ON THE FUČÍK SPECTRUM OF THE p-LAPLACIAN WITH NO-FLUX BOUNDARY CONDITION

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Abstract. In this paper, we study the quasilinear elliptic problem

$$\begin{aligned} -\Delta_p u &= a\left(u^+\right)^{p-1} - b\left(u^-\right)^{p-1} && \text{in } \Omega,\\ u &= \text{constant} && \text{on } \partial\Omega,\\ 0 &= \int_{\partial\Omega} |\nabla u|^{p-2}\,\nabla u\cdot\nu\,\mathrm{d}\sigma, \end{aligned}$$

where the operator is the p-Laplacian and the boundary condition is of type no-flux. In particular, we consider the Fučík spectrum of the p-Laplacian with no-flux boundary condition which is defined as the set Π_p of all pairs $(a,b)\in\mathbb{R}^2$ such that the problem above has a nontrivial solution. It turns out that this spectrum has a first nontrivial curve $\mathcal{C}$ being Lipschitz continuous, decreasing and with a certain asymptotic behavior. Since (λ_2,λ_2) lies on this curve $\mathcal{C}$, with λ_2 being the second eigenvalue of the corresponding no-flux eigenvalue problem for the p-Laplacian, we get a variational characterization of λ_2. This paper extends corresponding works for Dirichlet, Neumann, Steklov and Robin problems.

1. Introduction

In this paper, we are interested in the so-called Fučík spectrum of the p-Laplacian with no-flux boundary condition which is defined as the set Π_p of all pairs $(a,b)\in\mathbb{R}^2$ such that the problem

$$\begin{aligned} -\Delta_p u &= a\left(u^+\right)^{p-1} - b\left(u^-\right)^{p-1} && \text{in } \Omega,\\ u &= \text{constant} && \text{on } \partial\Omega,\\ 0 &= \int_{\partial\Omega} |\nabla u|^{p-2}\,\nabla u\cdot\nu\,\mathrm{d}\sigma \end{aligned} \tag{1.1}$$

has a nontrivial weak solution, where $\Omega\subset\mathbb{R}^N$, $N\geq 2$, is a bounded domain with smooth boundary $\partial\Omega$, $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$ is the p-Laplace differential operator with $1<p<+\infty$, $\nu(x)$ denotes the outer unit normal of Ω at the point $x\in\partial\Omega$ and $u^{\pm}=\max\{\pm u,0\}$ are the positive and negative parts of u, respectively. The boundary condition is of type no-flux and such problems have their origin in plasma physics. Temam [25] studied the problem of the equilibrium of a plasma in a cavity

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which occurred for the first time in Mercier [20] and has the form

$$\begin{aligned}
&\mathfrak{L}u = -\lambda bu && \text{in } \Omega_\rho,\\
&\mathfrak{L}u = 0 && \text{in } \Omega_\nu = \Omega - \overline{\Omega}_\rho \text{ (the vacuum)},\\
&\ u = 0 && \text{on } \Gamma_\rho = \partial\Omega_\rho,\\
&\frac{\mathrm{d}u}{\mathrm{d}\nu} \text{ is continuous} && \text{on } \Gamma_\rho,\\
&\ u = \text{constant} = \gamma && \text{on } \Gamma \ (\gamma \text{ unknown}),\\
&\ I = \int_{\Gamma_\rho} \frac{1}{x_1}\frac{\mathrm{d}u}{\mathrm{d}\nu}\,\mathrm{d}\Gamma,\\
&\ u \text{ does not vanish} && \text{in } \Omega_\rho,
\end{aligned} \tag{1.2}$$

where $I > 0$ is given, u, λ and Ω_ρ are the unknowns, while λ plays the role of an eigenvalue of the self-adjoint operator $\mathfrak{L}$. The solution of (1.2) determines the shape at equilibrium of a confined plasma. A simplified model of (1.2) has been presented by the same author in [26] given by

$$\begin{aligned}
-\Delta u &= -\lambda u^- && \text{in } \Omega,\\
u &= \text{constant} = \gamma && \text{on } \partial\Omega,\\
I &= \int_{\partial\Omega} \frac{\mathrm{d}u}{\mathrm{d}\nu}\,\mathrm{d}\sigma.
\end{aligned} \tag{1.3}$$

In (1.3) the region $u < 0$ is the region filled by the plasma and the region $u > 0$ corresponds to the vacuum. These regions can be found when we solve problem (1.3). The region $u = 0$ corresponds to the free boundary which separates the plasma and the vacuum. For other models of type (1.3) we refer to the works of Berestycki-Brézis [3], Gourgeon-Mossino [15], Kinderlehrer-Spruck[16], Puel [23], Schaeffer [24], Zou [28, 29] and the references therein. A nice overview about no-flux problems also in the case of variable exponent problems can be found in the book chapter of Boureanu [4].

In (1.1) we assume that $I = 0$ and so it corresponds to nonresonant surfaces called no-flux surfaces on which the wave number of the perturbation parallel to the equilibrium magnetic field is zero, see Afrouzi-Mirzapour- Rădulescu [1]. Note that when $N = 1$ and $\Omega = (a, b)$, problem (1.1) becomes the periodic boundary value problem

$$\begin{aligned}
-\left(|u'|^{p-2}u'\right)' &= \lambda|u|^{p-2}u \quad \text{in } (a,b),\\
u(a) &= u(b),\\
u'(a) &= u'(b).
\end{aligned}$$

In this paper, we are interested in the nontrivial parts of Π_p and we show that there exists a first nontrivial curve $\mathcal{C} \subset \Pi_p$ which turns out to be Lipschitz continuous, decreasing and with a certain asymptotic behavior. With this work we close the gap in the literature where the Fučík spectrum of the p-Laplacian has been already studied for Dirichlet, Neumann, Steklov and Robin boundary condition, respectively.

The idea of considering the set Σ of all pairs $(a, b) \in \mathbb{R}^2$ such that

$$Tu = au^+ - bu^-$$

has a nontrivial solution with T being self-adjoint, goes back to Fučík [12] (see also Dancer [9]) who recognized that the set Σ plays an important role in the study of semilinear equations of type

$$Tu = f(x,u),$$

where $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with jumping nonlinearities satisfying

$$\frac{f(x,s)}{s} \to a \quad \text{as } s \to +\infty, \qquad \frac{f(x,s)}{s} \to b \quad \text{as } s \to -\infty.$$

Indeed, a systematic study of this spectrum for the one-dimensional Laplacian with periodic boundary condition has been done by Fučík [13] who proved that this spectrum is composed of two families of curves in $\mathbb{R}^2$ emanating from the points (λ_k, λ_k) determined by the eigenvalues λ_k. After this, several works on this spectrum have been published for the negative Laplacian with Dirichlet boundary condition on bounded domains. In particular, Dancer [9] showed that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in Σ_2, where Σ_2 is the Fučík spectrum of $-\Delta$ with Dirichlet condition and $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$. A starting work on the Fučík spectrum of the p-Laplacian with Dirichlet condition has been done by Cuesta-de Figueiredo-Gossez [8] who proved the existence of a first nontrivial curve in this spectrum, see also a similar result for $-\Delta$ by de Figueiredo-Gossez [10]. These results have been transferred to Neumann, Steklov and Robin boundary conditions by Arias-Campos-Gossez [2], Martínez-Rossi [19] and Motreanu-Winkert [21], respectively. We refer to the book chapter of Motreanu-Winkert [22] concerning the differences in these works.

In our work, we are going to transfer the techniques of [2], [8], [19] and [21] to our problem (1.1) with no-flux boundary condition. One difference is that in our problem the first eigenvalue of the corresponding eigenvalue problem is zero. Indeed, if $a = b = \lambda$, problem (1.1) becomes the following no-flux eigenvalue problem for the p-Laplacian

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u && \text{in } \Omega,\\ u &= \text{constant} && \text{on } \partial\Omega,\\ 0 &= \int_{\partial\Omega} |\nabla u|^{p-2}\, \nabla u \cdot \nu \,\mathrm{d}\sigma, \end{aligned} \tag{1.4}$$

which has been treated by Lê [17]. Since the first eigenvalue λ_1 in (1.4) is zero, all nonzero constants are corresponding eigenfunctions. Thus, λ_1 is simple. Furthermore, from Lê [17] we know that λ_1 is isolated, the spectrum of (1.4) is closed and each eigenfunction corresponding to an eigenvalue $\lambda > 0$ changes sign in Ω. The first eigenfunction can be given as L^p-normalized constant by $\varphi_1 = \frac{1}{|\Omega|^{\frac{1}{p}}}$. As a consequence of our results, we obtain a variational characterization of the second eigenvalue λ_2 of (1.4) by

$$\lambda_2 = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \left[\int_\Omega |\nabla u|^p \,\mathrm{d}x \right],$$

where

$$\begin{aligned} \Gamma &= \left\{ \gamma \in C\left([-1,1], S\right) \,:\, \gamma(-1) = -\varphi_1,\, \gamma(1) = \varphi_1 \right\},\\ S &= \left\{ u \in V \,:\, \|u\|_p = 1 \right\}, \end{aligned}$$

$$V = \left\{ u \in W^{1,p}(\Omega) \, : \, u \mid_{\partial\Omega} = \text{constant} \right\}.$$

It turns out that the point (λ_2, λ_2) lies on the first nontrivial curve $\mathcal{C}$ of Π_p, see Figure 1.

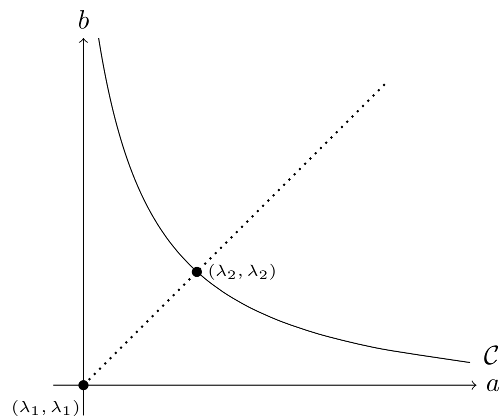

FIGURE 1. The curve $\mathcal{C}$

Finally, we mention some existence results for elliptic problems with no-flux boundary condition. As we already noted, there are only few works in this direction. We refer to Le-Schmitt [18] for a sub-supersolution approach involving general nonhomogeneous operators, Zhao-Zhao-Xie [27] for a mountain-pass solution, Fan-Deng [11] for an application on a variational principle due to Ricceri in variable exponent Sobolev spaces and Boureanu-Udrea [5, 6] for isotropic and anisotropic variable exponent problems. Other references can be found in the book chapter of Boureanu [4].

The paper is organized as follows. In Section 2 we present some results on the function spaces, the p-Laplacian and state the weak formulation of problem (1.1). Moreover, we recall the mountain-pass theorem for manifolds. In Section 3 we describe the Fučík spectrum Π_p via critical points of the corresponding functional and show the existence of a curve of elements of Π_p. In Section 4 we prove that this curve is indeed the first nontrivial curve in Π_p. As a consequence we derive a variational characterization of the second eigenvalue λ_2 of (1.4), see Corollary 4.4. Finally, in Section 5, we prove that this first nontrivial curve is Lipschitz continuous, decreasing and converging in the cases $p \leq N$ and $p > N$ separately, see Proposition 5.1 and Theorems 5.2 and 5.4.

2. Preliminaries

In this section we recall some facts about the function space, the operator and tools from critical point theory. To this end, let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$ and let $1 \leq p < \infty$. We denote by $L^p(\Omega) := L^p(\Omega; \mathbb{R})$ and $L^p(\Omega; \mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\| \cdot \|_p$ while $W^{1,p}(\Omega)$ stands for the Sobolev space endowed with the norm $\| \cdot \|_{1,p}$, namely,

$$\|u\|_{1,p} := \left(\int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega |u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

Let

$$V=\left\{u\in W^{1,p}(\Omega)\,:\,u\left|_{\partial\Omega}=\text{constant}\right.\right\}.$$

Then V is a closed subspace of $W^{1,p}(\Omega)$ and so a reflexive Banach space with norm $\|\cdot\|_{1,p}$, see Le-Schmitt [18] or Zhao-Zhao-Xie [27, Lemma 2.1]. Note that for any $v\in V$ we have that $v^+,v^-\in V$.

A function $u\in V$ is said to be a weak solution of (1.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x=\int_\Omega a\left(\left(u^+\right)^{p-1}-b\left(u^-\right)^{p-1}\right)v\,\mathrm{d}x \tag{2.1}$$

is satisfied for all $v\in V$.

For $1<p<\infty$, we consider the nonlinear operator $A\colon V\to V^*$ defined by

$$\langle A(u),v\rangle:=\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x \tag{2.2}$$

for $u,v\in V$ with $\langle\cdot,\cdot\rangle$ being the duality pairing between V and its dual space V^*. The properties of the operator $A\colon V\to V^*$ can be summarized as follows, see, for example, Carl-Le-Motreanu [7, Lemma 2.111].

Proposition 2.1. *The operator A defined by (2.2) is bounded, continuous, monotone (hence maximal monotone) and of type (S_+), that is,*

$$u_n\rightharpoonup u\quad\text{in } V\quad\text{and}\quad\limsup_{n\to\infty}\langle Au_n,u_n-u\rangle\le 0,$$

imply $u_n\to u$ in V.

Let X be a reflexive Banach space, let X^* be its dual space and let $\varphi\in C^1(X,\mathbb{R})$. We say that $\{u_n\}_{n\in\mathbb{N}}\subset X$ is a Palais-Smale sequence ((PS)-sequence for short) for φ if $\{\varphi(u_n)\}_{n\in\mathbb{N}}\subseteq\mathbb{R}$ is bounded and

$$\varphi'(u_n)\to 0\quad\text{in } X^*\quad\text{as } n\to\infty.$$

We say that φ satisfies the Palais-Smale condition ((PS)-condition for short) if any (PS)-sequence $\{u_n\}_{n\in\mathbb{N}}$ of φ admits a convergent subsequence in X.

The following version of the mountain-pass theorem in the sense of manifolds will be used in the sequel. We refer to Ghoussoub [14, Theorem 3.2].

Theorem 2.2. *Let X be a Banach space and let $g,\ f\in C^1(X,\mathbb{R})$. Further, suppose that 0 is a regular value of g and let $M=\{u\in X\,:\,g(u)=0\}$, $u_0,\ u_1\in M$ and $\varepsilon>0$ such that $\|u_1-u_0\|_X>\varepsilon$ and*

$$\inf\{f(u)\,:\,u\in M\text{ and }\|u-u_0\|_X=\varepsilon\}>\max\{f(u_0),f(u_1)\}.$$

Assume that f satisfies the (PS)-condition on M and that

$$\Gamma=\{\gamma\in C([-1,1],M)\,:\,\gamma(-1)=u_0\text{ and }\gamma(1)=u_1\}$$

is nonempty. Then

$$c=\inf_{\gamma\in\Gamma}\max_{u\in\gamma[-1,1]}f(u),$$

is a critical value of $f_{|_M}$.

3. The Fučík spectrum through critical points

In this section, we are going to determine the elements of the Fučík spectrum Π_p through critical points.

Let $s \in \mathbb{R}$ be a real nonnegative parameter and consider the functional $J_s \colon V \to \mathbb{R}$ defined by

$$J_s(u) = \int_\Omega |\nabla u|^p \, \mathrm{d}x - s \int_\Omega \left(u^+\right)^p \, \mathrm{d}x. \tag{3.1}$$

It is clear that $J_s \in C^1(V, \mathbb{R})$. Recall that

$$S = \left\{ u \in V \,:\, I(u) = \int_\Omega |u|^p \, \mathrm{d}x = 1 \right\}.$$

We know that S is a smooth submanifold of V and so, $\tilde{J}_s = J_{s|_S}$ is a C^1-function in the sense of manifolds.

Applying the Lagrange multiplier rule, we note that $u \in S$ is a critical point of $\tilde{J}_s$ (in the sense of manifolds) if and only if there exists $t \in \mathbb{R}$ such that $J_s'(u) = tI'(u)$, that is

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x - s \int_\Omega \left(u^+\right)^{p-1} v \, \mathrm{d}x = t \int_\Omega |u|^{p-2} u v \, \mathrm{d}x \tag{3.2}$$

for all $v \in V$.

First, we investigate the relationship between the critical points of $\tilde{J}_s$ and the Fučík spectrum Π_p.

Lemma 3.1. *Let s be a nonnegative real parameter. The point $(s+t, t) \in \mathbb{R}^2$ belongs to the spectrum Π_p if and only if there exists a critical point $u \in S$ of $\tilde{J}_s$ such that $t = J_s(u)$.*

Proof. From the definition of a weak solution of (1.1), see (2.1), we observe that $(t+s, t) \in \Pi_p$ if and only if there exists $u \in S$ that solves the following no-flux problem

$$\begin{aligned} -\Delta_p u &= (t+s)\left(u^+\right)^{p-1} - t\left(u^-\right)^{p-1} && \text{in } \Omega,\\ u &= \text{constant} && \text{on } \partial\Omega,\\ 0 &= \int_{\partial\Omega} |\nabla u|^{p-2} \nabla u \cdot \nu \, \mathrm{d}\sigma. \end{aligned}$$

However, the corresponding weak solution of the problem above is given in (3.2). Taking $v = u$ in (3.2) we have that $t = J_s(u)$ and the proof is complete. $\square$

Lemma 3.1 allows us to find points in Π_p by the critical points of $\tilde{J}_s$. Next we are going to look for minimizers of $\tilde{J}_s$.

Proposition 3.2. *There hold:*

(i) *the first eigenfunction $\varphi_1 = \frac{1}{|\Omega|^{\frac{1}{p}}}$ is a global minimizer of $\tilde{J}_s$;*

(ii) *the point $(0, -s) \in \mathbb{R}^2$ belongs to Π_p.*

Proof. (i) Since $s \geq 0$ we have for $u \in S$

$$\tilde{J}_s(u) = \int_\Omega |\nabla u|^p \, \mathrm{d}x - s \int_\Omega \left(u^+\right)^p \, \mathrm{d}x \geq -s \int_\Omega \left(u^+\right)^p \, \mathrm{d}x \geq -s = J_s(\varphi_1)$$

for all $u \in S$. Hence, the first eigenfunction $\varphi_1 = \frac{1}{|\Omega|^{\frac{1}{p}}} \in V$ is a global minimizer of $\tilde{J}_s$.

(ii) From (i) and Lemma 3.1 we get the assertion. □

Now we obtain a second critical point of $\tilde{J}_s$ as local minimizer.

Proposition 3.3. *There hold:*

(i) *the negative eigenfunction $-\varphi_1 = -\frac{1}{|\Omega|^{\frac{1}{p}}}$ is a strict local minimizer of $\tilde{J}_s$;*

(ii) *the point $(s, 0) \in \mathbb{R}^2$ belongs to Π_p.*

Proof. (i) Suppose by contradiction that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset S$ with $u_n \neq -\varphi_1$, $u_n \to -\varphi_1$ in V and

$$\tilde{J}_s(u_n) \leq 0 = \lambda_1 = \tilde{J}_s(-\varphi_1). \tag{3.3}$$

We claim that u_n changes sign for n sufficiently large. Observe that, since $u_n \to -\varphi_1$, u_n must be < 0 somewhere. Suppose that $u_n \leq 0$ for a. a. $x \in \Omega$. Then we obtain

$$\tilde{J}_s(u_n) = \int_\Omega |\nabla u_n|^p \, \mathrm{d}x > 0 = \lambda_1,$$

since $u_n \neq -\varphi_1$ and $u_n \neq \varphi_1$ contradicting $\tilde{J}_s(u_n) \leq 0 = \lambda_1$. Therefore, u_n changes sign. We set

$$w_n = \frac{u_n^+}{\left\|u_n^+\right\|_p} \quad \text{and} \quad r_n = \|\nabla w_n\|_p \,. \tag{3.4}$$

Claim: $r_n \to +\infty$ as $n \to +\infty$

Arguing by contradiction, suppose $\{r_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded. Then from (3.4) we know that $\{w_n\}_{n\in\mathbb{N}}$ is bounded in V. Hence we find a subsequence (still denoted by $\{w_n\}_{n\in\mathbb{N}}$) such that $w_n \to w$ in $L^p(\Omega)$ for some $w \in X$. Since $\|w_n\|_p = 1$ and $w_n \geq 0$ for a. a. $x \in \Omega$, we see that $\|w\|_p = 1$ and $w \geq 0$. Therefore, the Lebesgue measure of the set $\{x \in \Omega : u_n(x) > 0\}$ does not approach 0 when $n \to +\infty$. However, this contradicts the assumption that $u_n \to -\varphi_1$ in $L^p(\Omega)$ which means that $\{x \in \Omega : u_n(x) > 0\} \to 0$. This proves the Claim.

From (3.3) and (3.4) we get that

$$\begin{aligned} 0 \geq \tilde{J}_s(u_n) &= \int_\Omega \left|\nabla u_n^+\right|^p \, \mathrm{d}x + \int_\Omega \left|\nabla u_n^-\right|^p \, \mathrm{d}x - s \int_\Omega \left(u_n^+\right)^p \, \mathrm{d}x \\ &\geq (r_n - s) \int_\Omega \left(u_n^+\right)^p \, \mathrm{d}x. \end{aligned}$$

Hence, $0 \geq r_n - s$ which contradicts the Claim. This completes the proof of (i).

(ii) This follows from Lemma 3.1 since $J_s(-\varphi_1) = 0$. □

Using the two local minima from Proposition 3.2 and 3.3 we are looking for a third critical point of $\tilde{J}_s$ by using the mountain-pass theorem in its version on C^1-manifolds.

First, we define a norm of the derivative of the restriction $\tilde{J}_s$ of J_s to S at the point $u \in S$ by

$$\left\|\tilde{J}_s'(u)\right\|_* = \min\left\{ \|J_s'(u) - tT'(u)\|_* \, : \, t \in \mathbb{R}\right\}$$

with $T(\cdot) = \|\cdot\|_p^p$ and $\|\cdot\|_*$ being the norm in the dual space V^* of V.

Lemma 3.4. *The functional $\tilde{J}_s\colon S \to \mathbb{R}$ satisfies the* (PS)*-condition on S in the sense of manifolds.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq S$ be a (PS)-sequence, that is, $\{\tilde{J}_s(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\|\tilde{J}_s'(u_n)\|_* \to 0$ as $n \to \infty$. Then we find a sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that

$$\begin{aligned}&\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v \,\mathrm{d}x - s\int_\Omega \left(u_n^+\right)^{p-1} v\,\mathrm{d}x - t_n\int_\Omega |u_n|^{p-2}u_n v\,\mathrm{d}x\right|\\ &\leq \varepsilon_n \|v\|_{1,p},\end{aligned} \tag{3.5}$$

for all $v \in V$ with $\varepsilon_n \to 0^+$.

Since $\{u_n\}_{n\in\mathbb{N}} \subseteq S$ we have $J_s(u_n) \geq \|\nabla u_n\|_p^p - s$ and because $\{J_s(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in V. So we may assume, for a subsequence if necessary, that

$$u_n \rightharpoonup u \quad \text{in } V \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega).$$

We choose $v = u_n$ in (3.5) and note again that $\{u_n\}_{n\in\mathbb{N}} \subseteq S$. Hence, the sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded. Taking $v = u_n - u$ in (3.5) we obtain that

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n-u)\,\mathrm{d}x\\ &= s\int_\Omega \left(u_n^+\right)^{p-1}(u_n-u)\,\mathrm{d}x + t_n\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,\mathrm{d}x + O(\varepsilon_n),\end{aligned} \tag{3.6}$$

where the right-hand side of (3.6) goes to zero as $n \to \infty$. Hence, we have

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n-u)\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

From the (S_+)-property of $-\Delta_p$ (see Proposition 2.1), we conclude that $u_n \to u$ in V. Thus, $\tilde{J}_s$ fulfills the (PS)-condition. □

Now we prove the existence of a third critical point of $\tilde{J}_s$ which is different from φ_1 and $-\varphi_1$.

Proposition 3.5.

(i) *Let*

$$\Gamma = \left\{\gamma \in C\left([-1,1],S\right) \,:\, \gamma(-1) = -\varphi_1,\, \gamma(1) = \varphi_1\right\}.$$

For each $s \geq 0$ we have that

$$c(s) =: \inf_{\gamma\in\Gamma}\max_{u\in\gamma[-1,1]} J_s(u) \tag{3.7}$$

is a critical value of $\tilde{J}_s$ such that $c(s) > \max\{\tilde{J}_s(-\varphi_1), \tilde{J}_s(\varphi_1)\} = 0$.

(ii) *The point $(s + c(s), c(s))$ belongs to Π_p.*

Proof. (i) First note that $-\varphi_1$ is a strict local minimizer of $\tilde{J}_s$ with $\tilde{J}_s(-\varphi_1) = 0$ by Proposition 3.3 and φ_1 is a global minimizer of $\tilde{J}_s$ with $\tilde{J}_s(\varphi_1) = -s$ by Proposition 3.2. Similar to the proof of Lemma 2.9 in Cuesta-de Figueiredo-Gossez [8] we can show by using Ekeland's variational principle that

$$\inf\left\{\tilde{J}_s(u) \,:\, u \in S \text{ and } \|u-(-\varphi_1)\|_{1,p} = \varepsilon\right\} > \max\{\tilde{J}_s(-\varphi_1), \tilde{J}_s(\varphi_1)\} = \lambda_1,$$

with small $\varepsilon > 0$. We choose $\varepsilon > 0$ small enough such that

$$2\|\varphi_1\|_{1,p} = \|\varphi_1 - (-\varphi_1)\|_{1,p} > \varepsilon.$$

Moreover, from Lemma 3.4 we know that $\tilde{J}_s\colon S \to \mathbb{R}$ satisfies the (PS)-condition on the manifold S. Therefore, we can apply the mountain-pass theorem, stated as Theorem 2.2, which guarantees that $c(s)$ introduced in (3.7) is a critical value of $\tilde{J}_s$ with $c(s) > 0$. Hence, we have a third critical point different from $-\varphi_1$ and φ_1.

(ii) Using the fact that $c(s)$ given in (3.7) is a critical value of $\tilde{J}_s$ in combination with Lemma 3.1 shows that $(s + c(s), c(s)) \in \Pi_p$. $\square$

4. The first nontrivial curve

In Proposition 3.5 (ii) we have shown that the point $(s + c(s), c(s))$ belongs to Π_p for $s \geq 0$. Since Π_p is symmetric with respect to the diagonal, we can complete it with its symmetric part and obtain the following curve in Π_p

$$\mathcal{C} = \big\{(s + c(s), c(s)), (c(s), s + c(s)) \,:\, s \geq 0\big\}. \tag{4.1}$$

In this section, we are going to prove that the curve $\mathcal{C}$ is the first nontrivial curve in Π_p. We start by showing that the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ are isolated in Π_p.

Proposition 4.1. *There is no sequence $\{a_n, b_n\}_{n\in\mathbb{N}} \in \Pi_p$ with $a_n > 0$ and $b_n > 0$ such that $\{a_n, b_n\}_{n\in\mathbb{N}} \to \{a, b\}$ with $a = 0$ or $b = 0$.*

Proof. We argue by contradiction and suppose there exist sequences $\{a_n, b_n\}_{n\in\mathbb{N}} \subseteq \Pi_p$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq V$ with $a_n \to 0$, $b_n \to b$, $a_n > 0$, $b_n > 0$, $\|u_n\|_p = 1$ and

$$\begin{aligned}
-\Delta_p u_n &= a_n \left(u_n^+\right)^{p-1} - b_n \left(u_n^-\right)^{p-1} && \text{in } \Omega,\\
u_n &= \text{constant} && \text{on } \partial\Omega,\\
0 &= \int_{\partial\Omega} |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nu \,\mathrm{d}\sigma.
\end{aligned} \tag{4.2}$$

The weak formulation of (4.2) is given by

$$\int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla v \,\mathrm{d}x = a_n \int_\Omega \left(u_n^+\right)^{p-1} v \,\mathrm{d}x - b_n \int_\Omega \left(u_n^-\right)^{p-1} v \,\mathrm{d}x \tag{4.3}$$

for all $v \in V$. We first test (4.3) with $v = u_n$ and obtain

$$\begin{aligned}
\|\nabla u_n\|_p^p &= a_n \int_\Omega \left(u_n^+\right)^{p-1} u_n \,\mathrm{d}x - b_n \int_\Omega \left(u_n^-\right)^{p-1} u_n \,\mathrm{d}x\\
&= a_n \int_\Omega \left(u_n^+\right)^{p} \,\mathrm{d}x + b_n \int_\Omega \left(u_n^-\right)^{p} \,\mathrm{d}x \leq a_n + b_n.
\end{aligned}$$

Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in V. We may assume, for a subsequence if necessary, that

$$u_n \rightharpoonup u \quad \text{in } V \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega).$$

Testing (4.3) with $v = u_n - u$ gives

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla (u_n - u) \,\mathrm{d}x\\
&= a_n \int_\Omega \left(u_n^+\right)^{p-1} (u_n - u) \,\mathrm{d}x - b_n \int_\Omega \left(u_n^-\right)^{p-1} (u_n - u) \,\mathrm{d}x.
\end{aligned}$$

This implies

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla (u_n - u) \,\mathrm{d}x = 0.$$

From the (S_+)-property of $-\Delta_p$ (see Proposition 2.1), we conclude that $u_n \to u$ in V. Hence, u solves the equation

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \,\mathrm{d}x = -b \int_\Omega \left(u^-\right)^{p-1} v \,\mathrm{d}x, \tag{4.4}$$

for all $v \in V$. If we take $v = u^+$ in (4.4), we see that

$$\int_\Omega \left|\nabla u^+\right|^p \,\mathrm{d}x = 0.$$

This means that either $u^+ = 0$ or $u^+ = \varphi_1$ since $\|u\|_p = 1$.

Let us first suppose that $u^+ = 0$. Then $u \leq 0$ and from (4.3) we know that u is an eigenfunction of the p-Laplacian with no-flux boundary condition, see (1.4). Therefore, $u = -\varphi_1$ since the only eigenfunctions that have constant sign are those related to $\lambda_1 = 0$. We conclude that $\{u_n\}_{n\in\mathbb{N}}$ converges either to φ_1 or to $-\varphi_1$ in $L^p(\Omega)$. This implies that either

$$|\{x \in \Omega : u_n(x) < 0\}| \to 0 \quad \text{or} \quad |\{x \in \Omega : u_n(x) > 0\}| \to 0, \tag{4.5}$$

respectively, with $|\cdot|$ being the Lebesgue measure.

Taking $v = u_n^+$ as test function in (4.3) along with Hölder's inequality and the continuous embedding $V \hookrightarrow L^r(\Omega)$ for any $r \in (p, p^*]$ with embedding constant $C > 0$ we get

$$\begin{aligned}
&\int_\Omega \left|\nabla u_n^+\right|^p \,\mathrm{d}x + \int_\Omega \left(u_n^+\right)^p \,\mathrm{d}x \\
&= a_n \int_\Omega \left(u_n^+\right)^p \,\mathrm{d}x + \int_\Omega \left(u_n^+\right)^p \,\mathrm{d}x \\
&= (a_n + 1) \int_\Omega \left(u_n^+\right)^p \,\mathrm{d}x \\
&\leq (a_n + 1) C^p \left|\{x \in \Omega : u_n(x) > 0\}\right|^{1-\frac{p}{r}} \left\|u_n^+\right\|_{1,p}^p .
\end{aligned}$$

From this we conclude that

$$|\{x \in \Omega : u_n(x) > 0\}|^{1-\frac{p}{r}} \geq (a_n + 1)^{-1} C^{-p} \tag{4.6}$$

Similarly, if we use $v = u_n^-$ in (4.3) we obtain

$$|\{x \in \Omega : u_n(x) < 0\}|^{1-\frac{p}{r}} \geq (b_n + 1)^{-1} C^{-p}. \tag{4.7}$$

Because $\{a_n, b_n\}_{n\in\mathbb{N}} \subseteq \Pi_p$ does not belong to the trivial lines of Π_p, we have that u_n changes sign. Hence, from (4.6) and (4.7) we reach a contradiction to (4.5). This completes the proof. □

Before we state the main result in this section, we need the following lemma.

Lemma 4.2. *For every $r > \inf_S J_s = -s$, each connected component of $\{u \in S : J_s(u) < r\}$ contains a critical point which is a local minimizer of $\tilde{J}_s$.*

Proof. Let C be a connected component of $\{u \in S : J_s(u) < r\}$ and let $d = \inf\{J_s(u) : u \in \overline{C}\}$.

Claim: There exists $u_0 \in \overline{C}$ such that $\tilde{J}_s(u_0) = d$.

Let $\{u_n\}_{n\in\mathbb{N}} \subset C$ be a sequence such that $\tilde{J}_s(u_n) \leq d + \frac{1}{n^2}$. From Ekeland's variational principle applied to $\tilde{J}_s$ on $\overline{C}$ we get a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \overline{C}$ such that

$$\tilde{J}_s(v_n) \leq \tilde{J}_s(u_n), \tag{4.8}$$

$$\|u_n - v_n\|_{1,p} \le \frac{1}{n}, \tag{4.9}$$

$$\tilde{J}_s(v_n) \le \tilde{J}_s(v) + \frac{1}{n}\|v - v_n\|_{1,p}, \tag{4.10}$$

for all $v \in \overline{C}$.

From (4.8) and n sufficiently large we have that

$$\tilde{J}_s(v_n) \le \tilde{J}_s(u_n) \le d + \frac{1}{n^2} < r.$$

Moreover, applying (4.10), we are able to show that $\{v_n\}_{n\in\mathbb{N}}$ is a (PS)-sequence for $\tilde{J}_s$. Then by Lemma 3.4 and (4.9) we conclude, for a subsequence if necessary, that $u_n \to u_0$ in V with $u_0 \in \overline{C}$ and $\tilde{J}_s(u_0) = d$. Finally, note that $u_0 \notin \partial C$ since otherwise the maximality of C as a connected component would be contradicted. Thus, u_0 is a local minimizer of $\tilde{J}_s$. □

The next results show that $\mathcal{C}$ is the first nontrivial curve in Π_p.

Theorem 4.3. *Let $s \ge 0$. Then $(s + c(s), c(s)) \in \mathcal{C}$ is the first nontrivial point of Π_p in the intersection between Π_p and the line $(s, 0) + t(1, 1)$ with $t > 0$.*

Proof. We are going to show the assertion by contradiction. Let $0 < \mu < c(s)$ and suppose that $(s + \mu, \mu) \in \Pi_p$. Taking Proposition 4.1 and the closedness of Π_p into account, we may suppose that μ is the minimum number with the required property. By using Lemma 3.1 it is clear that μ is a critical value of the functional $\tilde{J}_s$ and there is no critical value of $\tilde{J}_s$ in the interval $(0, \mu)$.

Let $u \in S$ be a critical point of $\tilde{J}_s$ at level μ. We have for all $v \in V$

$$\int_\Omega |\nabla u|^{p-2}\,\nabla u\cdot\nabla v\,\mathrm{d}x = (s+\mu)\int_\Omega \left(u^+\right)^{p-1} v\,\mathrm{d}x - \mu\int_\Omega \left(u^-\right)^{p-1} v\,\mathrm{d}x,$$

see Lemma 3.1. Choosing $v = u^+$ gives

$$\int_\Omega \left|\nabla u^+\right|^p\,\mathrm{d}x = (s+\mu)\int_\Omega \left(u^+\right)^p\,\mathrm{d}x. \tag{4.11}$$

Similarly, if we take $v = -u^-$ we obtain

$$\int_\Omega \left|\nabla u^-\right|^p\,\mathrm{d}x = \mu\int_\Omega \left(u^-\right)^p\,\mathrm{d}x. \tag{4.12}$$

Using (4.11) and (4.12) we see that

$$\tilde{J}_s\left(\frac{u^+}{\|u^+\|_p}\right) = \tilde{J}_s\left(\frac{-u^-}{\|u^-\|_p}\right) = \mu,$$

and

$$\tilde{J}_s\left(\frac{u^-}{\|u^-\|_p}\right) = \mu - s. \tag{4.13}$$

Now, we introduce for all $t \in [0, 1]$ the following paths defined by

$$u_1(t) = \frac{(1-t)u + tu^+}{\|(1-t)u + tu^+\|_p},$$

$$u_2(t) = \frac{tu^+ + (1-t)u^-}{\|tu^+ + (1-t)u^-\|_p},$$

$$u_3(t) = \frac{-tu^- + (1-t)u}{\|-tu^- + (1-t)u\|_p}.$$

Note that these paths are well-defined in S. It is easy to see that $u_1(t)$ goes from u to $\frac{u^+}{\|u^+\|_p}$, $u_2(t)$ goes from $\frac{u^+}{\|u^+\|_p}$ to $\frac{u^-}{\|u^-\|_p}$ and $u_3(t)$ goes from u to $\frac{-u^-}{\|u^-\|_p}$.

By means of (4.11) and (4.12) it is easy to see that

$$\begin{aligned}
&\tilde{J}_s(u_1(t)) = \mu = \tilde{J}_s(u_3(t)),\\
&\tilde{J}_s(u_2(t)) = \mu - st^p \frac{\|u^-\|_p^p}{\|tu^+ + (1-t)u^-\|_p^p} \le \mu
\end{aligned}$$

for all $t \in [0,1]$.

From this we know that we can move from u to $\frac{u^-}{\|u^-\|_p}$ via $u_1(t)$ and $u_2(t)$ which lies at level $\mu - s$, so we stay at level $\le \mu$. Let us investigate the levels below $\mu - s$. We introduce

$$\Upsilon = \{v \in S \,:\, \tilde{J}_s(v) < \mu - s\}.$$

We observe that $\varphi_1 \in \Upsilon$ and $-\varphi_1 \in \Upsilon$ if $\mu > s$. Due to the minimality property of μ, we know that φ_1 and $-\varphi_1$ are the only possible critical points of $\tilde{J}_s$ in Υ. Since $\frac{u^-}{\|u^-\|_p}$ does not change sign and vanishes on a set of positive measure, it cannot be a critical point of $\tilde{J}_s$. Hence, we find a path $\beta\colon [-\varepsilon,\varepsilon] \to S$ of class C^1 with $\beta(0) = \frac{u^-}{\|u^-\|_p}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{J}_s(\beta(t))|_{t=0} \neq 0$. Using this path and (4.13) we can move from $\frac{u^-}{\|u^-\|_p}$ to a point v by a path in S such that $\tilde{J}_s(v) < \mu - s$. In particular, we have $v \in \Upsilon$.

Applying Lemma 4.2 we obtain that the connected component of Υ containing v crosses $\{\varphi_1, -\varphi_1\}$. Let us suppose that we can continue from v to φ_1, the case continuing to $-\varphi_1$ can be argued similarly. Therefore, there exists a path $u_4(t)$ in Υ from $\frac{u^-}{\|u^-\|_p}$ to φ_1, whose symmetric path $-u_4(t)$ goes from $-\frac{u^-}{\|u^-\|_p}$ to $-\varphi_1$. As $u_4(t) \in S$, we have that

$$\tilde{J}_s(-u_4(t)) \le \tilde{J}_s(u_4(t)) + s < \mu - s + s = \mu,$$

since for each $\hat{u} \in S$ it holds

$$\left|\tilde{J}_s(\hat{u}) - \tilde{J}_s(-\hat{u})\right| \le s.$$

We already observed that we go from $-\varphi_1$ to $\frac{-u^-}{\|u^-\|_p}$ via $-u_4(t)$ by staying at level lower then μ. Finally from the path $u_3(t)$ we go from u to $\frac{-u^-}{\|u^-\|_p}$ by staying at level μ.

In summary, we have shown that we constructed a path joining u and φ_1 via $u_1(t)$, $u_2(t)$ as well as $u_4(t)$ and we have a path joining u and $-\varphi_1$ via $u_3(t)$ and $-u_4(t)$. Putting these paths together we have a path $\gamma(t)$ on S joining φ_1 and $-\varphi_1$ with $\tilde{J}_s(\gamma(t)) \le \mu$. In particular we have that $\tilde{J}_s$ has a critical value μ with $\lambda_1 < \mu < c(s)$, but there is no critical value in the interval $]\lambda_1, \mu[$ and this contradicts the definition of $c(s)$ in (3.7). $\square$

A direct consequence of Theorem 4.3 is a variational characterization of the second eigenvalue λ_2 of problem (1.4).

Corollary 4.4. *The second eigenvalue λ_2 of (1.4) has the following variational characterization*

$$\lambda_2 = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \left[\int_\Omega |\nabla u|^p \, dx\right].$$

Proof. We apply Theorem 4.3, Proposition 3.5 (i) and (3.1) for $s = 0$ in order to get

$$c(0) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} J_0(u) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \left[\int_\Omega |\nabla u|^p \, dx\right].$$

□

5. Properties of the first curve

In this section, we are going to prove some properties of the curve $\mathcal{C}$ defined in (4.1) and we study its asymptotic behavior.

Proposition 5.1. *The curve $s \mapsto (s + c(s), c(s))$ is Lipschitz continuous with Lipschitz constant $L \leq 1$ and decreasing.*

Proof. Let s_1 and s_2 be such that $s_1 < s_2$. Then we have $\tilde{J}_{s_1}(u) \geq \tilde{J}_{s_2}(u)$ for all $u \in S$ and so $c(s_1) \geq c(s_2)$.

For every $\varepsilon > 0$ we find a path $\gamma \in \Gamma$ such that

$$\max_{u\in\gamma[-1,1]} \tilde{J}_{s_2}(u) \leq c(s_2) + \varepsilon,$$

This implies

$$0 \leq c(s_1) - c(s_2) \leq \max_{u\in\gamma[-1,1]} \tilde{J}_{s_1}(u) - \max_{u\in\gamma[-1,1]} \tilde{J}_{s_2}(u) + \varepsilon.$$

Let $u_0 \in \gamma[-1, 1]$ be such that

$$\max_{u\in\gamma[-1,1]} \tilde{J}_{s_1}(u) = \tilde{J}_{s_1}(u_0),$$

from which we conclude that

$$0 \leq c(s_1) - c(s_2) \leq \tilde{J}_{s_1}(u_0) - \tilde{J}_{s_2}(u_0) + \varepsilon = s_1 - s_2 + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we obtain that the curve $s \mapsto (s + c(s), c(s))$ is Lipschitz continuous with Lipschitz constant $L \leq 1$.

Let us prove that the curve is decreasing. To this end, let $0 < s_1 < s_2$. Theorem 4.3 implies that $s_1+c(s_1) < s_2+c(s_2)$ since $(s_1+c(s_1), c(s_1)), (s_2+c(s_2), c(s_2)) \in \Pi_p$. From the first part of the proof, we already mentioned that $c(s_1) \geq c(s_2)$. This completes the proof. □

Next, we study the asymptotic behavior of the curve $\mathcal{C}$. Since $c(s)$ is decreasing and positive, there exists $\lim_{s\to\infty} c(s)$. As it was done in [2], [19] and [21], we distinguish between the two cases $p \leq N$ and $p > N$. We define for $1 < p < \infty$

$$\overline{\lambda}(N, p) = \inf\left\{\int_\Omega |\nabla u|^p \, dx \, : \, u \in S \text{ and } u \text{ changes sign in } \Omega\right\}$$

and for $p > N$

$$\overline{\lambda} = \inf\left\{\int_\Omega |\nabla u|^p \, dx \, : \, u \in S \text{ and } u \text{ vanishes somewhere in } \overline{\Omega}\right\}. \tag{5.1}$$

Since $W_0^{1,p}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ when $p > N$, the definition (5.1) makes sense and the infimum is achieved. So, $\overline{\lambda} > 0$. Moreover, we see that $\overline{\lambda}(N,p) = \overline{\lambda}$ when $p > N$ and $\overline{\lambda}(N,p) = 0$ when $p \leq N$, see Arias-Campos-Gossez [2]. Note that the sequences defined in [2, Remark 2.7] can be also used in our setting.

We start with the case $p \leq N$.

Theorem 5.2. *Let $p \leq N$. Then*

$$\lim_{s\to+\infty} c(s) = 0.$$

Proof. Arguing by contradiction we assume that there exists $\varepsilon > 0$ such that

$$\max_{u\in\gamma[-1,1]} \tilde{J}_s(u) \geq \varepsilon \tag{5.2}$$

for all $\gamma \in \Gamma$ and for all $s \geq 0$. Since $p \leq N$, we can choose a function $\phi \in V$ which is unbounded from above. Consider the path $\gamma \in \Gamma$ defined by

$$\gamma(t) = \frac{t\varphi_1 + (1-|t|)\phi}{\|t\varphi_1 + (1-|t|)\phi\|_p}$$

for $t \in [-1,1]$. The maximum of $\tilde{J}_s$ on $\gamma[-1,1]$ is achieved at $t_s \in [-1,1]$, that is

$$\max_{u\in\gamma[-1,1]} \tilde{J}_s(\gamma(t)) = \tilde{J}_s(\gamma(t_s)).$$

Taking $v_s = t_s\varphi_1 + (1-|t_s|)\phi$ we obtain from (5.2) that

$$\tilde{J}_s(v_s) \geq \varepsilon \|v_s\|_p^p,$$

that is

$$\int_\Omega |\nabla v_s|^p \,\mathrm{d}x - s\int_\Omega \left(v_s^+\right)^p \,\mathrm{d}x \geq \varepsilon \int_\Omega |v_s|^p \,\mathrm{d}x. \tag{5.3}$$

If we let $s \to +\infty$, we may assume that $t_s \to \hat{t} \in [-1,1]$ (for a subsequence if necessary). Since v_s is bounded in V, from (5.3) we have that

$$\int_\Omega \left(v_s^+\right)^p \,\mathrm{d}x \to 0 \quad \text{as } s \to +\infty,$$

from which we conclude that

$$\hat{t}\varphi_1 + (1-|\hat{t}|)\phi \leq 0.$$

Since ϕ is unbounded from above, this is only possible for $\hat{t} = -1$. Then taking $\hat{t} = -1$ and passing to the limit in (5.3) we get

$$0 = \int_\Omega |\nabla\varphi_1|^p \,\mathrm{d}x \geq \varepsilon \int_\Omega |\varphi_1|^p \,\mathrm{d}x.$$

This implies $\varepsilon \leq 0$ and so we have a contradiction. $\square$

Let $\tilde{\Pi}_p$ be the nontrivial part of Π_p, that is, $\tilde{\Pi}_p = \Pi_p \setminus \{(0\times\mathbb{R}) \cup (\mathbb{R}\times 0)\}$. Theorem 5.2 implies the following corollary.

Corollary 5.3. *Let $p \leq N$. Then there does not exist $\varepsilon > 0$ such that $\tilde{\Pi}_p$ is contained in the set $\{(a,b) \in \mathbb{R}^2 : a \text{ and } b > \varepsilon\}$.*

Let us now study the case $p > N$.

Theorem 5.4. *Let $p > N$. Then*

$$\lim_{s\to+\infty} c(s) = \overline{\lambda} > 0, \tag{5.4}$$

where $\overline{\lambda}$ is defined in (5.1).

Proof. By contradiction we suppose that there exists $\varepsilon > 0$ such that

$$\max_{u\in\gamma[-1,1]} \tilde{J}_s(u) > \overline{\lambda} + \varepsilon \tag{5.5}$$

for all $\gamma \in \Gamma$ and for all $s \geq 0$. Let u be a minimizer of (5.1) and consider the path $\gamma \in \Gamma$ defined by

$$\gamma(t) = \frac{t\varphi_1 + (1-|t|)u}{\|t\varphi_1 + (1-|t|)u\|_p}$$

for $t \in [-1, 1]$. The path is well defined because u vanishes somewhere, but φ_1 does not and it belongs to Γ.

As in the proof of Theorem 5.2, for every $s > 0$, we fix $t_s \in [-1, 1]$ such that

$$\max_{u\in\gamma[-1,1]} \tilde{J}_s\left(\gamma(t)\right) = \tilde{J}_s\left(\gamma(t_s)\right).$$

Denoting $v_s = t_s\varphi_1 + (1 - |t_s|)u$, from (5.5) it follows

$$\tilde{J}_s\left(v_s\right) \geq \left(\overline{\lambda} + \varepsilon\right) \|v_s\|_p^p,$$

that is,

$$\int_\Omega |\nabla v_s|^p \,\mathrm{d}x - s\int_\Omega (v_s^+)^p \,\mathrm{d}x \geq \left(\overline{\lambda} + \varepsilon\right) \int_\Omega |v_s|^p \,\mathrm{d}x. \tag{5.6}$$

Letting $s \to +\infty$, we can assume, for a subsequence, $t_s \to \overline{t} \in [-1, 1]$. The uniform boundedness of v_s implies $\int_\Omega (v_s^+)^p \,\mathrm{d}x \to 0$ due to (5.6). Since $v_s \to v_{\hat{t}}$ in V, we have $v_{\hat{t}}^+ = 0$ in $\overline{\Omega}$, then

$$\hat{t}\varphi_1 \leq -(1 - |\hat{t}|)u \quad \text{in } \overline{\Omega}. \tag{5.7}$$

Since u vanishes somewhere in $\overline{\Omega}$ and $\varphi_1 \equiv \frac{1}{|\Omega|^{\frac{1}{p}}} > 0$, from (5.7) we obtain that $\hat{t} \leq 0$. Passing to the limit in (5.6) we obtain

$$\int_\Omega \left|\nabla\left(\hat{t}\varphi_1 + (1 - |\hat{t}|)u\right)\right|^p \,\mathrm{d}x \geq \left(\overline{\lambda} + \varepsilon\right) \int_\Omega |\hat{t}\varphi_1 + (1 - |\hat{t}|)u|^p \,\mathrm{d}x.$$

Since $\nabla\varphi_1 \equiv 0$ and due to $(c + d)^p \geq c^p + d^p$ for c, $d \geq 0$, we arrive at

$$\begin{aligned}\left(1 - |\hat{t}|\right)^p \int_\Omega |\nabla u|^p \,\mathrm{d}x &\geq \left(\overline{\lambda} + \varepsilon\right) \int_\Omega |\hat{t}\varphi_1 + (1 - |\hat{t}|)u|^p \,\mathrm{d}x \\ &\geq \left(\overline{\lambda} + \varepsilon\right) \left[|\hat{t}|^p \int_\Omega \varphi_1^p \,\mathrm{d}x + (1 - |\hat{t}|)^p \int_\Omega |u|^p \,\mathrm{d}x\right].\end{aligned} \tag{5.8}$$

If $\hat{t} = -1$, (5.8) becomes

$$0 \geq \left(\overline{\lambda} + \varepsilon\right) \int_\Omega \varphi_1^p \,\mathrm{d}x,$$

Thus, $\overline{\lambda} + \varepsilon \leq 0$ which is a contradiction.

If $\hat{t} \in]-1, 0]$, since u is a minimizer of (5.1), (5.8) becomes

$$\left(1 - |\hat{t}|\right)^p \overline{\lambda} \geq \left(\overline{\lambda} + \varepsilon\right) \left(1 - |\hat{t}|\right)^p.$$

So, $\varepsilon \leq 0$, a contradiction. This shows (5.4). □

As a consequence of Theorem 5.4, we have the following result.

Proposition 5.5. *Let $p > N$. Then $\tilde{\Pi}_p$ is contained in the open set $\{(a,b) \in \mathbb{R}^2 : a \text{ and } b > \overline{\lambda}\}$, where $\overline{\lambda}$ is the largest number such that this inclusion holds. In particular, $\lambda_2 > \overline{\lambda}$.*

First, we prove the following lemma.

Lemma 5.6. *Let $p > N$ and let u be a minimizer of (5.1). Then u does not change sign in Ω and u vanishes at exactly one point in $\overline{\Omega}$.*

Proof. Let u be a minimizer of (5.1), let $x_0 \in \overline{\Omega}$ and let

$$V_{x_0} = \{v \in V : v(x_0) = 0\}.$$

We are going to show that, if u vanishes at x_0, then

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x = \overline{\lambda} \int_\Omega |u|^{p-2} uv \, \mathrm{d}x \tag{5.9}$$

for all $v \in V_{x_0}$. We have that

$$\overline{\lambda} = \inf \left\{ \int_\Omega |\nabla v|^p \, \mathrm{d}x : v \in S \text{ and } v \in V_{x_0} \right\}$$

and the infimum is achieved at u. The Lagrange multiplier rule implies that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x = \lambda \int_\Omega |u|^{p-2} uv \, \mathrm{d}x \tag{5.10}$$

for all $v \in V_{x_0}$ and for some $\lambda \in \mathbb{R}$. If we take $v = u$ in (5.10), we obtain that $\lambda = \overline{\lambda}$ and so (5.9) is true.

Let us now assume that u vanishes in at least two points $x_1, x_2 \in \overline{\Omega}$. The function $w = |u|$ is also a minimizer in (5.1) which vanishes at x_1 and x_2, that is, w fulfills (5.9) for all $v \in V_{x_1}$ and also for all $v \in V_{x_2}$. Note that any $v \in V$ can be written as $v = v_1 + v_2$ with $v_1 \in V_{x_1}$ and $v_2 \in V_{x_2}$. Therefore, w satisfies (5.9) for all $v \in V$. If we then choose $v = 1$ in (5.9), we see that $w \geq 0$ changes sign which is a contradiction.

Finally, we want to show that the minimizer u does not change sign. Let $u^+ \not\equiv 0$ with $u(x_0) = 0$. This implies $u^+(x_0) = 0$. Taking $v = u^+$ in (5.9) we see that $\frac{u^+}{\|u^+\|_p}$ is a minimizer in (5.1). Hence, due to the first part of the proof, u^+ vanishes only at x_0 and so $u \geq 0$. □

Now we can prove Proposition 5.5.

Proof of Proposition 5.5. Let $(a,b) \in \tilde{\Pi}_p$ and let $u \not\equiv 0$ be a corresponding solution of (1.1). Choosing $v = 1$ as test function in (2.1) we obtain that

$$\int_\Omega \left(a \left(u^+\right)^{p-1} - b(u^-)^{p-1} \right) \mathrm{d}x = 0.$$

Hence, u changes sign in Ω. Note that u^+ and u^- both vanish somewhere since u changes sign. Testing (2.1) with $v = u^+$ and $v = u^-$ we get that

$$a = \frac{\displaystyle\int_\Omega |\nabla u^+|^p \, \mathrm{d}x}{\displaystyle\int_\Omega |u^+|^p \, \mathrm{d}x} \geq \overline{\lambda} \qquad \text{and} \qquad b = \frac{\displaystyle\int_\Omega |\nabla u^-|^p \, \mathrm{d}x}{\displaystyle\int_\Omega |u^-|^p \, \mathrm{d}x} \geq \overline{\lambda}. \tag{5.11}$$

Next, we want to show that $a, b > \overline{\lambda}$. Let us assume that $a = \overline{\lambda}$. Then we see from (5.11) that $\frac{u^+}{\|u+\|_p}$ is a minimizer in (5.1). Since u changes sign, u^+ vanishes in many points (at least in more than one point) which contradicts Lemma 5.6. Hence $a > \overline{\lambda}$ and in the same way we can show that $b > \overline{\lambda}$. Therefore, $c(s) > \overline{\lambda}$ and from Theorem 5.4 we know that $\lim_{s\to+\infty} c(s) = \overline{\lambda}$.

Proposition 3.5 (ii) implies that $(s + c(s), c(s)) \in \tilde{\Pi}_p \subset \Pi_p$ and in particular, $(c(0), c(0)) = (\lambda_2, \lambda_2) \in \tilde{\Pi}_p$. Since $c(s) > \overline{\lambda}$ from the first part of the proof, it follows that $\overline{\lambda} < \lambda_2$. □

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References

[1] G.A. Afrouzi, M. Mirzapour, V.D. Rădulescu, *The variational analysis of a nonlinear anisotropic problem with no-flux boundary condition*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **109** (2015), no. 2, 581–595.

[2] M. Arias, J. Campos, J.-P. Gossez, *On the antimaximum principle and the Fučik spectrum for the Neumann p-Laplacian*, Differential Integral Equations **13** (2000), no. 1-3, 217–226.

[3] H. Berestycki, H. Brézis, *On a free boundary problem arising in plasma physics*, Nonlinear Anal. **4** (1980), no. 3, 415–436.

[4] M.-M. Boureanu, *On some variable exponent problems with no-flux boundary condition*, Current trends in mathematical analysis and its interdisciplinary applications, 253–285, Birkhäuser/Springer, Cham, 2019.

[5] M.-M. Boureanu, D.N. Udrea, *Existence and multiplicity results for elliptic problems with $p(\cdot)$-growth conditions*, Nonlinear Anal. Real World Appl. **14** (2013), no. 4, 1829–1844.

[6] M.-M. Boureanu, C. Udrea, *No-flux boundary value problems with anisotropic variable exponents*, Commun. Pure Appl. Anal. **14** (2015), no. 3, 881–896.

[7] S. Carl, V.K. Le, D. Motreanu, "Nonsmooth Variational Problems and Their Inequalities" Springer, New York, 2007.

[8] M. Cuesta, D. de Figueiredo, J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations **159** (1999), no. 1, 212–238.

[9] E.N. Dancer, *On the Dirichlet problem for weakly non-linear elliptic partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **76** (1976/77), no. 4, 283–300.

[10] D.G. de Figueiredo, J.-P. Gossez, *On the first curve of the Fučik spectrum of an elliptic operator*, Differential Integral Equations **7** (1994), no. 5-6, 1285–1302.

[11] X. Fan, S.-G. Deng, *Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations*, Nonlinear Anal. **67** (2007), no. 11, 3064–3075.

[12] S. Fučík, *Boundary value problems with jumping nonlinearities*, Časopis Pěst. Mat. **101** (1976), no. 1, 69–87.

[13] S. Fučík, "Solvability of Nonlinear Equations and Boundary Value Problems", D. Reidel Publishing Co., Dordrecht-Boston, Mass, 1980.

[14] N. Ghoussoub, "Duality and Perturbation Methods in Critical Point Theory", Cambridge University Press, Cambridge, 1993.

[15] H. Gourgeon, J. Mossino, *Sur un problème à frontière libre de la physique des plasmas*, Ann. Inst. Fourier (Grenoble) **29** (1979), no. 4, 127–141.

[16] D. Kinderlehrer, J. Spruck, *The shape and smoothness of stable plasma configurations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), no. 1, 131–148.

[17] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal. **64** (2006), no. 5, 1057–1099.

[18] V.K. Le, K. Schmitt, *Sub-supersolution theorems for quasilinear elliptic problems: a variational approach*, Electron. J. Differential Equations **2004**, No. 118, 7 pp.

[19] S.R. Martínez, J.D. Rossi, *On the Fučik spectrum and a resonance problem for the p-Laplacian with a nonlinear boundary condition*, Nonlinear Anal. **59** (2004), no. 6, 813–848.

[20] C. Mercier, *The magnetohydrodynamic approach of the problem of plasma confinment in closed magnetic configurations*, EURATOM-CEA, Comm. of the European Communities, Luxembourg, 1974.
[21] D. Motreanu, P. Winkert, *On the Fučik spectrum for the p-Laplacian with Robin boundary condition*, Nonlinear Anal. **74** (2011), no. 14, 4671–4681.
[22] D. Motreanu, P. Winkert, *The Fučík spectrum for the negative p-Laplacian with different boundary conditions*, Nonlinear analysis, 471–485, Springer Optim. Appl. **68**, Springer, New York, 2012.
[23] J.-P. Puel, *Sur un problème de valeur propre non linéaire et de frontière libre*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 15, A861–A863.
[24] D.G. Schaeffer, *Non-uniqueness in the equilibrium shape of a confined plasma*, Comm. Partial Differential Equations **2** (1977), no. 6, 587–600.
[25] R. Temam, *A non-linear eigenvalue problem: the shape at equilibrium of a confined plasma*, Arch. Rational Mech. Anal. **60** (1975/76), no. 1, 51–73.
[26] R. Temam, *Remarks on a free boundary value problem arising in plasma physics*, Comm. Partial Differential Equations **2** (1977), no. 6, 563–585.
[27] L. Zhao, P. Zhao, X. Xie, *Existence and multiplicity of solutions for divergence type elliptic equations*, Electron. J. Differential Equations **2011**, No. 43, 9 pp.
[28] W. Zou, F. Li, M. Liu, B. Lv, *Existence of solutions for a nonlocal problem arising in plasma physics*, J. Differential Equations **256** (2014), no. 4, 1653–1682.
[29] W. Zou, F. Li, B. Lv, *On a nonlocal problem for a confined plasma in a Tokamak*, Appl. Math. **58** (2013), no. 6, 609–642.

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