GAIN AND LOSS ON CRITICAL LOGARITHMIC DOUBLE PHASE EQUATIONS

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Abstract. This paper is concerned with the study of the following double phase equation with logarithmic nonlinearity

$$\begin{aligned}&-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)+|u|^{p-2}u+\mu(x)|u|^{q-2}u\\&=K_1(x)|u|^{p^*-2}u+\lambda K_2(x)|u|^{r-2}u\log(|u|)\quad\text{in } \mathbb{R}^N,\end{aligned}$$

with dimension $N\geq 2$, parameter $\lambda>0$, $1<p<q<N$, $\mu\colon\mathbb{R}^N\to[0,\infty)$ is a Lipschitz continuous function and $\max\{p,N(p-1)/(N-p)\}<r<p^*=Np/(N-p)$. Here, the weight function K_1 is positive, while K_2 may change sign on $\mathbb{R}^N$. By a different variational approach, we prove an existence result which in some aspects improves our contribution in [A. Bahrouni, A. Fiscella, P. Winkert, J. Math. Anal. Appl. **547** (2025), no. 2, Paper No. 129311, 24 pp.]. For this, we need some restrictive assumptions on the weights $\mu(\cdot)$, K_1 and K_2.

1. Introduction

In our paper [7], we mainly studied the following quasilinear equation

$$\begin{aligned}&-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)+|u|^{p-2}u+\mu(x)|u|^{q-2}u\\&=K_1(x)|u|^{p^*-2}u+\lambda K_2(x)|u|^{r-2}u\log(|u|)+\gamma K_3(x)|u|^{\beta-2}u,\quad\text{in } \mathbb{R}^N,\end{aligned}\tag{1.1}$$

driven by an operator of double phase type. In particular, in [7, Theorem 4.1] we proved the existence of a mountain pass solution of (1.1) in a superlinear logarithmic setting with exponents $1<p<q<\beta<r<p^*$, where $p^*=Np/(N-p)$, and considering $\gamma=\lambda$ with λ sufficiently large. In order to deal with the logarithmic term, we strongly used the nonlinear perturbation with exponent β. Indeed, to get a mountain pass solution for (1.1), we needed an important asymptotic property of the mountain pass level itself, as λ goes to ∞. The proof of this asymptotic condition was obtained by a challenging combination of the superlinear logarithmic term and of the β-nonlinearity, explicitly highlighted in the assumption

$$K_2(x)\leq\frac{e\,(r-\beta)\,r\,(\beta-\sigma)}{\beta(r-\sigma)}K_3(x),\quad\text{for any } x\in\mathbb{R}^N,$$

with $q<\sigma<\beta$, strongly requested in [7, Theorem 4.1].

2020 *Mathematics Subject Classification.* 35B08, 35B33, 35J15, 35J20, 35J62.

Key words and phrases. double phase equation, logarithmic nonlinearity, critical nonlinearity, variational methods.

In the present paper, we want to face (1.1) without the help of the β-perturbation, that is considering $\gamma = 0$. For this, we study the equation

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + |u|^{p-2}u + \mu(x)|u|^{q-2}u \\ &= K_1(x)|u|^{p^*-2}u + \lambda K_2(x)|u|^{r-2}u\log(|u|), \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.2}$$

with the following structural assumptions, similar to the ones in [7, Theorem 4.1]:

(H_1) $1 < p < q < N$, $q < p^*$ and $\mu\colon \mathbb{R}^N \to \mathbb{R}_+ = [0,\infty)$ is Lipschitz continuous such that $\mu(\cdot) \in L^\infty(\mathbb{R}^N)$.

(H_2) $K_1 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $K_1(x) > 0$ for all $x \in \mathbb{R}^N$ and if $\{A_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure $|A_n| \leq C$ for all $n \in \mathbb{N}$ and some $C > 0$, then

$$\lim_{n\to\infty} \int_{A_n \cap B_\rho^c(0)} K_1(x)\,\mathrm{d}x = 0,$$

for some $\rho > 0$.

(H_3) $K_2 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $|K_2| \leq K_1$ on $\mathbb{R}^N$.

We point out that we still suppose that K_2 can change sign in $\mathbb{R}^N$. However, in order to handle a superlinear logarithmic term, we need a further condition for the weight functions appearing in (1.2):

(H_4) there exist $R > 0$ and $\kappa > 0$ such that $\mu(x) = 0$, $K_1(x) = \|K_1\|_\infty$ and $K_2(x) = \|K_2\|_\infty$ for a.a. $x \in B_R(0)$.

The requirement in (H_4) for K_1 and K_2 is quite standard when working with critical equations in $\mathbb{R}^N$, as shown in [17]. The restriction on the double phase weight $\mu(\cdot)$ is crucial to exploit the explicit expression of the extremal functions for the Sobolev inequality into $L^{p^*}(\mathbb{R}^N)$, as used in [11].

Our main result is the following theorem.

Theorem 1.1. *Let* (H_1)–(H_4) *be satisfied and let* r *be such that*

$$\max\left\{p, \frac{N(p-1)}{N-p}\right\} < r < p^*.$$

Then, equation (1.2) *admits at least one nontrivial weak solution for any* $\lambda > 0$.

We strongly point out that in Theorem 1.1 we are able to cover the situation when $p < r \leq q$, remain unanswered in [7, Theorem 4.1]. Indeed, we can guarantee that $N(p-1)/(N-p) < p < r < p^*$ whenever $N > p^2$. Also, in Theorem 1.1 we can consider any generic value for the parameter $\lambda > 0$. However, technically speaking, we are not able to get a mountain pass solution. More precisely, by the mountain pass theorem we construct a Palais-Smale sequence at the critical mountain pass level. But this sequence admits a subsequence which just converges weakly to a nontrivial critical point of the energy functional related to (1.2). That is, we cannot prove the strong convergence of the Palais-Smale subsequence, which guarantees the attainability of the critical mountain pass level.

Thus, comparing Theorem 1.1 with [7, Theorem 4.1], we have the following gains:

(i) we do not need to add any β-perturbation to control the logarithmic term, as in (1.1);
(ii) we cover a strongly superlinear logarithmic situation, with possibly $p < r \leq q$;
(iii) the parameter $\lambda > 0$ is generic.

However, we need to pay some information in exchange:

(i) we have a new restrictive assumption for the weights $\mu(\cdot)$, K_1 and K_2 as given in (H_4);
(ii) formally, we do not get a mountain pass solution for (1.2).

The double phase operator given in problems (1.1) and (1.2) is associated to the energy functional

$$\Psi(u) = \int_{\mathbb{R}^N} \left(\frac{1}{p} |\nabla u|^p + \frac{\mu(x)}{q} |\nabla u|^q \right) \mathrm{d}x, \tag{1.3}$$

which was first introduced in [37, 38, 39] to provide models for strongly anisotropic materials in the framework of homogenization. A distinguishing feature of the double phase functional (1.3) is the variation in its ellipticity depending on the behavior of the function $\mu(\cdot)$. Specifically, the energy density exhibits ellipticity of order q in regions where $\mu(x) > \varepsilon$ for any fixed $\varepsilon > 0$, while it has ellipticity of order p at points where $\mu(x) = 0$. Consequently, the integrand in (1.3) switches between two distinct types of elliptic behavior. A first mathematical treatment of functionals of type (1.3) has been done in a number of papers in [8, 9, 10, 12, 13, 15, 29, 30, 31, 36] related to regularity properties of local minimizers.

Over the past 10 years, there have been several contributions dealing with double phase problems in the whole space $\mathbb{R}^N$. We refer to [2, 4, 6, 22, 23, 25, 26, 28, 33], see also the references therein. Only the authors in [25] do allow a critical growth, in addition to the unboundedness of the domain. For bounded domains and critical growth for double phase problems we mention the papers by [5, 11, 18, 19, 27, 32]. None of these works, however, consider the presence of a logarithmic term on the right-hand side of the equation. For double phase problems involving nonlinearities of logarithmic type on the right-hand side there are only few works. In addition to the authors' aforementioned work [7], we can simply make reference to [1] who proved the existence of a nonnegative solution based on the Nehari manifold method of the problem

$$\begin{aligned} &- \operatorname{div} \left(|\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right) + V(x) |u|^{p-2} u \\ &= \lambda K(x) |u|^{r-2} u \log(|u|) \quad \text{in } \mathcal{D}, \quad u\big|_{\partial D} = 0, \end{aligned}$$

where $\mathcal{D} \subset \mathcal{M}$ is an open bounded subset of a smooth complete compact Riemannian N-manifold and $r \in (1, p)$. Very recently, the authors in [3] considered logarithmic type double phase problems where the logarithm appears not only on the right-hand side but also in the operator. However, due to the different operator, the function space and the variational setting are different to the present work. In summary, our work combines several important aspects: critical growth, the presence of a logarithmic term, and the unboundedness of the domain. Furthermore, we improve upon the results from our earlier work in [7] in a nontrivial way.

The paper is organized as follows. In Section 2 we introduce the solution space, the energy functional of (1.2) and some preliminary results. We give the proof of Theorem 1.1 in Section 3, by using several auxiliary lemmas.

2. Variational setting

In this section, we first state some known results about Musielak-Orlicz spaces in $\mathbb{R}^N$. By $L^\ell(\mathbb{R}^N)$ we denote the usual Lebesgue space endowed with the norm

$\|\cdot\|_\ell$ for $1 \le \ell \le \infty$. While $W^{1,\ell}(\mathbb{R}^N)$ stands for the Sobolev spaces equipped with the norm $\|\nabla \cdot\|_\ell + \|\cdot\|_\ell$, for any $1 < \ell < \infty$.

Supposing assumption (H_1), we consider the nonlinear function $\mathcal{H}\colon \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ given by

$$\mathcal{H}(x,t) := t^p + \mu(x) t^q.$$

Denoting by $M(\mathbb{R}^N)$ the set of all measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$, we then introduce the Musielak-Orlicz Lebesgue space $L^{\mathcal{H}}(\mathbb{R}^N)$ by

$$L^{\mathcal{H}}(\mathbb{R}^N) := \left\{ u \in M(\mathbb{R}^N)\colon \varrho_{\mathcal{H}}(u) := \int_{\mathbb{R}^N} \mathcal{H}(x,|u|)\,\mathrm{d}x < \infty \right\}$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf\left\{ \tau > 0\colon \varrho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \le 1 \right\},$$

where the modular function is given by

$$\varrho_{\mathcal{H}}(u) := \int_{\mathbb{R}^N} \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_{\mathbb{R}^N} \left(|u|^p + \mu(x)\,|u|^q \right) \mathrm{d}x.$$

By $L^q_\mu(\mathbb{R}^N)$ we denote the weighted space given by

$$L^q_\mu(\mathbb{R}^N) := \left\{ u \in M(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} \mu(x)|u|^q\,\mathrm{d}x < \infty \right\}$$

equipped with the seminorm

$$\|u\|_{q,\mu} := \left(\int_{\mathbb{R}^N} \mu(x)|u|^q\,\mathrm{d}x \right)^{\frac{1}{q}}.$$

Moreover, the corresponding Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\mathbb{R}^N)$ is defined by

$$W^{1,\mathcal{H}}(\mathbb{R}^N) := \left\{ u \in L^{\mathcal{H}}(\mathbb{R}^N)\colon |\nabla u| \in L^{\mathcal{H}}(\mathbb{R}^N) \right\}$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}} := \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. In the following, we equip the space $W^{1,\mathcal{H}}(\mathbb{R}^N)$ with the equivalent norm

$$\|u\| := \inf\left\{ \tau > 0\colon \int_{\mathbb{R}^N} \left[\left(\frac{|\nabla u|}{\tau}\right)^p + \mu(x) \left(\frac{|\nabla u|}{\tau}\right)^q + \left|\frac{u}{\tau}\right|^p + \mu(x) \left|\frac{u}{\tau}\right|^q \right] \mathrm{d}x \le 1 \right\},$$

whereby the corresponding modular is defined by

$$\varrho(u) := \int_{\mathbb{R}^N} \left[|\nabla u|^p + \mu(x)\,|\nabla u|^q + |u|^p + \mu(x)|u|^q \right] \mathrm{d}x.$$

Both spaces $L^{\mathcal{H}}(\mathbb{R}^N)$ and $W^{1,\mathcal{H}}(\mathbb{R}^N)$ are separable reflexive Banach spaces, see [26, Theorem 2.7].

Next, we recall the relations between the norm $\|\cdot\|$ and the associated modular $\varrho(\cdot)$. We refer to [26, Proposition 2.6] for its proof, see also [14].

Lemma 2.1. *Let* (H_1) *be satisfied,* $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ *and* $c > 0$. *Then the following hold:*

(i) *for* $u \neq 0$ *we have* $\|u\| = c$ *if and only if* $\varrho(\frac{u}{c}) = 1$;
(ii) $\|u\| < 1$ *implies* $\|u\|^q \le \varrho(u) \le \|u\|^p$;

(iii) $\|u\| > 1$ *implies* $\|u\|^p \le \varrho(u) \le \|u\|^q$;
(iv) $\varrho(u) \to 0$ *if and only if* $\|u\| \to 0$;
(v) $\varrho(u) \to \infty$ *if and only if* $\|u\| \to \infty$.

The following result is taken from [26, Theorem 2.7].

Lemma 2.2. *Let* (H_1) *be satisfied. Then, the embedding* $W^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^\ell(\mathbb{R}^N)$ *is continuous for any* $\ell \in [p, p^*]$*. Also,* $W^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^\ell_{\mathrm{loc}}(\mathbb{R}^N)$ *is compact for any* $\ell \in [1, p^*)$*.*

Furthermore, we recall the continuous and the compact embedding of $W^{1,\mathcal{H}}(\mathbb{R}^N)$ into the Lebesgue space

$$L^s_{K_1}(\mathbb{R}^N) := \left\{ u \in M(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} K_1(x)|u|^s \,\mathrm{d}x < \infty \right\},$$

where $1 < s < \infty$ and K_1 fulfills (H_2). Then, in [7, Proposition 3.1] we proved the following result.

Lemma 2.3. *Let* (H_2) *be satisfied. Then,* $W^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^s_{K_1}(\mathbb{R}^N)$ *is compact for any* $s \in (p, p^*)$*.*

A function $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ is called a weak solution of (1.2) if

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + \mu(x) |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi \right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left(|u|^{p-2} u\varphi + \mu(x) |u|^{q-2} u\varphi \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} K_1(x)|u|^{p^*-2} u\varphi \,\mathrm{d}x + \lambda \int_{\mathbb{R}^N} K_2(x)|u|^{r-2} u \log(|u|)\varphi \,\mathrm{d}x,
\end{aligned}$$

is satisfied for any $\varphi \in W^{1,\mathcal{H}}(\mathbb{R}^N) \setminus \{0\}$. Moreover, the corresponding energy functional $I_\lambda \colon W^{1,\mathcal{H}}(\mathbb{R}^N) \to \mathbb{R}$ of problem (1.2) is given by

$$\begin{aligned}
I_\lambda(u) =& \frac{1}{p}\left(\|\nabla u\|_p^p + \|u\|_p^p \right) + \frac{1}{q}\left(\|\nabla u\|_{q,\mu}^q + \|u\|_{q,\mu}^q \right) - \int_{\mathbb{R}^N} K_1(x) \frac{|u|^{p^*}}{p^*} \,\mathrm{d}x \\
& - \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r} |u|^r \log(|u|) \,\mathrm{d}x + \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r^2} |u|^r \,\mathrm{d}x.
\end{aligned}$$

By [7, Lemma 3.3], we know that I_λ is well defined and of class $C^1(W^{1,\mathcal{H}}(\mathbb{R}^N), \mathbb{R})$. Also, it is clear that weak solutions of (1.2) are critical points of I_λ.

Finally, we recall the following technical lemma which allows us to deal with the logarithmic nonlinearity in (1.2), see [35] for its proof.

Lemma 2.4.

(i) *For any* $\sigma > 0$*, we have*

$$\log(t) \le \frac{1}{e\sigma} t^\sigma \quad \textit{for any } t \in [1, \infty).$$

(ii) *For any* $\sigma > 0$*, we have*

$$t^\sigma |\log(t)| \le \frac{1}{e\sigma} \quad \textit{for any } t \in (0, 1).$$

(iii) *For any* $\sigma \in (0, 1)$ *and* $s > 1$*, there exists* $C_\sigma > 0$ *such that*

$$t^s |\log(t)| \le C_\sigma \left(t^{s(1-\sigma)} + t^{s(1+\sigma)} \right) \quad \textit{for any } t > 0.$$

3. The existence result

We first study the compactness property for the functional I_λ under a suitable threshold $\overline{c}$, set as

$$\overline{c} := \left(\frac{1}{p} - \frac{1}{p^*}\right) \frac{S^{\frac{p^*}{p^*-p}}}{\|K_1\|_\infty^{\frac{p}{p^*-p}}} > 0, \tag{3.1}$$

where $S > 0$ is the best constant of the Sobolev embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, given as

$$S := \inf_{u \in W^{1,p}(\mathbb{R}^N)} \frac{\|\nabla u\|_p^p + \|u\|_p^p}{\|u\|_{p^*}^p}. \tag{3.2}$$

For this, we say that $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ is a Palais-Smale sequence for I_λ at level $c \in \mathbb{R}$ if

$$I_\lambda(u_n) \to c \quad \text{and} \quad I_\lambda'(u_n) \to 0 \quad \text{in } \left(W^{1,\mathcal{H}}(\mathbb{R}^N)\right)^* \quad \text{as } n \to \infty. \tag{3.3}$$

Then, by [7, Lemma 4.5] we have the following result.

Lemma 3.1. *Let* (H_1)–(H_3) *be satisfied and let* $\lambda > 0$. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ *be a bounded* $(\mathrm{PS})_c$ *sequence with* $c \in \mathbb{R}$. *Then, up to a subsequence,* $\nabla u_n(x) \to \nabla u(x)$ *a.e. in* $\mathbb{R}^N$ *as* $n \to \infty$.

In what follows, we provide a technical result for the logarithmic term.

Lemma 3.2. *Let* (H_1)–(H_3) *be satisfied and let* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ *be a sequence satisfying*

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\mathbb{R}^N), \quad u_n \to u \text{ in } L^s_{K_1}(\mathbb{R}^N), \quad u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N, \tag{3.4}$$

for any $s \in (p, p^*)$. *Then, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K_2(x)|u_n|^r \log(|u_n|)\,\mathrm{d}x = \int_{\mathbb{R}^N} K_2(x)|u|^r \log(|u|)\,\mathrm{d}x \tag{3.5}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K_2(x)|u_n|^{r-2}u_n \log(|u_n|)\varphi\,\mathrm{d}x = \int_{\mathbb{R}^N} K_2(x)|u|^{r-2}u \log(|u|)\varphi\,\mathrm{d}x, \tag{3.6}$$

for any $\varphi \in W^{1,\mathcal{H}}(\mathbb{R}^N)$.

Proof. We only prove (3.5), equation (3.6) can be proved in a similar way. By Lemma 2.4 and (H_3), for any Lebesgue measurable set $U \subset \mathbb{R}^N$ and for any $\sigma > 0$ such that $r\sigma < \min\{r - p, p^* - r\}$, we have

$$\int_U K_2(x)|u_n|^r \log(|u_n|)\,\mathrm{d}x \le C_\sigma \int_U K_1(x)\left(|u_n|^{r(1-\sigma)} + |u_n|^{r(1+\sigma)}\right)\mathrm{d}x.$$

Thus, considering (3.4) and Vitali's convergence theorem we get the assertion. $\square$

We are now ready to study the compactness of I_λ under the threshold $\overline{c}$.

Lemma 3.3. *Let* (H_1)–(H_3) *be satisfied and let* $\lambda > 0$. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ *be a* $(\mathrm{PS})_c$ *sequence with* $0 < c < \overline{c}$ *as given in* (3.1). *Then there exists* $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ *being a nontrivial critical point for* I_λ *such that, up to a subsequence,* $u_n \rightharpoonup u$ *in* $W^{1,\mathcal{H}}(\mathbb{R}^N)$ *as* $n \to \infty$.

Proof. Let us fix $c < \bar{c}$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a $(\mathrm{PS})_c$ sequence in $W^{1,\mathcal{H}}(\mathbb{R}^N)$, that is, (3.3) is fulfilled. We first show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}}(\mathbb{R}^N)$ arguing by contradiction. Then, going to a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, we have $\lim\limits_{n\to\infty}\|u_n\| = \infty$ and $\|u_n\| \geq 1$ for any $n \in \mathbb{N}$. Let $\sigma > 0$ be such that $1 < p < \max\{r,q\} < \sigma < p^*$ and let $\varepsilon > 0$ be such that $r+\varepsilon \in (p,p^*)$. Thus, invoking Lemmas 2.1 and 2.4, we get

$$\begin{aligned}
&o_n(1) + c + C\|u_n\| \\
&= I(u_n) - \frac{1}{\sigma}\langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{p}-\frac{1}{\sigma}\right)\left(\|\nabla u_n\|_p^p + \|u_n\|_p^p\right) + \left(\frac{1}{q}-\frac{1}{\sigma}\right)\left(\|\nabla u_n\|_{q,\mu}^q + \|u_n\|_{q,\mu}^q\right) \\
&\quad + \left(\frac{1}{\sigma}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x - \lambda\left(\frac{1}{r}-\frac{1}{\sigma}\right)\int_{\mathbb{R}^N} K_2(x)|u_n|^r\log(|u_n|)\,\mathrm{d}x \\
&\geq \left(\frac{1}{q}-\frac{1}{\sigma}\right)\|u_n\|^p + \left(\frac{1}{\sigma}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x \\
&\quad - \lambda\left(\frac{1}{r}-\frac{1}{\sigma}\right)\int_{\{x\in\mathbb{R}^N\,:\,|u_n(x)|>1|\}} K_2(x)|u_n|^r\log(|u_n|)\,\mathrm{d}x \\
&\quad - \lambda\left(\frac{1}{r}-\frac{1}{\sigma}\right)\int_{\{x\in\mathbb{R}^N\,:\,|u_n|<1|\}} K_2(x)|u_n|^r\log(|u_n|)\,\mathrm{d}x \\
&\geq \left(\frac{1}{q}-\frac{1}{\sigma}\right)\|u_n\|^p + \left(\frac{1}{\sigma}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x \\
&\quad - \frac{\lambda}{e(r+\varepsilon)}\left(\frac{1}{r}-\frac{1}{\sigma}\right)\int_{\mathbb{R}^N}|K_2(x)||u_n|^{r+\varepsilon}\,\mathrm{d}x.
\end{aligned}$$

Now, we can find a suitable constant $C_\lambda > 0$ such that

$$\frac{\lambda}{e(r+\varepsilon)}\left(\frac{1}{r}-\frac{1}{\sigma}\right)|t|^{r+\varepsilon} \leq \left(\frac{1}{\sigma}-\frac{1}{p^*}\right)|t|^{p^*} + C_\lambda, \quad \text{for any } t \in \mathbb{R},$$

and so, from (H_3), we obtain

$$o(1) + c + C\|u_n\| \geq \left(\frac{1}{q}-\frac{1}{\sigma}\right)\|u_n\|^p - C_\lambda\int_{\mathbb{R}^N}|K_2(x)|\,\mathrm{d}x.$$

This leads to a contradiction.

Hence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}}(\mathbb{R}^N)$. By Lemmas 2.3 and 3.1, there exists a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, and $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ such that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\mathbb{R}^N), && u_n \rightharpoonup u \quad \text{in } L^{p^*}(\mathbb{R}^N), \\
&\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \mathbb{R}^N, && u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N, \\
&u_n \to u \text{ in } L^s_{K_1}(\mathbb{R}^N) \quad \text{for any } s \in (p,p^*).
\end{aligned} \tag{3.7}$$

From (3.3), (3.6) and (3.7) we see that $\langle I'_\lambda(u), \varphi\rangle = 0$ for any $\varphi \in W^{1,\mathcal{H}}(\mathbb{R}^N)$, which means that u is a critical point of I_λ.

Now, let us prove by contradiction that u is nontrivial. If $u \equiv 0$, by (3.5) we have

$$\int_{\mathbb{R}^N} K_2(x)|u_n|^r\log(|u_n|)\,\mathrm{d}x = o_n(1),$$

from which, by using also (3.3) and (3.7), we obtain

$$\begin{aligned} c+o_n(1)=I_\lambda(u_n)&=\frac{1}{p}\left(\|\nabla u_n\|_p^p+\|u_n\|_p^p\right)+\frac{1}{q}\left(\|\nabla u_n\|_{q,\mu}^q+\|u_n\|_{q,\mu}^q\right)\\ &\quad-\frac{1}{p^*}\int_{\mathbb{R}^N}K_1(x)|u_n|^{p^*}\,\mathrm{d}x+o_n(1) \end{aligned}\tag{3.8}$$

and

$$\begin{aligned} o_n(1)=\langle I_\lambda'(u_n),u_n\rangle&=\|\nabla u_n\|_p^p+\|u_n\|_p^p+\|\nabla u_n\|_{q,\mu}^q+\|u_n\|_{q,\mu}^q\\ &\quad-\int_{\mathbb{R}^N}K_1(x)|u_n|^{p^*}\,\mathrm{d}x+o_n(1). \end{aligned}\tag{3.9}$$

Taking (3.2) and (3.9) into account, it follows

$$o_n(1)\ge\varrho(u_n)\left[1-\|K_1\|_\infty\frac{\left(\|\nabla u_n\|_p^p+\|u_n\|_p^p\right)^{\frac{p^*}{p}-1}}{S^{\frac{p^*}{p}}}\right].\tag{3.10}$$

If $\varrho(u_n)\to 0$, by Lemma 2.1 we have that $\|u_n\|\to 0$ so that by (3.8) we obtain $c=0$, a contradiction.

Thus, (3.10) implies that

$$\|\nabla u_n\|_p^p+\|u_n\|_p^p\ge\frac{S^{\frac{p^*}{p^*-p}}}{\|K_1\|_\infty^{\frac{p}{p^*-p}}}+o_n(1),$$

so that by (3.8) and (3.9) we get

$$\begin{aligned} c&=\left(\frac{1}{p}-\frac{1}{p^*}\right)\left(\|\nabla u_n\|_p^p+\|u_n\|_p^p\right)+\left(\frac{1}{q}-\frac{1}{p^*}\right)\left(\|\nabla u_n\|_{q,\mu}^q+\|u_n\|_{q,\mu}^q\right)+o_n(1)\\ &\ge\left(\frac{1}{p}-\frac{1}{p^*}\right)\frac{S^{\frac{p^*}{p^*-p}}}{\|K_1\|_\infty^{\frac{p}{p^*-p}}}+o_n(1), \end{aligned}$$

which contradicts $c<\overline{c}$ and (3.1). □

In order to apply Lemma 3.3, we need to guarantee that $I_\lambda(u)$ falls into the range of validity $0<c<\overline{c}$, for a suitable $u\in W^{1,\mathcal{H}}(\mathbb{R}^N)$. To this end, the idea is to employ a suitable truncation of the function

$$U_\varepsilon(x)=\frac{C_{N,p}\,\varepsilon^{(N-p)/p(p-1)}}{\left(\varepsilon^{p/(p-1)}+|x|^{p/(p-1)}\right)^{(N-p)/p}},\quad\text{with }\varepsilon>0,\tag{3.11}$$

which belongs to $W^{1,p}(\mathbb{R}^N)$. Here, the best constant of the Sobolev embedding (3.2) is attained, considering a normalization constant $C_{N,p}>0$ given by

$$C_{N,p}=\left[N\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{(N-p)/p^2}.$$

Let us consider $B_R(0)$ as in (H_4), then we can introduce a cut-off function $\phi_R\in C_0^\infty(B_R(0))$ such that

$$0\le\phi_R\le 1\quad\text{and}\quad\phi_R(x)=1\text{ for }x\in B_{R/2}(0).\tag{3.12}$$

For any $\varepsilon>0$, we set

$$u_\varepsilon=\phi_R\,U_\varepsilon\quad\text{and}\quad v_\varepsilon=\frac{u_\varepsilon}{\|u_\varepsilon\|_{p^*}}.\tag{3.13}$$

Then, considering S as in (3.2), we can prove the following crucial estimates for v_ε.

Lemma 3.4. *Let v_ε be as defined in (3.13). Then, for any $r > N(p-1)/(N-p)$ and as $\varepsilon \to 0^+$, we have:*

(i) $\displaystyle\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} |v_\varepsilon|^p \,\mathrm{d}x = S + \mathrm{O}(\varepsilon^{(N-p)/(p-1)})$;

(ii) $\displaystyle\int_{\mathbb{R}^N} |v_\varepsilon|^r \,\mathrm{d}x = C_1 \varepsilon^{N-r(N-p)/p} + \mathrm{O}(\varepsilon^{N-r(N-p)/p})$;

(iii) $\displaystyle\int_{\mathbb{R}^N} |v_\varepsilon|^r \,|\log(|v_\varepsilon|)| \,\mathrm{d}x = C_2 \varepsilon^{N-r(N-p)/p} \log\left(\frac{1}{\varepsilon}\right) + \mathrm{O}(\varepsilon^{N-r(N-p)/p})$.

Proof. Assertions (i) and (ii) follow directly from [20, Theorem 8.4], see also [17, 21, 24]. We just prove (iii), inspired by [16, Lemmas 3.2 and 3.4].

First, recall the estimate given in [20, Lemma 7.1]

$$\int_{\mathbb{R}^N} |u_\varepsilon|^{p^*} \,\mathrm{d}x = S^{\frac{N}{p}} + \mathrm{O}(\varepsilon^{\frac{N}{p-1}}). \tag{3.14}$$

By (3.13) we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_\varepsilon|^r \,|\log(|v_\varepsilon|)| \,\mathrm{d}x &= \int_{\mathbb{R}^N} \left(\frac{\phi_R U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^r |\log(\phi_R)| \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{\phi_R U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^r \left|\log\left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)\right| \mathrm{d}x \\
&=: A_1 + A_2.
\end{aligned} \tag{3.15}$$

We begin by evaluating A_1, taking into account (3.11), (3.12), (3.14) as well as Lemma 2.4, so that

$$\begin{aligned}
|A_1| &= \int_{B_R(0)\setminus B_{R/2}(0)} \left(\frac{\phi_R U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^r |\log(\phi_R)| \,\mathrm{d}x \\
&\le \frac{1}{erS^{\frac{rN}{p}}} \int_{B_R(0)\setminus B_{R/2}(0)} U_\varepsilon^r \,\mathrm{d}x \\
&= \frac{C_{N,p}}{erS^{\frac{rN}{p}}} \varepsilon^{\frac{(N-p)r}{p(p-1)} - \frac{(N-p)r}{p-1}} \int_{B_R(0)\setminus B_{R/2}(0)} \frac{1}{\left(1 + \left|\frac{x}{\varepsilon}\right|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}} \,\mathrm{d}x \\
&= \frac{C_{N,p}}{erS^{\frac{rN}{p}}} \varepsilon^{\frac{(N-p)r}{p(p-1)} - \frac{(N-p)r}{p-1} + N} \int_{B_{R/\varepsilon}(0)\setminus B_{R/2\varepsilon}(0)} \frac{1}{\left(1 + |y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}} \,\mathrm{d}y \\
&\le \frac{C_{N,p}}{erS^{\frac{rN}{p}}} \varepsilon^{\frac{(N-p)r}{p(p-1)} - \frac{(N-p)r}{p-1} + N} \int_{R/2\varepsilon}^{R/\varepsilon} \frac{t^{N-1}}{t^{\frac{(N-p)r}{p-1}}} \,\mathrm{d}t \\
&= C\varepsilon^{\frac{(N-p)r}{p(p-1)}} = \mathrm{O}(\varepsilon^{N - \frac{r(N-p)}{p}}),
\end{aligned} \tag{3.16}$$

for a suitable $C > 0$, where the last identity comes from $r > N(p-1)/(N-p)$. In order to estimate A_2 we first split as

$$\begin{aligned}
A_2 &= \int_{\mathbb{R}^N\setminus B_{R/2}(0)} \left(\frac{\phi_R U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^r \left|\log\left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)\right| \mathrm{d}x \\
&\quad + \int_{B_{R/2}(0)} \left(\frac{\phi_R U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^r \left|\log\left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)\right| \mathrm{d}x \\
&=: A_3 + A_4.
\end{aligned} \tag{3.17}$$

By (3.14) and Lemma 2.4, we obtain

$$
\begin{aligned}
|A_3| &\leq \int_{\mathbb{R}^N\setminus B_{R/2}(0)} \left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^r \left|\log\left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)\right| \,\mathrm{d}x\\
&\leq C_\sigma \int_{\mathbb{R}^N\setminus B_{R/2}(0)} \left(\left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^{r(1-\sigma)} + \left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^{r(1-\sigma)}\right) \mathrm{d}x\\
&\leq \frac{C_\sigma}{S^{\frac{r(1-\sigma)N}{p}}}\,\varepsilon^{\frac{(N-p)r(1-\sigma)}{p(p-1)}-\frac{(N-p)r(1-\sigma)}{p-1}+N}\\
&\qquad\times \int_{\mathbb{R}^N\setminus B_{R/2\varepsilon}(0)} \frac{1}{\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r(1-\sigma)}{p}}}\,\mathrm{d}x\\
&\quad+ \frac{C_\sigma}{S^{\frac{r(1+\sigma)N}{p}}}\,\varepsilon^{\frac{(N-p)r(1+\sigma)}{p(p-1)}-\frac{(N-p)r(1+\sigma)}{p-1}+N}\\
&\qquad\times \int_{\mathbb{R}^N\setminus B_{R/2\varepsilon}(0)} \frac{1}{\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r(1+\sigma)}{p}}}\,\mathrm{d}x\\
&\leq \frac{C_\sigma}{S^{\frac{r(1-\sigma)N}{p}}}\,\varepsilon^{\frac{(N-p)r(1-\sigma)}{p(p-1)}-\frac{(N-p)r(1-\sigma)}{p-1}+N} \int_{R/2\varepsilon}^\infty \frac{t^{N-1}}{t^{\frac{(N-p)r(1-\sigma)}{p}}}\,\mathrm{d}t\\
&\quad+ \frac{C_\sigma}{S^{\frac{r(1+\sigma)N}{p}}}\,\varepsilon^{\frac{(N-p)r(1+\sigma)}{p(p-1)}-\frac{(N-p)r(1+\sigma)}{p-1}+N} \int_{R/2\varepsilon}^\infty \frac{t^{N-1}}{t^{\frac{(N-p)r(1+\sigma)}{p}}}\,\mathrm{d}t\\
&\leq C\varepsilon^{\frac{(N-p)r(1-\sigma)}{p(p-1)}} = \mathrm{O}(\varepsilon^{N-\frac{r(N-p)}{p}}),
\end{aligned}
\tag{3.18}
$$

for a suitable $C>0$, by taking $\varepsilon>0$ and above all $\sigma>0$ sufficiently small such that

$$
\frac{(N-p)r(1-\sigma)}{p-1} > N > Np-(N-p)r.
$$

On the other hand, we have

$$
\begin{aligned}
A_4 &= \int_{B_{R/2}(0)} \left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)^r \left|\log\left(\frac{U_\varepsilon}{\|u_\varepsilon\|_{p^*}}\right)\right| \,\mathrm{d}x\\
&= C_{N,p}^r \int_{B_{R/2}(0)} \frac{\varepsilon^{\frac{(N-p)r}{p(p-1)}}}{\|u_\varepsilon\|_{p^*}^r \left(\varepsilon^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\\
&\qquad\times \left|\log\left(\frac{C_{N,p}\varepsilon^{\frac{(N-p)}{p(p-1)}}}{\|u_\varepsilon\|_{p^*}\left(\varepsilon^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}}\right)^{\frac{(N-p)}{p}}}\right)\right| \mathrm{d}x\\
&= C_{N,p}^r\,\varepsilon^{\frac{(N-p)r}{p(p-1)}-\frac{(N-p)r}{p-1}+N} \int_{B_{R/2\varepsilon}(0)} \frac{1}{\|u_\varepsilon\|_{p^*}^r\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\\
&\qquad\qquad\times \left|\log\left(\frac{C_{N,p}\varepsilon^{\frac{-(N-p)}{p}}}{\|u_\varepsilon\|_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)}{p}}}\right)\right| \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{C^r_{N,p}(N-p)}{p}\,\varepsilon^{\frac{(N-p)r}{p(p-1)}-\frac{(N-p)r}{p-1}+N}\log\left(\frac{1}{\varepsilon}\right)\\
&\qquad\times\int_{B_{R/2\varepsilon}(0)}\frac{1}{\|u_\varepsilon\|^r_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\,\mathrm{d}y\\
&\quad+C^r_{N,p}\,\varepsilon^{\frac{(N-p)r}{p(p-1)}-\frac{(N-p)r}{p-1}+N}\int_{B_{R/2\varepsilon}(0)}\frac{1}{\|u_\varepsilon\|^r_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\\
&\qquad\qquad\times\left|\log\left(\frac{C_{N,p}}{\|u_\varepsilon\|_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)}{p}}}\right)\right|\,\mathrm{d}y\\
&= \frac{C^r_{N,p}(N-p)}{p}\,\varepsilon^{\frac{(N-p)r}{p(p-1)}-\frac{(N-p)r}{p-1}+N}\log\left(\frac{1}{\varepsilon}\right)\\
&\qquad\times\int_{\mathbb{R}^N}\frac{1}{\|u_\varepsilon\|^r_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\,\mathrm{d}y\\
&\quad+\frac{C^r_{N,p}(N-p)}{p}\,\varepsilon^{\frac{(N-p)r}{p(p-1)}-\frac{(N-p)r}{p-1}+N}\log\left(\frac{1}{\varepsilon}\right)\\
&\qquad\times\int_{\mathbb{R}^N\setminus B_{R/2\varepsilon}(0)}\frac{1}{\|u_\varepsilon\|^r_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\,\mathrm{d}y\\
&\quad+C^r_{N,p}\,\varepsilon^{\frac{(N-p)r}{p(p-1)}-\frac{(N-p)r}{p-1}+N}\int_{B_{R/2\varepsilon}(0)}\frac{1}{\|u_\varepsilon\|^r_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\\
&\qquad\qquad\times\left|\log\left(\frac{C_{N,p}}{\|u_\varepsilon\|_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)}{p}}}\right)\right|\,\mathrm{d}y.
\end{aligned}
$$

By (3.14), we observe that

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus B_{R/2\varepsilon}(0)}\frac{1}{\|u_\varepsilon\|^r_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\,\mathrm{d}y &\le \frac{1}{S^{\frac{rN}{p}}}\int_{R/2\varepsilon}^{\infty}\frac{t^{N-1}}{t^{\frac{(N-p)r}{p-1}}}\,\mathrm{d}t\\
&\le C\varepsilon^{\frac{(N-p)r}{p-1}-N}=\mathrm{O}(\varepsilon^{\frac{(N-p)r}{p-1}-N}),
\end{aligned}
$$

for a suitable $C>0$. From (3.14) and Lemma 2.4, for a suitable $C>0$ independent by ε, we have

$$
\left|\int_{B_{R/2\varepsilon}(0)}\frac{1}{\|u_\varepsilon\|^r_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\right.
$$

$$
\begin{aligned}
&\times \left|\log\left(\frac{C_{N,p}}{\|u_\varepsilon\|_{p^*}\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)}{p}}}\right)\right| \mathrm{d}y\Bigg| \\
&\leq \frac{(|\log(C_{N,p})|+|\log(\|u_\varepsilon\|_{p^*})|)}{\|u_\varepsilon\|_{p^*}^r}\int_{\mathbb{R}^N}\frac{1}{\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\,\mathrm{d}y \\
&\quad + \frac{N-p}{p}\int_{\mathbb{R}^N}\frac{1}{\|u_\varepsilon\|_{p^*}^r\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\left|\log\left(1+|y|^{\frac{p}{p-1}}\right)\right|\mathrm{d}y \\
&\leq C\int_{\mathbb{R}^N}\frac{1}{\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}}}\,\mathrm{d}y \\
&\quad + \frac{N-p}{S^{\frac{N}{p}}pe\sigma}\int_{\mathbb{R}^N}\frac{1}{\left(1+|y|^{\frac{p}{p-1}}\right)^{\frac{(N-p)r}{p}-\sigma}}\,\mathrm{d}y<\infty
\end{aligned}
$$

while the last inequality holds true if we choose $\sigma>0$ small enough such that

$$
N-\frac{(N-p)r}{p-1}+\frac{p\sigma}{p-1}<0.
$$

Hence, combining the above calculations and taking into account (3.14), we obtain

$$
A_4 = C\varepsilon^{N-\frac{(N-p)r}{p}}\log\left(\frac{1}{\varepsilon}\right)+\mathrm{O}(\varepsilon^{N-\frac{r(N-p)}{p}}). \tag{3.19}
$$

Summing up (3.15), (3.16), (3.17), (3.18), and (3.19), we get

$$
\int_{\mathbb{R}^N}|u_\varepsilon|^r\,|\log(|u_\varepsilon|)|\,\mathrm{d}x = C\varepsilon^{N-\frac{(N-p)r}{p}}\log\left(\frac{1}{\varepsilon}\right)+\mathrm{O}(\varepsilon^{N-\frac{r(N-p)}{p}}).
$$

This completes the proof. □

We are now able to prove the estimate for $I_\lambda(tv_\varepsilon)$, with a suitable $t\geq 0$, which allows us to apply Lemma 3.3.

Lemma 3.5. *Let* (H_1)–(H_4) *be satisfied, let* $\lambda>0$ *and let* v_ε *be as in* (3.13)*. Then, then there exists* $\varepsilon>0$ *sufficiently small such that*

$$
\sup_{t\geq 0} I_\lambda(tv_\varepsilon)<\overline{c} \quad \textit{for any } \lambda>0.
$$

Proof. Let $\varepsilon>0$ and $\lambda>0$. By (H_4) and (3.13), we have

$$
\begin{aligned}
I_\lambda(tv_\varepsilon) &= \frac{t^p}{p}\left(\|\nabla v_\varepsilon\|_p^p+\|v_\varepsilon\|_p^p\right)-\|K_1\|_\infty\frac{t^{p^*}}{p^*} \\
&\quad -\lambda\|K_2\|_\infty\frac{t^r}{r}\int_{\mathbb{R}^N}|v_\varepsilon|^r\log(t|v_\varepsilon|)\,\mathrm{d}x+\lambda\|K_2\|_\infty\frac{t^r}{r^2}\int_{\mathbb{R}^N}|v_\varepsilon|^r\,\mathrm{d}x.
\end{aligned} \tag{3.20}
$$

Since $p<r<p^*$, we easily see that $\lim_{t\to 0}I_\lambda(tv_\varepsilon)=0$ and $\lim_{t\to\infty}I_\lambda(tv_\varepsilon)=-\infty$. Hence, there exists $t_\varepsilon\geq 0$ such that

$$
\sup_{t\geq 0}I_\lambda(tv_\varepsilon)=I_\lambda(t_\varepsilon v_\varepsilon).
$$

If $t_\varepsilon = 0$, the proof of the lemma follows immediately. On the other hand, if $t_\varepsilon > 0$, then using the fact that $\frac{\mathrm{d}}{\mathrm{d}t} I_\lambda(t_\varepsilon v_\varepsilon) = 0$, we obtain

$$\begin{aligned}
0 &= t_\varepsilon^{p-1}\left(\|\nabla v_\varepsilon\|_p^p + \|v_\varepsilon\|_p^p\right) - \|K_1\|_\infty t_\varepsilon^{p^*-1} \\
&\quad - \lambda \|K_2\|_\infty t_\varepsilon^{r-1} \log(t_\varepsilon) \int_{\mathbb{R}^N} |v_\varepsilon|^r \,\mathrm{d}x \\
&\quad - \lambda \|K_2\|_\infty t_\varepsilon^{r-1} \int_{\mathbb{R}^N} |v_\varepsilon|^r \log(|v_\varepsilon|) \,\mathrm{d}x.
\end{aligned} \tag{3.21}$$

Clearly, $\{t_\varepsilon\}_{\varepsilon>0}$ is bounded. Indeed, if $t_\varepsilon < e$ the claim holds trivially, while if $t_\varepsilon > e$ by (3.21), we have

$$\begin{aligned}
& t_\varepsilon^{p-1}\left(\|\nabla v_\varepsilon\|_p^p + \|v_\varepsilon\|_p^p\right) - \|K_1\|_\infty t_\varepsilon^{p^*-1} \\
&= \lambda \|K_2\|_\infty t_\varepsilon^{r-1} \int_{\mathbb{R}^N} |v_\varepsilon|^r \log(t_\varepsilon |v_\varepsilon|) \,\mathrm{d}x \\
&= \lambda \|K_2\|_\infty t_\varepsilon^{r-1} \int_{\mathbb{R}^N} |v_\varepsilon|^r \log(t_\varepsilon) \,\mathrm{d}x + \lambda \|K_2\|_\infty t_\varepsilon^{r-1} \int_{\mathbb{R}^N} |v_\varepsilon|^r \log(|v_\varepsilon|) \,\mathrm{d}x \\
&\geq \lambda \|K_2\|_\infty e^{r-1} \int_{\mathbb{R}^N} |v_\varepsilon|^r \,\mathrm{d}x + \lambda \|K_2\|_\infty e^{r-1} \int_{\mathbb{R}^N} |v_\varepsilon|^r \log(|v_\varepsilon|) \,\mathrm{d}x,
\end{aligned}$$

which gives the required boundedness. Furthermore, combining (3.21) with Lemma 3.4, for $\varepsilon > 0$ sufficiently small, we get

$$\begin{aligned}
0 &= t_\varepsilon^{p-1}\left(S + \mathrm{O}(\varepsilon^{(N-p)/(p-1)})\right) - \|K_1\|_\infty t_\varepsilon^{p^*-1} \\
&\quad - \lambda \|K_2\|_\infty t_\varepsilon^{r-1} \log(t_\varepsilon) C_1 \, \varepsilon^{N-r(N-p)/p} - \lambda \|K_2\|_\infty t_\varepsilon^{r-1} C_2 \, \varepsilon^{N-r(N-p)/p} \log\left(\frac{1}{\varepsilon}\right) \\
&\quad + \mathrm{O}(\varepsilon^{N-r(N-p)/p}),
\end{aligned}$$

from which, up to a subsequence, we have either $t_\varepsilon \to 0$ and the proof of the lemma is immediate, or

$$t_\varepsilon \to \left(\frac{S}{\|K_1\|_\infty}\right)^{1/(p^*-p)} \qquad \text{as } \varepsilon \to 0^+. \tag{3.22}$$

On the other hand, by setting

$$h(t) = \frac{S}{p} t^p - \frac{\|K_1\|_\infty}{p^*} t^{p^*}, \quad t > 0,$$

by direct calculation we have

$$\max_{t>0} h(t) = h\left(\left(\frac{S}{\|K_1\|_\infty}\right)^{1/(p^*-p)}\right) = \left(\frac{1}{p} - \frac{1}{p^*}\right) \frac{S^{\frac{p^*}{p^*-p}}}{\|K_1\|_\infty^{\frac{p}{p^*-p}}}. \tag{3.23}$$

Thus, for $\varepsilon > 0$ sufficiently small, by (H_4), (3.20), (3.22), (3.23) and Lemma 3.4, we obtain

$$\begin{aligned}
\sup_{t\geq 0} I_\lambda(t v_\varepsilon) &= I_\lambda(t_\varepsilon v_\varepsilon) \\
&\leq \frac{S}{p} t_\varepsilon^p + C \varepsilon^{(N-p)/(p-1)} - \frac{\|K_1\|_\infty}{p^*} t_\varepsilon^{p^*} \\
&\quad + \lambda \|K_2\|_\infty t_\varepsilon^r \left(\frac{1}{r^2} - \frac{\log(t_\varepsilon)}{r}\right) \int_{\mathbb{R}^N} |v_\varepsilon|^r \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \lambda\|K_2\|_\infty \frac{t_\varepsilon^r}{r}\int_{\mathbb{R}^N} |v_\varepsilon|^r \log(|v_\varepsilon|)\,\mathrm{d}x\\
&\leq \left(\frac{1}{p}-\frac{1}{p^*}\right)\frac{S^{\frac{p^*}{p^*-p}}}{\|K_1\|_\infty^{\frac{p}{p^*-p}}} + C\varepsilon^{(N-p)/(p-1)} + C\varepsilon^{N-r(N-p)/p}\\
&\quad - C\varepsilon^{N-r(N-p)/p}\log\left(\frac{1}{\varepsilon}\right)\\
&< \left(\frac{1}{p}-\frac{1}{p^*}\right)\frac{S^{\frac{p^*}{p^*-p}}}{\|K_1\|_\infty^{\frac{p}{p^*-p}}}
\end{aligned}
$$

with $C>0$ and $r>N(p-1)/(N-p)$ in the last inequality. This completes the proof. □

We conclude by studying the mountain pass geometry for I_λ in correspondence of v_ε, as set in Lemma 3.5.

Lemma 3.6. *Let* (H_1)–(H_3) *be satisfied, let $\lambda>0$ and let $\varepsilon>0$ be as set in Lemma 3.5. Then we have the following statements:*

- (i) *there exist $\delta>0$ and $\alpha>0$ such that $I_\lambda(u)\geq\alpha$ for any $u\in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with $\|u\|=\delta$;*
- (ii) *there exist $\tau_\varepsilon>0$ sufficiently large such that $\|\tau_\varepsilon v_\varepsilon\|>\delta$ and $I_\lambda(\tau_\varepsilon v_\varepsilon)<0$.*

Proof. Let $\lambda>0$ and let $\varepsilon>0$ be as set in Lemma 3.5. Let $u\in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with $\|u\|\leq 1$ and let $s>0$ such that $r+s\in(q,p^*)$. By Lemmas 2.1, 2.3 and 2.4 along with Hölder's and Young's inequalities, we get

$$
\begin{aligned}
I_\lambda(u) &\geq \frac{1}{q}\varrho(u) - \frac{\lambda}{r}\int_{\{x\in\mathbb{R}^N:\,|u(x)|>1\}} K_2(x)|u|^r\log(|u|)\,\mathrm{d}x - \frac{1}{p^*}\int_{\mathbb{R}^N} K_1(x)|u|^{p^*}\,\mathrm{d}x\\
&\geq \frac{1}{q}\|u\|^q - \frac{d_1}{p^*}\|u\|^{p^*} - \frac{d_2\lambda}{r}\|u\|^{r+s},
\end{aligned}
$$

where d_1, d_2 are positive constants. Since $q<r+s<p^*$, we can easily get (i) assuming $\|u\|$ sufficiently small.

On the other hand, we have

$$\lim_{t\to\infty} I_\lambda(tv_\varepsilon) = -\infty$$

from which we can conclude the proof. □

Proof of Theorem 1.1. Let $\lambda>0$ and let $\varepsilon>0$ be as set in Lemma 3.5. By Lemma 3.6 together with the mountain pass theorem without (PS) condition, see [34, Theorems 1.15 and 2.8], there exists a $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n\}_{n\in\mathbb{N}}\subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ of I_λ, at the positive critical mountain pass value given by

$$c_\lambda := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\lambda(\gamma(t))$$

with

$$\Gamma := \left\{\gamma\in C\left([0,1],W^{1,\mathcal{H}}(\mathbb{R}^N)\right) : \gamma(0)=0,\ \gamma(1)=t_\varepsilon v_\varepsilon\right\}.$$

By Lemma 3.4 we have

$$0<c_\lambda\leq\sup_{t\geq 0} I_\lambda(tv_\varepsilon)<\overline{c}.$$

Thus, we can apply Lemma 3.3 to $\{u_n\}_{n\in\mathbb{N}}$, so that there exists a nontrivial weak solution $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ of (1.2). □

Acknowledgments

A. Fiscella is member of *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). A. Fiscella realized the manuscript within the auspices of the CNPq project titled *Variational methods for problems with mixed nature and non-standard growth* (303986/2024-7), of the FAEPEX - PIND project titled *Problems and functionals with non-standard growth* (2583/25) and of the FAPESP project titled *Nonuniformly elliptic problems* (2024/04156-0).

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