu [8], [10], [9] and Vetro [12] for nonlinear Robin problems and by Wang [13] for nonlinear Neumann problems. All the aforementioned results treat the superlinear case and impose more restrictive conditions on the reaction $g\colon \overline{\Omega} \times [0, \infty) \to \mathbb{R}$. Moreover, our work here complements the recent works by Bahrouni, Ounaies and Rădulescu [2], [3], Bahrouni, Rădulescu and Winkert [4] and Kajikiya [7] where the authors prove for Dirichlet problems an existence theorem for small values of $\lambda > 0$.

The paper is organized as follows. In Section 2 we state the main notations and the main results which will be used later. The auxiliary problem (1.2) is then considered in Section 3 by applying critical point theory and Ekeland's variational principle, see Theorem 2.2. Taking into account this result, we are going to prove Theorem 1.1 in Section 4.

2. Preliminaries

In the whole paper we suppose that Ω is a bounded domain in $\mathbb{R}^N$, $N > 2$. Given $1 \leq r \leq \infty$, $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ stand for the usual Lebesgue spaces equipped with the norm $\|\cdot\|_r$ while $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ denote the Sobolev spaces endowed with the norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0}$, respectively. By r', we denote the conjugate of $r \in (1,\infty)$, that is, $\frac{1}{r} + \frac{1}{r'} = 1$.

On the boundary $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ, by which we can define in the usual way the boundary Lebesgue space $L^r(\partial\Omega)$, with norm $\|\cdot\|_{r,\partial\Omega}$. It is known that there exists a unique continuous linear operator $\gamma\colon W^{1,r}(\Omega) \to L^r(\partial\Omega)$, called trace map, such that

$$\gamma(u) = u\big|_{\partial\Omega} \quad \text{for all } u \in W^{1,r}(\Omega) \cap C^0(\overline{\Omega}).$$

Henceforth, although all restrictions of Sobolev functions to $\partial\Omega$ are understood in the sense of traces, we will avoid the usage of the trace operator γ to simplify notation.

In the following we will equip the space $E = W^{1,p}(\Omega)$ with the norm

$$\|u\|_E = \left(\|\nabla u\|_p^p + \|u\|_{p,\beta,\partial\Omega}^p\right)^{\frac{1}{p}} \quad \text{with } \|u\|_{p,\beta,\partial\Omega} = \left(\int_{\partial\Omega} \beta(x)|u|^p \, d\sigma\right)^{\frac{1}{p}},$$

which is equivalent to the standard one $\|\cdot\|_{1,p}$, see Papageorgiou and Winkert [11]. Note that the critical Sobolev exponent to $r \in (1,\infty)$ is given by

$$r^* := \begin{cases} \dfrac{Nr}{N-r} & \text{if } r < N, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

The following definition is important in our treatment.

Definition 2.1. *Let X be a real Banach space, let $c \in \mathbb{R}$ and let $F \subset X$ be a closed subset. We say that $I \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ on F ($(\mathrm{PS})_{F,c}$-condition for short), if any subsequence $(u_n)_{n\in\mathbb{N}} \subseteq F$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in X^*, has a convergent subsequence to some $u \in F$. If $F = X$ we write $(\mathrm{PS})_c$. If $F = X$ and I satisfies the $(\mathrm{PS})_c$-condition at every level $c \in \mathbb{R}$ we say that I satisfies the (PS)-condition.*

This compactness-type condition on I is crucial in deriving the minimax theory of the critical values.

Let us recall the following version of Ekeland's variational principle established by Ekeland [5] or Gonçalves and Miyagaki [6].

Theorem 2.2. *Let X be a real Banach space. If $I \in C^1(X,\mathbb{R})$ is bounded from below on a closed subset $F \subset X$ with a nonempty interior and if*

$$I(v) < 0 < \inf_{u\in\partial F} I(u) \quad \textit{for some } v \in F^\circ, \tag{2.2}$$

then

$$c := \inf_{u\in F} I(u) \tag{2.3}$$

is a critical value of I provided that I fulfills the $(\mathrm{PS})_{F,c}$-condition.

Definition 2.3. *Let X be a real Banach space and let $I \in C^1(X,\mathbb{R})$.*

(1) We say that u is a c-Ekeland solution of I if $I(u) = 0$ and $I'(u) = c$, where c is given in (2.3).
(2) We say that I has the Ekeland geometry if I satisfies property (2.2).

3. Study of the auxiliary problem

Our aim in this section is the proof of Theorem 1.3. First we give the precise definition of a weak solution of problem (1.2).

Definition 3.1. *We call $u \in E$ a weak solution of problem (1.1) if*

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx + \int_\Omega V(x)uv\,dx + \int_{\partial\Omega} \beta(x)|u|^{p-2}uv\,d\sigma\\&= \int_\Omega a(x)|u|^{q-1}uv\,dx + \int_\Omega f(x)v\,dx\end{aligned}$$

is satisfied for all $v \in E$.

In order to find weak solutions we are going to study the corresponding energy functional $J\colon E \to \mathbb{R}$ of (1.2) given by

$$\begin{aligned}J(u) = &\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega V(x)|u|^2\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)|u|^p\,d\sigma\\&- \frac{1}{q+1}\int_\Omega a(x)|u|^{q+1}\,dx - \int_\Omega f(x)u\,dx.\end{aligned}$$

Lemma 3.2. *Suppose that H(a), H(β), H(f) and H(V') are satisfied and let $0 < q < p-1$. Let $d \in \mathbb{R}$ and let $F \subset E$ be a closed subset. Then J satisfies the $(\mathrm{PS})_{F,d}$-condition.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a (PS)-sequence of J, then there exists $c_1 > 0$ such that $J(u_n) \leq c_1$. We claim that $(u_n)_{n\in\mathbb{N}}$ is bounded in E.

Case 1: $p = 2$. Note that in this case we have $2^* = \frac{2N}{N-2}$ since $2 < N$ by assumption, see also (2.1).

Applying Hölder's inequality, hypotheses H(a), H(β), H(f) and H(V') along with the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\begin{aligned}c_1 &\geq J(u_n)\\&= \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \frac{1}{2}\int_\Omega V(x)|u_n|^2\,dx + \frac{1}{2}\int_{\partial\Omega}\beta(x)|u_n|^2\,d\sigma\\&\quad - \frac{1}{q+1}\int_\Omega a(x)|u_n|^{q+1}\,dx - \int_\Omega f(x)u_n\,dx\\&\geq \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \frac{1}{2}\int_\Omega V^+(x)|u_n|^2\,dx - \frac{1}{2}\int_\Omega V^-(x)|u_n|^2\,dx\\&\quad + \frac{1}{2}\int_{\partial\Omega}\beta(x)|u_n|^2\,d\sigma - \int_\Omega f(x)u_n\,dx\\&\geq \left(\frac{1}{2} - \frac{\|V^-\|_{\frac{N}{2}}}{2S}\right)\|\nabla u_n\|_2^2 + \frac{1}{2}\int_{\partial\Omega}\beta(x)|u_n|^2\,d\sigma - \|f\|_2\|u_n\|_2\\&\geq \left(\frac{1}{2} - \frac{\|V^-\|_{\frac{N}{2}}}{2S}\right)\|u_n\|_E^2 - C_1\|f\|_2\|u_n\|_E,\end{aligned} \tag{3.1}$$

with a positive constant C_1. Thus, there exists $C_2 > 0$ such that

$$\|u_n\|_E \leq C_2 \quad \text{for all } n \in \mathbb{N}.$$

Case 2: $p > 2$**.** Note that in this case we do not need to know if $N > p$ or $N \leq p$.

Again, by applying Hölder's inequality, conditions H(a), H(β), H(f) and H(V') and the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ we have

$$\begin{aligned}
c_1 &\geq J(u_n) \\
&\geq \frac{1}{p}\int_\Omega |\nabla u_n|^p\,dx + \frac{1}{2}\int_\Omega V^+(x)|u_n|^2\,dx - \frac{1}{2}\int_\Omega V^-(x)|u_n|^2\,dx \\
&\quad + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u_n|^p\,d\sigma - \int_\Omega f(x)u_n\,dx \\
&\geq \frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(x)|u_n|^p\,d\sigma - C_3\|V^-\|_\infty\|u_n\|^2 - C_4\|f\|_{p'}\|u_n\|_p \\
&\geq \frac{1}{p}\|u_n\|_E^p - C_5\|V^-\|_\infty\|u_n\|_E^2 - C_6\|f\|_{p'}\|u_n\|_E,
\end{aligned} \tag{3.2}$$

where C_5, C_6 are positive constants. Since $p > 2 > 1$, it is easy to see that $(u_n)_{n\in\mathbb{N}}$ is also bounded in this case. The proof that the sequence $(u_n)_{n\in\mathbb{N}}$ is strongly convergent in E is standard and is omitted. ☐

Lemma 3.3. *Assume that the hypotheses of Theorem 1.3 are fulfilled. Then, problem (1.2) has the Ekeland geometry property.*

Proof. First we are going to show that there exist $\rho, \gamma > 0$ such that

$$J(u) \geq \gamma \quad \text{for all } u \in E \text{ with } \|u\|_E = \rho. \tag{3.3}$$

Case 1: $p = 2$**.**

Let $u \in E$. From hypotheses H(a), H(β), H(f) and H(V') and (3.1) it follows that

$$\begin{aligned}
J(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega V^+(x)|u|^2\,dx - \frac{1}{2}\int_\Omega V^-(x)|u|^2\,dx \\
&\quad + \frac{1}{2}\int_{\partial\Omega}\beta(x)|u|^2\,d\sigma - \int_\Omega f(x)u\,dx \\
&\geq \left(\frac{1}{2} - \frac{\|V^-\|_{\frac{N}{2}}}{2S}\right)\|u\|_E^2 - \|f\|_2\|u\|_E.
\end{aligned} \tag{3.4}$$

We set

$$\rho = \frac{2\|f\|_2}{\left(\frac{1}{2} - \frac{\|V^-\|_{\frac{N}{2}}}{2S}\right)}$$

Then, by (3.4), we derive that

$$J(u) \geq \left(\frac{1}{4} - \frac{\|V^-\|_{\frac{N}{2}}}{4S}\right) = \gamma.$$

This proves (3.3) if $p = 2$. The case $p > 2$ works in the same way by using (3.2) instead of (3.1).

Now let $\varphi \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(\varphi) \subset B(x_0, R_0)$. This yields, for t sufficiently small enough,

$$\begin{aligned}
J(t\varphi) &= \frac{t^p}{p}\int_\Omega |\nabla\varphi|^p\,dx + \frac{t^p}{p}\int_\Omega V(x)|\varphi|^p\,dx + \frac{t^p}{p}\int_{\partial\Omega}\beta(x)|\varphi|^p\,d\sigma\\
&\quad - \frac{t^{q+1}}{q+1}\int_\Omega a(x)|\varphi|^{q+1}\,dx - t\int_\Omega f(x)\varphi\,dx\\
&= t\Bigg(\frac{t^{p-1}}{p}\int_\Omega |\nabla\varphi|^p\,dx + \frac{t^{p-1}}{p}\int_\Omega V(x)|\varphi|^p\,dx + \frac{t^{p-1}}{p}\int_{\partial\Omega}\beta(x)|\varphi|^p\,d\sigma\\
&\qquad - \frac{t^q}{q+1}\int_\Omega a(x)|\varphi|^{q+1}\,dx - \int_\Omega f(x)\varphi\,dx\Bigg)\\
&< 0.
\end{aligned} \tag{3.5}$$

since $q < p-1$.

Combining (3.3) and (3.5), we obtain the desired conclusion. □

Proof of Theorem 1.3. Next, we consider the minimization problem

$$c = \inf_{u \in \overline{B}_0(\rho)} J(u).$$

It is clear that $-\infty < c < 0$. Then by Theorem 2.2 as well as Lemmas 3.2 and 3.3, there exists $u_0 \in E$ such that u_0 is a nontrivial weak solution of problem (1.2). □

In the next step we are going to prove that every nontrivial weak solution of problem (1.2) belongs to $L^\infty(\Omega)$.

Lemma 3.4. *Suppose that the hypotheses of Theorem 1.3 are fulfilled. Moreover, assume that H(V) holds. Then, for every weak solution u of problem (1.2), we have*

$$-M = -\|f\|_\infty \le u(x) \le \|f\|_\infty = M \quad \text{for all } x \in \overline{\Omega}.$$

Proof. Let $u \in E$ be a solution of problem (1.2). This leads to

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx + \int_\Omega V(x)uv\,dx\\
&\quad + \int_{\partial\Omega}\beta(x)|u|^{p-2}uv d\sigma - \int_\Omega a(x)|u|^{q-1}uv\,dx\\
&\le \int_\Omega \|f\|_\infty|v|\,dx \quad \text{for all } v \in E.
\end{aligned} \tag{3.6}$$

Choosing $v = (u - \|f\|_\infty)^+ \in E$ in (3.6) gives

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\left(u-\|f\|_\infty\right)^+\,dx + \int_\Omega V(x)u\left(u-\|f\|_\infty\right)^+\,dx\\
&\quad + \int_{\partial\Omega}\beta(x)|u|^{p-2}u\left(u-\|f\|_\infty\right)^+\,d\sigma - \int_\Omega a(x)|u|^{q-1}u\left(u-\|f\|_\infty\right)^+\,dx\\
&\le \int_\Omega \|f\|_\infty\left(u-\|f\|_\infty\right)^+\,dx
\end{aligned}$$

Hence, by H(a) and H(V), we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla (u - \|f\|_\infty)^+ \, dx + \int_\Omega (u - \|f\|_\infty)(u - \|f\|_\infty)^+ \, dx + \int_{\partial\Omega} \beta(x)|u|^{p-2}u\,(u - \|f\|_\infty)^+ \, d\sigma \le 0.$$

From this we conclude that

$$\|\nabla(u - \|f\|_\infty)^+\|_p^p + \|(u - \|f\|_\infty)^+\|_2^2 + \int_{\partial\Omega} \beta(x)|u|^{p-2}u\,(u - \|f\|_\infty)^+ \, d\sigma \le 0,$$

which implies that

$$u(x) \le \|f\|_\infty \quad \text{for all } x \in \overline{\Omega}.$$

Similarly, choosing $v = (-\|f\|_\infty - u)^+ \in E$, we obtain

$$-\|f\|_\infty \le u.$$

The proof is now complete. □

4. Proof of the main result

In this section we are going to prove Theorem 1.1.

Definition 4.1. *We call $u \in E$ a weak solution of problem (1.1) if*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_\Omega V(x)uv \, dx + \int_{\partial\Omega} \beta(x)|u|^{p-2}uv \, d\sigma = \int_\Omega a(x)|u|^{q-1}uv \, dx + \lambda \int_\Omega g(x,u)v \, dx + \int_\Omega f(x)v \, dx$$

is satisfied for all $v \in E$.

The corresponding energy functional $J_\lambda \colon E \to \mathbb{R}$ of (1.1) is given by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{2}\int_\Omega V(x)|u|^2 \, dx + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u|^p \, d\sigma - \frac{1}{q+1}\int_\Omega a(x)|u|^{q+1} \, dx - \lambda \int_\Omega G(x,u) \, dx - \int_\Omega f(x)u \, dx,$$

where $G(x,s) = \int_0^s g(x,t)\,dt$.

Now, we choose a function $h \in \mathcal{D}(\Omega, \mathbb{R})$ such that $0 \le h \le 1$ in Ω, $h(x) = 1$ for $|x| \le 2\|f\|_\infty$ and $h(x) = 0$ for $|x| \ge 4\|f\|_\infty$, where $\mathcal{D}(\Omega, \mathbb{R})$ is the space of all smooth functions with compact support. Then the function

$$\overline{G}(x,u) := h(x)G(x,u(x)) = h(x)\int_0^{u(x)} g(x,s)\,ds$$

is of class C^1 in $\Omega \times \mathbb{R}$. Hence, by H(g), we see that $\overline{G}(x,u)$ and $\overline{G}_u(x,u)$ are bounded on $\Omega \times \mathbb{R}$.

Next, we define $\hat{J}_\lambda \colon E \to \mathbb{R}$ by

$$\hat{J}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{2}\int_\Omega V(x)u^2 \, dx + \frac{1}{p}\int_{\partial\Omega} \beta(x)u^p \, d\sigma - \frac{1}{q+1}\int_\Omega a(x)|u|^{q+1} dx - \lambda \int_\Omega h(u(x))G(x,u(x)) dx - \int_\Omega f(x)u dx.$$

It is easy to see that a critical point of $\hat{J}_\lambda$ is a solution of the problem

$$\begin{aligned} -\Delta_p u + V(x)u &= a(x)|u|^{q-1}u + H(x,u) + f(x) \qquad &&\text{in } \Omega,\\ |\nabla u|^{p-2}\nabla u \cdot \nu + \beta(x)|u|^{p-2}u &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

with

$$H(x,u) = \lambda h(u) G_u(x,u) + \lambda h'(u) G(x,u).$$

Consider the following minimization problem

$$c_\lambda = \inf_{u \in \overline{B}_0(\rho)} \hat{J}_\lambda(u),$$

where ρ is as defined in the proof of Lemma 3.3.

Our idea is the following: First, we find an Ekeland solution v_λ of $\hat{J}_\lambda$. Then, we prove that $\|v_\lambda\|_\infty \leq 2M$ for $|\lambda|$ small enough. Then $h'(v_\lambda) = 0$, $h(v_\lambda) = 1$ and therefore v_λ becomes a solution of (1.1).

Lemma 4.2. *Under assumptions of Theorem 1.1, for each $\lambda \in \mathbb{R}$, $\hat{J}_\lambda$ satisfies the* PS-*condition.*

Proof. Applying the hypotheses H(a), H(β), H(f), H(g) and H(V) and the fact that $\overline{G}(x,u)$ and $\overline{G}_u(x,u)$ are bounded on $\Omega \times \mathbb{R}$, the proof is similar to that one of Lemma 3.2. □

Lemma 4.3. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then there exists $\lambda_0 > 0$ such that $\hat{J}_\lambda$ has the Ekeland geometry property when $|\lambda| \leq |\lambda_0|$.*

Proof. Due to the boundedness of $\overline{G}(x,u)$, we get

$$J(u) - C\lambda \leq \hat{J}_\lambda(u) \leq J(u) + C\lambda \quad \text{for all } u \in E, \tag{4.1}$$

where $C > 0$ is independent of λ and u. Let $\overline{B}_0(\rho)$ be as in the proof of Lemma 3.3. Applying (4.1) for $|\lambda|$ small enough yields

$$-\infty < \inf_{u \in \overline{B}_0(\rho)} \hat{J}_\lambda(u) < 0$$

and

$$0 < \inf_{u \in \partial B_0(\rho)} J(u) - C\lambda < \inf_{u \in \partial B_0(\rho)} \hat{J}_\lambda(u).$$

This completes the proof. □

Lemma 4.4. *Under conditions H(a), H(β), H(f), H(g) and H(V), let $\lambda_n \in \mathbb{R}$ be a sequence converging to zero and let u_n be an Ekeland solution of $\hat{J}_{\lambda_n}$. Then, up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges to an Ekeland solution $v \in E$ of J.*

Proof. Using again (4.1), we have

$$J(u) - C\lambda_n \leq \hat{J}_{\lambda_n}(u) \leq J(u) + C\lambda_n \quad \text{for all } u \in E,$$

and so

$$\inf_{u \in \overline{B}_0(\rho)} J(u) - C\lambda_n \leq \inf_{u \in \overline{B}_0(\rho)} \hat{J}_{\lambda_n}(u) \leq \inf_{u \in \overline{B}_0(\rho)} J(u) + C\lambda_n \quad \text{for all } u \in E.$$

Therefore

$$c_{\lambda_n} \to c \quad \text{as } n \to +\infty. \tag{4.2}$$

Invoking condition H(g) and using the fact that $\overline{G}(x,u)$ and $(\overline{G})_u(x,u)$ are bounded on $\Omega \times \mathbb{R}$ as well as $\lambda_n \to 0$, we deduce that $(u_n)_{n\in\mathbb{N}}$ is a (PS)-sequence of J. So, by Lemma 3.2, $u_n \to v$ in E. This fact along with (4.2) yield

$$J(v) = c \quad \text{and} \quad J'(v) = 0.$$

This completes the proof. □

Lemma 4.5. *Assume that H(a), H(β), H(f), H(g) and H(V) hold. Let $\lambda_n \in \mathbb{R}$ be a sequence converging to zero and let u_n be an Ekeland solution of $\hat{J}_{\lambda_n}$. Then, up to a subsequence, u_n converges to u in $L^\infty(\Omega)$, where u is an Ekeland solution of J.*

Proof. Since $N < p$ we know that E is continuously embedded into $L^\infty(\Omega)$. It follows, because of Lemma 4.4, up to a subsequence, that $u_n \to u$ in $L^\infty(\Omega)$. This shows the assertion. □

Lemma 4.6. *Under the same assumptions of Theorem 1.1, there is a positive constant λ_0 such that any Ekeland solution $v \in E$ of $\hat{J}_\lambda$ with $|\lambda| \leq \lambda_0$ satisfies*

$$\|v\|_\infty \leq 2M.$$

Proof. We argue by contradiction and suppose there exist $\lambda_n \in \mathbb{R}$, $u_n \in E$ such that $\lambda_n \to 0$, u_n is an Ekeland solution of $\hat{J}_{\lambda_n}$ and $\|u_n\|_\infty > 2M$. By Lemma 4.5, $(u_n)_{n\in\mathbb{N}}$ converges to an Ekeland solution $w \in L^\infty(\Omega)$ of J. Using Lemma 3.4, it follows that $\|w\|_\infty < M$. Then, by Lemma 4.5, $\|u_n\|_\infty < 2M$ for n large enough which is a contradiction. □

Proof of Theorem 1.1. We choose $\lambda_0 > 0$ which satisfies Lemmas 4.3 and 4.6. Then, by Lemmas 4.2, 4.3 and Theorem 2.2, there exists $u_\lambda \in E$ such that u_λ is a critical point of $\hat{J}_\lambda$ and $c_\lambda = \hat{J}_\lambda(u_\lambda)$ with $|\lambda| < |\lambda_0|$. From Lemma 4.6, we have $\|u_\lambda\|_\infty < 2M$. Thus, $h'(u_\lambda) = 0$ and $h(u_\lambda) = 1$. Therefore, u_λ is a nontrivial weak solution of problem (1.1). □

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