STURM-LIOUVILLE EQUATIONS INVOLVING DISCONTINUOUS NONLINEARITIES

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Abstract. This paper deals with equations of Sturm-Liouville-type having nonlinearities on the right-hand side being possibly discontinuous. We present different existence results of such equations under various hypotheses on the nonlinearities. Our approach relies on critical point theory for locally Lipschitz functionals. In particular, under suitable assumptions, an existence result of a non-zero local minimum for locally Lipschitz functionals is established.

1. Introduction

The present paper is concerned with the existence of solutions to equations of Sturm-Liouville type having discontinuous nonlinearities on the right-hand side. Such equations are second-order differential equations of the form

$$-(\hat{p}u')' + \hat{q}u = \lambda \hat{r} f(u) \quad \text{in }]a,b[, \tag{1.1}$$

satisfying a boundary condition, for example of Dirichlet type, i.e. $u(a) = u(b) = 0$. Here, the given functions $\hat{p}, \hat{r} > 0$ and $\hat{q}$ are supposed to be integrable, λ is a parameter and $f : \mathbb{R} \to \mathbb{R}$ is a function. When $f(u) = u$, the function $\hat{r}$ is also known as the density or weight function and, as it is well known, equations of this type arise in different areas of pure and applied mathematics as well as quantum mechanics. For instance, the one-dimensional time-dependent Schrödinger equation is a special case of a Sturm-Liouville equation (see Teschl [23]). In addition, when f possesses discontinuous nonlinearities, equations of type (1.1) are special prototypes of several problems in Mechanics and Engineering (see, for instance, Motreanu-Panagiotopoulos [17] and Panagiotopoulos [19]).

The aim of this work is to present existence results to equations of type (1.1) where the functions $\hat{p}$, $\hat{q}$, $\hat{r}$ are supposed to be essentially bounded on $[a,b]$ and f is assumed to be almost everywhere continuous which allows functions having an uncountable set of points of discontinuity. The proof of our main result is based on an abstract nonsmooth critical point result developed in Section 2 which ensures the existence of at least one nontrivial local minimum. It should be pointed out that the derivation and application of such critical point results in recent years has been initiated by works of Ricceri [21] and [22], which can be seen as a starting point in that direction, and it has been developed in the papers of Marano-Motreanu [15] and [16], Bonanno-Candito [4] and Bonanno [2].

It is in the nature of things that Sturm-Liouville equations of type (1.1) have been treated by a wide range of authors in the past under different conditions

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on the nonlinearity. We mention the works of Afrouzi-Heidarkhani [1], Bonanno-Buccellato [3], Bonanno-Sciammetta [6], He-Ge [12], Henderson-Thompson [13] and the references therein. Indeed, our results extend those ones in [6] requiring L^1-Carathéodory functions to the case of discontinuous right-hand sides.

As a particular case of our main theorem we have the following result if $\hat{p} = \hat{r} = 1$ and $\hat{q} = 0$.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function satisfying $\inf_{\mathbb{R}} f > 0$. Then there exists a number $\overline{\lambda} > 0$ such that, for each $\lambda \in]0, \overline{\lambda}[$, problem*

$$\begin{aligned} -u'' &= \lambda f(u) \qquad \text{in }]a,b[, \\ u(a) &= u(b) = 0 \end{aligned} \tag{1.2}$$

admits at least one nontrivial positive solution.

We recall that a solution of (1.2) is a function $u \in C^1([a,b])$ such that u' is absolutely continuous, $u(a) = u(b) = 0$ and $-u''(t) = \lambda f(u(t))$ for almost every $t \in [a,b]$ (see Section 3). The proof of this theorem follows directly by applying Corollary 3.5 (see Section 3, Remark 3.6).

The paper is arranged as follows. In Section 2, we recall some basic facts about nonsmooth analysis and we establish an existence result of a non-zero local minimum for locally Lipschitz functionals (see Theorem 2.3). Moreover, two consequences (Theorems 2.4 and 2.5) are pointed out. In particular, Theorem 2.5 is a useful tool in order to establish nontrivial solutions to differential problems having discontinuous nonlinearities. Section 3 is devoted to a boundary value problem with Sturm-Liouville equation involving discontinuous nonlinearities, where the main result is Theorem 3.1 which ensures the existence of a nontrivial solution. Finally, in the same section, two helpful corollaries (Corollaries 3.4 and 3.5) and some concrete examples (Examples 3.7 and 3.8) to illustrate the applicability of our results, are pointed out.

2. Preliminaries

Let us start by recalling some basic notions in nonsmooth analysis that are required in the sequel. For a real Banach space $(X, \|\cdot\|_X)$, we denote by X^* its dual space and by $\langle \cdot, \cdot \rangle$ the duality pairing between X and X^*. A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exist a neighborhood U_x of x and a constant $L_x \geq 0$ such that

$$|f(y) - f(z)| \leq L_x \|y - z\|_X \quad \text{ for all } y, z \in U_x.$$

For a locally Lipschitz function $f : X \to \mathbb{R}$ on a Banach space X, the generalized directional derivative of f at the point $x \in X$ along the direction $y \in X$ is defined by

$$f^\circ(x; y) := \limsup_{z \to x, t \to 0^+} \frac{f(z + ty) - f(z)}{t}$$

(see Clarke [10, Chapter 2]). Note that if $f : X \to \mathbb{R}$ is strictly differentiable, that is, for all $x \in X$, $f'(x) \in X^*$ exists such that

$$\lim_{\substack{z \to x \\ t \to 0^+}} \frac{f(z + ty) - f(z)}{t} = \langle f'(x), y \rangle \quad \text{for all } y \in X,$$

then the usual directional derivative $f'(x;y)$ given by

$$f'(x;y) = \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$$

exists and coincides with the generalized directional derivative $f^\circ(x;y)$.

If $f_1,\ f_2 : X \to \mathbb{R}$ are locally Lipschitz functions, then we have

$$(f_1+f_2)^\circ(x;y) \le f_1^\circ(x;y) + f_2^\circ(x;y) \quad \text{for all } x,y\in X. \tag{2.1}$$

The generalized gradient of a locally Lipschitz function $f : X \to \mathbb{R}$ at $x \in X$ is the set

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y\rangle \le f^\circ(x;y) \quad \text{for all } y \in X\}.$$

Based on the Hahn-Banach theorem we easily verify that $\partial f(x)$ is nonempty. An element $x \in X$ is said to be a critical point of a locally Lipschitz function $f : X \to \mathbb{R}$ if there holds

$$f^\circ(x;y) \ge 0 \quad \text{for all } y \in X$$

or, equivalently, $0 \in \partial f(x)$ (see Chang [9]).

Let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions. We put

$$I = \Phi - \Psi.$$

We further fix two numbers $r_1, r_2 \in [-\infty, +\infty]$ such that $r_1 < r_2$. The following definition is a special version of the Palais-Smale condition ((PS) for short).

Definition 2.1. *We say that the function $I : X \to \mathbb{R}$ fulfills the Palais-Smale condition cut off lower at r_1 and upper at r_2 (${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition for short) if any sequence $(u_n) \subseteq X$ satisfying*

(1) *$I(u_n)$ is bounded;*
(2) *there exists a sequence $(\varepsilon_n) \subset \mathbb{R}_+, \varepsilon_n \to 0^+$ such that*

$$I^\circ(u_n; v) \ge -\varepsilon_n \|v\|_X \quad \textit{for all } v \in X;$$

(3) *$r_1 < \Phi(u_n) < r_2$ for all $n \in \mathbb{N}$;*

has a convergent subsequence. If $r_1 = -\infty$, $r_2 \in \mathbb{R}$, we write $(\mathrm{PS})^{[r_2]}$ and the case $r_1 \in \mathbb{R}$, $r_2 = +\infty$ will be denoted by ${}^{[r_1]}(\mathrm{PS})$.

It is easy to see that if $r_1 = -\infty$ and $r_2 = +\infty$, the definition above reduces to the well-known (PS)-condition for locally Lipschitz continuous functions (see Motreanu-Rădulescu [18, Definition 1.7]). We should also mention that if I fulfills the ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition, then it satisfies the ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition for all $s_1, s_2 \in [-\infty, +\infty]$ such that $r_1 \le s_1 < s_2 \le r_2$. Particularly, if I satisfies the usual (PS)-condition for locally Lipschitz continuous functions, then it fulfills the ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition for all $s_1, s_2 \in [-\infty, +\infty]$ with $s_1 < s_2$.

The following result is due to Bonanno [2, Lemma 3.1]

Lemma 2.2. *Let X be a real Banach space and let $I : X \to \mathbb{R}$ be a locally Lipschitz continuous function being bounded from below. Then, for all minimizing sequence $(u_n) \subseteq X$ of I, there exists a minimizing sequence $(v_n) \subseteq X$ of I such that*

(1) *$I(v_n) \le I(u_n)$ for all $n \in \mathbb{N}$;*
(2) *$I^\circ(v_n; h) \ge -\varepsilon_n \|h\|_X$ for all $h \in X$, for all $n \in \mathbb{N}$ and with $\varepsilon_n \to 0^+$.*

Now, we can prove the following.

Theorem 2.3. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions. Put*

$$I = \Phi - \Psi$$

and assume that there exist $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 < \Phi(x_0) < r_2$ such that

$$\sup_{u \in \Phi^{-1}(]r_1,r_2[)} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \tag{2.2}$$

$$\sup_{u \in \Phi^{-1}(]-\infty,r_1])} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \tag{2.3}$$

Furthermore, suppose that I satisfies the ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition.

Then, there exists $u_0 \in \Phi^{-1}(]r_1, r_2[)$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ with u_0 being a critical point of I.

Proof. We first put

$$C := r_2 - \Phi(x_0) + \Psi(x_0) \tag{2.4}$$

to define

$$\Phi_{r_1} := \max\{\Phi(u), r_1\}, \qquad \Psi_C := \min\{\Psi(u), C\}, \tag{2.5}$$

and

$$J := \Phi_{r_1} - \Psi_C.$$

Note that J is locally Lipschitz continuous and bounded from below. Taking into account Lemma 2.2, let $(u_n) \subseteq X$ be a minimizing sequence of J, i.e. $\lim_{n\to\infty} J(u_n) = \inf_X J$, we find a sequence $(v_n) \subseteq X$ of J such that $\lim_{n\to\infty} J(v_n) = \inf_X J$, and $J^\circ(v_n; h) \geq -\varepsilon_n \|h\|_X$ for all $h \in X$, for all $n \in \mathbb{N}$ and with $\varepsilon_n \to 0^+$.

If $J(x_0) = \inf_X J$, then from (2.2) and (2.5) we infer, for $u \in \Phi^{-1}(]r_1, r_2[)$, $\Psi(u) \leq C$ and therefore, $J(u) = I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$. Thus, $I(x_0) = J(x_0) \leq J(u) = I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$.

Let us now suppose that $\inf_X J < J(x_0)$. Then we find $n_0 > 0$ such that $J(v_n) < J(x_0)$ for all $n > n_0$.

Claim: $r_1 < \Phi(v_n) < r_2$ for all $n > n_0$

First observe

$$\Phi(v_n) - \Psi_C(v_n) \leq \Phi_{r_1}(v_n) - \Psi_C(v_n) < \Phi(x_0) - \Psi(x_0).$$

Hence

$$\Phi(v_n) < \Psi_C(v_n) + \Phi(x_0) - \Psi(x_0) \leq C + \Phi(x_0) - \Psi(x_0) = r_2 \tag{2.6}$$

implying $\Phi(v_n) < r_2$. Suppose now that $\Phi(v_n) \leq r_1$. This yields

$$r_1 - \Psi(v_n) = \Phi_{r_1}(v_n) - \Psi(v_n) < \Phi(x_0) - \Psi(x_0),$$

or equivalently

$$r_1 - \Phi(x_0) + \Psi(x_0) < \Psi(v_n).$$

Owing to (2.3) we obtain $\Phi(v_n) > r_1$ being a contradiction. This proves the claim.

Then, from the claim and (2.2) we obtain $J(v_n) = I(v_n)$ and $J^\circ(v_n; h) = I^\circ(v_n; h)$ for all $n > n_0$ and for all $h \in X$. Therefore, $\lim_{n\to\infty} I(v_n) = \lim_{n\to\infty} J(v_n) = \inf_X J$ and $I^\circ(v_n; h) \geq -\varepsilon_n \|h\|$ for all $h \in X$. Since I satisfies the ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition, we conclude that (v_n) admits a subsequence strongly converging to

$v^* \in X$. Thus, $I(v^*) = \inf_X J \leq J(u) = I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$. In summary we have

$$I(v^*) \leq I(u) \quad \text{for all } u \in \Phi^{-1}(]r_1, r_2[). \tag{2.7}$$

Due to the claim and because of the continuity of Φ there holds $v^* \in \Phi^{-1}([r_1, r_2])$. If $v^* \in \Phi^{-1}(]r_1, r_2[)$, the conclusion of the theorem follows directly from (2.7).

If $\Phi(v^*) = r_1$, then (2.3) implies

$$I(v^*) = r_1 - \Psi(v^*) \geq r_1 - \sup_{\Phi(u) \leq r_1} \Psi(u) \geq \Phi(x_0) - \Psi(x_0) = I(x_0).$$

This combined with (2.7) gives $I(x_0) \leq I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and the assertion of the theorem is proved.

If $\Phi(v^*) = r_2$, we have, due to $I(v^*) = J(v^*)$,

$$r_2 - \Psi(v^*) = r_2 - \Psi_C(v^*),$$

which implies $\Psi(v^*) = \Psi_C(v^*) \leq C$. Let us now suppose that $I(v^*) < I(x_0)$. Then (2.4) along with $\Psi(v^*) \leq C$ gives

$$I(v^*) = r_2 - \Psi(v^*) \geq r_2 - C = \Phi(x_0) - \Psi(x_0) = I(x_0),$$

a contradiction. Therefore, $I(v^*) = I(x_0)$ and (2.7) yields $I(x_0) \leq I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$. That finishes the proof of the theorem taking into account that each local minimum is also a critical point of I (see, for instance, [15, Proposition 2.1]). □

For a real Banach space X and locally Lipschitz continuous functions $\Phi, \Psi : X \to \mathbb{R}$ we define

$$I_\lambda = \Phi - \lambda\Psi$$

with $\lambda > 0$. Moreover, we put

$$\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup\limits_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)} \tag{2.8}$$

for all $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ and

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup\limits_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1} \tag{2.9}$$

for all $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$.

Theorem 2.4. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions. Suppose that there exist two numbers $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 < r_2$ such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2),$$

where β and ρ are as in (2.8) and (2.9), and for each $\lambda \in \left]\frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ fulfills the ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition.

Then, for each $\lambda \in \left]\frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right[$ there exists $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ with $u_{0,\lambda}$ being a critical point of I_λ.

Proof. Fixing $\lambda \in \left]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$, we have $\beta(r_1,r_2) < \frac{1}{\lambda} < \rho(r_1,r_2)$ which implies the existence of $v_1, v_2 \in \Phi^{-1}(]r_1,r_2[)$ such that

$$\frac{\sup\limits_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v_1)}{r_2 - \Phi(v_1)} < \frac{1}{\lambda} \quad \text{and} \quad \frac{1}{\lambda} < \frac{\Psi(v_2) - \sup\limits_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u)}{\Phi(v_2) - r_1}.$$

Now, let $x_0 \in \Phi^{-1}(]r_1,r_2[)$ be such that

$$\Phi(x_0) - \lambda\Psi(x_0) = \min\{\Phi(v_1) - \lambda\Psi(v_1), \Phi(v_2) - \lambda\Psi(v_2)\}.$$

This implies

$$\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \lambda\Psi(u) < r_2 - \Phi(x_0) + \lambda\Psi(x_0),$$

and

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1])} \lambda\Psi(u) < r_1 - \Phi(x_0) + \lambda\Psi(x_0).$$

Now, we may apply Theorem 2.3 to the function $I_\lambda = \Phi - \lambda\Psi$ which yields the assertion of the theorem. □

As a direct consequence of Theorem 2.4 we have the following result.

Theorem 2.5. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions satisfying $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Suppose that there exist $r \in \mathbb{R}$ and $\hat{u} \in X$ with $0 < \Phi(\hat{u}) < r$ such that*

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\hat{u})}{\Phi(\hat{u})} \tag{2.10}$$

and for each $\lambda \in \Lambda^{r,\hat{u}} := \left]\frac{\Phi(\hat{u})}{\Psi(\hat{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ fulfills the $(\mathrm{PS})^{[r]}$-condition.

Then, for each $\lambda \in \Lambda^{r,\hat{u}}$ there exists $u_\lambda \in \Phi^{-1}(]0,r[)$ such that $I_\lambda(u_\lambda) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]0,r[)$ with u_λ being a critical point of I_λ.

Proof. Our aim is to apply Theorem 2.4. To this end, let $r_1 = 0$ and $r_2 = r$. Then, owing to (2.10), we obtain

$$\beta(0,r) \le \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(\hat{u})}{r - \Phi(\hat{u})} < \frac{r\dfrac{\Psi(\hat{u})}{\Phi(\hat{u})} - \Psi(\hat{u})}{r - \Phi(\hat{u})} = \frac{\Psi(\hat{u})}{\Phi(\hat{u})} = \rho(0,r).$$

Now, let $(v_n) \subseteq \Phi^{-1}(]0,r[)$ such that $\lim_{n\to\infty} v_n = 0$. It follows

$$\beta(0,r) \le \frac{\sup\limits_{u\in\Phi^{-1}(]0,r[)} \Psi(u) - \Psi(v_n)}{r - \Phi(v_n)} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, since Φ, Ψ are continuous and $\Phi(0) = \Psi(0) = 0$, it results

$$\beta(0,r) \le \frac{\sup\limits_{u\in\Phi^{-1}(]0,r[)} \Psi(u)}{r}.$$

This finally gives

$$\left]\frac{\Phi(\hat{u})}{\Psi(\hat{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[\subseteq \left]\frac{1}{\rho(0,r)}, \frac{1}{\beta(0,r)}\right[.$$

Since I_λ satisfies the $(\mathrm{PS})^{[r]}$-condition, it fulfills the ${}^{[0]}(\mathrm{PS})^{[r]}$-condition as well. Hence, the assumptions of Theorem 2.4 are satisfied and the assertion of the theorem follows. □

Remark 2.6. In order to obtain the existence of multiple critical points for non-differentiable functionals, we refer to [7] and the references therein.

We say that a function $f : [a, b]\times\mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{H}$ if $x \to f(x, t)$ is measurable for every $t \in \mathbb{R}$, there exists a set $A \subset [a, b]$ with $m(A) = 0$ such that the set

$$D_f := \bigcup_{x\in[a,b]\setminus A} \{t \in \mathbb{R} : f(x, \cdot) \text{ is discontinuous at } t\} \tag{2.11}$$

has measure zero, f is locally essentially bounded, and the functions

$$f^-(x,t) := \lim_{\delta\to 0^+} \operatorname*{essinf}_{|t-z|<\delta} f(x,z), \qquad f^+(x,t) := \lim_{\delta\to 0^+} \operatorname*{esssup}_{|t-z|<\delta} f(x,z),$$

are superpositionally measurable, that is, $f^-(x, u(x))$ and $f^+(x, u(x))$ are measurable for all measurable functions $u : [a, b] \to \mathbb{R}$. Functions belonging to $\mathcal{H}$ are sometimes called highly discontinuous.

3. Main result

In this section, we are going to apply the abstract results of Section 2 to suitable differential equations. To this end, we consider the following Sturm-Liouville boundary value problem

$$\begin{aligned} -(pu')' + qu &= \lambda f(x, u) \quad \text{in }]a, b[, \\ u(a) = u(b) &= 0 \end{aligned} \tag{3.1}$$

where $p, q \in L^\infty([a, b])$ fulfilling $\operatorname{essinf}_{[a,b]} p > 0$, $\operatorname{essinf}_{[a,b]} q \geq 0$, $\lambda > 0$ is a parameter to be specified, and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function.

If f and q are continuous functions, then a classical solution of (3.1) is a function $u \in C^1([a, b])$ such that $pu' \in C^1([a, b])$, $u(a) = u(b) = 0$, and $-(p(x)u'(x))' + q(x)u(x) = \lambda f(x, u(x))$ for all $x \in [a, b]$. If $u \in AC([a, b])$ satisfies $pu' \in AC([a, b])$, $u(a) = u(b) = 0$, and $-(p(x)u'(x))'+q(x)u(x) = \lambda f(x, u(x))$ for almost all $x \in [a, b]$, then u is called a generalized solution of (3.1). Finally, we say that $u \in W^{1,2}_0([a, b])$ is a weak solution of (3.1) if

$$\int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx = \lambda \int_a^b f(x, u(x))v(x)dx$$

is satisfied for all test functions $v \in W^{1,2}_0([a, b])$. Note that a weak solution of (3.1) is also a generalized solution.

In what follows we identify by X the Sobolev space $W_0^{1,2}([a,b])$ equipped with the norm

$$\|u\|_X = \left(\int_a^b p|u'|^2 dx + \int_a^b q|u|^2 dx\right)^{\frac{1}{2}} \quad \text{for all } u \in X.$$

We set

$$p_0 := \operatorname*{essinf}_{x\in[a,b]} p(x) > 0, \qquad q_0 := \operatorname*{essinf}_{x\in[a,b]} q(x) \geq 0,$$

and

$$m := \frac{2p_0}{b-a}. \tag{3.2}$$

It is well known that X is compactly embedded into $C([a,b])$ with the estimate

$$\|u\|_\infty \leq \frac{1}{\sqrt{2m}}\|u\|_X. \tag{3.3}$$

Finally, we put

$$K := \frac{6p_0}{12\|p\|_\infty + (b-a)^2\|q\|_\infty}. \tag{3.4}$$

The main result in this paper is the following.

Theorem 3.1. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a function belonging to class $\mathcal{H}$. Put $F(x,s) := \int_0^s f(x,t)dt$ for all $(x,s) \in [a,b] \times \mathbb{R}$ and suppose that*

(H1) *there exist two positive constants c, d with $d < c$ such that*

$$F(x,s) \geq 0 \quad \text{for all } (x,s) \in [a,b] \times [0,d] \tag{3.5}$$

and

$$\frac{\int_a^b \max_{|s|\leq c} F(x,s)dx}{c^2} < K \frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{d^2}; \tag{3.6}$$

(H2) *there holds*

$$\lambda f^-(x,s) - q(x)s \leq 0 \leq \lambda f^+(x,s) - q(x)s \quad \text{implies} \quad \lambda f(x,s) - q(x)s = 0$$

for a.a. $x \in [a,b]$, for all $s \in D_f$ (see (2.11)), and for each $\lambda \in \Lambda_{c,d}$, where

$$\Lambda_{c,d} := \left] \frac{1}{K} \frac{md^2}{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}, \frac{mc^2}{\int_a^b \max_{|s|\leq c} F(x,s)dx} \right[.$$

Then, for each $\lambda \in \Lambda_{c,d}$, problem (3.1) admits at least one nontrivial weak solution u_λ such that $|u_\lambda(x)| < c$ for all $x \in [a,b]$.

Proof. First, we put

$$\Phi(u) := \frac{1}{2}\|u\|_X^2, \qquad \Psi(u) := \int_a^b F(x,u)dx$$

for all $u \in X$. It is clear that both Φ and Ψ are locally Lipschitz on X.

Fixing $\lambda \in \Lambda_{c,d}$, we are going to prove that critical points of $\Phi - \lambda\Psi$ are weak solutions of problem (3.1). For this purpose, let $u_0 \in X$ be a critical point of $\Phi - \lambda\Psi$, that is

$$(\Phi - \lambda\Psi)^\circ (u_0; v - u_0) \geq 0 \quad \text{for all } v \in X,$$

which implies $\Phi'(u_0)(w) + \lambda (-\Psi)^\circ (u_0; w) \geq 0$ for all $w \in X$ (see (2.1)), that is

$$-\left(\int_a^b p(x)u_0'(x)w'(x)dx + \int_a^b q(x)u_0(x)w(x)dx\right) \leq \lambda (-\Psi)^\circ (u_0; w) \quad \text{for all } w \in X.$$

Defining

$$T(w) := -\left(\int_a^b p(x)u_0'(x)w'(x)dx + \int_a^b q(x)u_0(x)w(x)dx\right) \quad \text{for all } w \in X$$

we easily verify that T is a linear and continuous operator on X satisfying $T \in \lambda\partial(-\Psi)(u_0)$. Note that Ψ is locally Lipschitz on $L^2([a,b])$ and X is densely embedded into $L^2([a,b])$. Hence, by virtue of Theorem 2.2 of Chang [9], we have $\partial(-\Psi)\big|_X(u_0) \subseteq \partial(-\Psi)\big|_{L^2([a,b])}(u_0)$ which implies that T is a linear and continuous operator on $L^2([a,b])$. Thus, we find an element $\hat{w} \in L^2([a,b])$ such that

$$T(w) = \int_a^b w(x)\hat{w}(x)dx \quad \text{for all } w \in L^2([a,b]).$$

Let us now consider the auxiliary problem given by

$$\begin{gathered} -(pu')' - qu = \hat{w} \qquad \text{in }]a,b[, \\ u(a) = u(b) = 0. \end{gathered} \tag{3.7}$$

It is known that problem (3.7) has a unique weak solution $\hat{u} \in W^{2,2}([a,b]) \cap X$ (see, for example, Brezis [8, Chapter VIII.4, Example 2]), that is

$$-\left(\int_a^b p(x)\hat{u}'(x)w'(x)dx + \int_a^b q(x)\hat{u}(x)w(x)dx\right) = \int_a^b \hat{w}(x)w(x)dx = T(w)$$

for all $w \in X$. Since a linear continuous operator on X is uniquely determined by a function in X (see, for example, Kufner-John-Fučík [14, Theorem 5.9.3]) we have $\hat{u} = u_0$ which gives $u_0 \in W^{2,2}([a,b])$ and

$$\begin{aligned} &\int_a^b (p(x)u_0'(x))'w(x)dx - \int_a^b q(x)u_0(x)w(x)dx \\ &= -\left(\int_a^b p(x)u_0'(x)w'(x)dx + \int_a^b q(x)u_0(x)w(x)dx\right) \\ &\leq \lambda(-\Psi)^\circ(u_0; w) \quad \text{for all } w \in X. \end{aligned}$$

Then, owing to Theorem 2.1 of Chang [9], one get

$$(p(x)u_0'(x))' - q(x)u_0(x) \in \left[(-\lambda f)^- (x, (u_0(x)), (-\lambda f)^+ (x, u_0(x))\right]$$

for almost all $x \in [a,b]$ which can be equivalently written as

$$-(p(x)u_0'(x))' \in \left[\lambda f^-(x, u_0(x)) - q(x)u_0(x), \lambda f^+(x, u_0(x)) - q(x)u_0(x)\right]$$

for almost all $x \in [a,b]$. Then, as $f \in \mathcal{H}$, it follows

$$-(p(x)u_0'(x))' + q(x)u_0(x) = \lambda f(x, u_0(x)) \quad \text{for almost all } x \in u_0^{-1}(D_f). \tag{3.8}$$

Indeed, since $m(D_f) = 0$, due to De Giorgi-Buttazzo-Dal Maso [11], we obtain $-(p(x)u_0'(x))' = 0$ for almost all $x \in u_0^{-1}(D_f)$. Now, from (3.8) and hypothesis (H2), we conclude that

$$\lambda f(x, u_0(x)) - q(x)u_0(x) = 0 \quad \text{for almost all } x \in u_0^{-1}(D_f).$$

Hence, (3.8) holds and therefore

$$-(p(x)u_0'(x))' + q(x)u_0(x) = \lambda f(x, u_0(x)) \quad \text{for almost all } x \in [a, b],$$

which ensures that a critical point of $\Phi - \lambda\Psi$ is a weak solution of (3.1).

Our aim is now to apply Theorem 2.5. First we are going to prove that inequality (2.10) is satisfied. To this end, let $r := mc^2$ and define

$$\hat{u}(x) = \begin{cases} \frac{4d}{b-a}(x-a) & \text{if } x \in \left[a, a+\frac{1}{4}(b-a)\right[, \\ d & \text{if } x \in \left[a+\frac{1}{4}(b-a), b-\frac{1}{4}(b-a)\right], \\ \frac{4d}{b-a}(b-x) & \text{if } x \in \left]b-\frac{1}{4}(b-a), b\right] \end{cases}$$

which belongs obviously to X. A simple calculation shows

$$4d^2\left(\frac{p_0}{b-a} + \frac{q_0}{12}(b-a)\right) \leq \frac{1}{2}\|\hat{u}\|_X^2 \leq 4d^2\left(\frac{\|p\|_\infty}{b-a} + \frac{\|q\|_\infty}{12}(b-a)\right). \tag{3.9}$$

Now, we claim that $\frac{1}{\sqrt{K}}d < c$. Arguing by contradiction, we assume that

$$c \leq \frac{1}{\sqrt{K}}d. \tag{3.10}$$

Since $d < c$ and due to (3.10) along with (3.5) we derive

$$\begin{aligned} \frac{\int_a^b \max_{|s|\leq c} F(x,s)dx}{c^2} &\geq \frac{\int_a^b F(x,d)dx}{c^2} \geq \frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{c^2} \\ &\geq \frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{\left(\frac{1}{\sqrt{K}}d\right)^2} = K\frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{d^2}, \end{aligned}$$

which contradicts (3.6). Thus, $\frac{1}{\sqrt{K}}d < c$.

This gives $\frac{1}{K}d^2 < c^2$, that is

$$\frac{12\|p\|_\infty + (b-a)^2\|q\|_\infty}{6p_0}d^2 < c^2.$$

The last inequality can be rewritten as

$$4\left(\frac{\|p\|_\infty}{b-a} + \frac{\|q\|_\infty}{12}(b-a)\right)d^2 < \frac{2p_0}{b-a}c^2$$

which implies, due to (3.9) and the choice $r = mc^2$ (see also (3.2)), that $\Phi(\hat{u}) < r$.

Furthermore, owing to (3.3) and the representation $r = mc^2$, there holds

$$\max_{x\in[a,b]} |u(x)| \leq c \quad \text{for all } \|u\|_X \leq \sqrt{2r}. \tag{3.11}$$

Then, we have

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} = \frac{\sup\limits_{\|u\|_X\le\sqrt{2r}}\displaystyle\int_a^b F(x,u(x))dx}{r} \le \frac{\displaystyle\int_a^b \max_{|s|\le c} F(x,s)dx}{mc^2}. \tag{3.12}$$

Moreover, by applying (3.5) and (3.9), we obtain

$$\begin{aligned}\frac{\Psi(\hat{u})}{\Psi(\hat{u})} &= \frac{\displaystyle\int_a^b F(x,\hat{u}(x))\,dx}{\frac{1}{2}\|\hat{u}\|_X^2} \ge \frac{\displaystyle\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)\,dx}{4d^2\left(\frac{\|p\|_\infty}{b-a}+\frac{\|q\|_\infty}{12}(b-a)\right)} \\ &= K\frac{\displaystyle\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)\,dx}{md^2}.\end{aligned} \tag{3.13}$$

Hence, due to (3.12) as well as (3.13) along with (3.6) we see that

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{\Psi(\hat{u})}{\Phi(\hat{u})} \quad \text{and} \quad \Lambda_{c,d} \subseteq \left]\frac{\Phi(\hat{u})}{\Psi(\hat{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[.$$

Consequently, condition (2.10) in Theorem 2.5 is satisfied. We only have to show that $\Phi-\lambda\Psi$ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $\lambda\in\Lambda^{r,\hat{u}}$. To this end, let $(u_n)\subseteq X$ be a sequence such that

(a) $(\Phi-\lambda\Psi)(u_n)$ is bounded;

(b) there exists a sequence $(\varepsilon_n)\subset\mathbb{R}_+, \varepsilon_n\to 0^+$ such that

$$(\Phi-\lambda\Psi)^\circ(u_n;v) \ge -\varepsilon_n\|v\|_X \quad \text{for all } v\in X;$$

(c) $\Phi(u_n)<r$ for all $n\in\mathbb{N}$.

From (c) we directly see that (u_n) is a bounded sequence in X. Therefore, we may assume that

$$u_n \rightharpoonup u \text{ in } X \quad \text{and} \quad u_n\to u \text{ in } L^2([a,b]) \tag{3.14}$$

because of the compact embedding $X\hookrightarrow L^2([a,b])$. Taking $v=u-u_n$ in (b), it gives

$$\Phi'(u_n)(u-u_n)+\lambda(-\Psi)^\circ(u_n;u-u_n) \ge -\varepsilon_n\|u-u_n\|_X. \tag{3.15}$$

By virtue of Young's inequality, we obtain

$$\begin{aligned}&\Phi'(u_n)(u-u_n)\\ &= \int_a^b p(x)u_n'(x)\left(u'(x)-u_n'(x)\right)dx + \int_a^b q(x)u_n(x)\left(u(x)-u_n(x)\right)dx\\ &\le \frac{1}{2}\left(\int_a^b \left[p(x)|u_n'(x)|^2+q(x)|u_n(x)|^2+p(x)|u'(x)|^2+q(x)|u(x)|^2\right]dx\right) - \|u_n\|_X^2\\ &= \frac{1}{2}\|u_n\|_X^2+\frac{1}{2}\|u\|_X^2-\|u_n\|_X^2 = \frac{1}{2}\|u\|_X^2-\frac{1}{2}\|u_n\|_X^2.\end{aligned}$$

Applying the last inequality to (3.15), one has

$$-\varepsilon_n\|u-u_n\|_X+\frac{1}{2}\|u_n\|_X^2 \le \frac{1}{2}\|u\|_X^2+\lambda(-\Psi)^\circ(u_n;u-u_n). \tag{3.16}$$

Note that Ψ is well-defined and locally Lipschitz on $L^2([a,b])$. Since $\left(-\Psi\big|_X\right)^\circ(u;v) \le (-\Psi)^\circ\big|_X(u;v)$ for all $u,v \in X$ (see Chang [9, Proof of Theorem 2.2, p.111], the upper semicontinuity of $(-\Psi)^\circ$ in the strong topology of $L^2([a,b]) \times L^2([a,b])$ (see, Clarke [10, Proposition 2.1.1]) implies that

$$\limsup_{n\to\infty} (-\Psi)^\circ (u_n; u-u_n) \le 0. \tag{3.17}$$

Passing to the upper limit and using (3.17), inequality (3.16) becomes

$$\limsup_{n\to\infty} \|u_n\|_X \le \|u\|_X. \tag{3.18}$$

Since X is uniformly convex and due to (3.14) and (3.18), we have $u_n \to u$ in X (see, Brezis [8, Proposition III.30]). Hence, the functional $\Phi-\lambda\Psi$ fulfills the $(\mathrm{PS})^{[r]}$-condition. Therefore, the assumptions of Theorem 2.5 are satisfied which implies the existence of a critical point $u_\lambda \in X$ of $\Phi-\lambda\Psi$. As shown in the beginning of this proof, we know that a critical point of $\Phi-\lambda\Psi$ is a weak solution of (3.1) as well. In addition, Theorem 2.5 implies that $u_\lambda \in \Phi^{-1}(]0,r[)$ which ensures that u_λ is nontrivial due to the choice of Φ, that is, $0<\|u_\lambda\|_X<\sqrt{2mc}$. On the other side (3.3) gives $|u_\lambda(x)| \le \frac{1}{\sqrt{2m}}\|u_\lambda\|_X$. This proves the assertion from the theorem. □

Remark 3.2. We observe that Theorem 3.1 and its consequences can be applied to study problems with a complete Sturm-Liouville equation. To be precise, given the problem

$$\begin{aligned} -(pu')' + ru' + qu &= \lambda f(x,u) \quad \text{in }]a,b[,\\ u(a) = u(b) &= 0 \end{aligned} \tag{3.19}$$

where $p,r,q \in L^\infty([a,b])$ fulfilling $\operatorname{essinf}_{[a,b]} p > 0$ and $\operatorname{essinf}_{[a,b]} q \ge 0$, we can apply Theorem 3.1 arguing exactly as in [5, Section 4]. Indeed, called R a primitive of $\frac{r}{p}$, it is enough to observe that the generalized solutions of the problem

$$\begin{aligned} -(e^{-R}pu')' + e^{-R}qu &= \lambda e^{-R} f(x,u) \quad \text{in }]a,b[,\\ u(a) = u(b) &= 0 \end{aligned}$$

are generalized solutions of the problem (3.19).

Remark 3.3. Note that if f is independent of x the hypothesis (H1) of Theorem 3.1 becomes

(H1') *there exist two positive constants c,d with $d<c$ such that*

$$F(s) \ge 0 \quad \textit{for all } s \in [0,d] \tag{3.20}$$

and

$$\frac{\max_{|s|\le c} F(s)}{c^2} < \frac{K}{2}\frac{F(d)}{d^2}. \tag{3.21}$$

The interval is then

$$\Lambda_{c,d} := \left] \frac{2}{K}\frac{m}{b-a}\frac{d^2}{F(d)}, \frac{m}{b-a}\frac{c^2}{\max_{|s|\le c} F(s)} \right[.$$

We also mention that if $f:\mathbb{R}\to\mathbb{R}$ is a locally essentially bounded and almost everywhere continuous function, then it belongs to the class $\mathcal{H}$.

A direct consequence of Theorem 3.1 is the following result.

Corollary 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is locally essentially bounded and almost everywhere continuous. Put $F(s) := \int_0^s f(t)dt$ and suppose that (H1') and*

(K2) *for each $\hat{s} \in D_f$ there exists a neighborhood U of $\hat{s}$ such that*

$$\inf_U f > \frac{p_0 K}{m^2} \frac{F(d)}{d^2} \max\{\|q\|_\infty \hat{s}, q_0 \hat{s}\};$$

are satisfied. Then, for each $\lambda \in \Lambda_{c,d}$, problem (3.1) admits at least one nontrivial weak solution u_λ such that $|u_\lambda(x)| < c$ for all $x \in [a, b]$.

Proof. Our aim is to apply Theorem 3.1. Note that f belongs to the class $\mathcal{H}$ and since f is independent of x, hypothesis (H1) of Theorem 3.1 is fulfilled (see Remark 3.3).

In order to prove (H2) of Theorem 3.1, let $\hat{s} > 0$. By virtue of hypothesis (K2) it follows

$$f(s) - \frac{p_0 K}{m^2} \frac{F(d)}{d^2} \|q\|_\infty \hat{s} \geq \inf_U f - \frac{p_0 K}{m^2} \frac{F(d)}{d^2} \|q\|_\infty \hat{s} =: h > 0 \quad \text{for all } s \in U,$$

which implies

$$\frac{2m}{(b-a)K} \frac{d^2}{F(d)} f(s) - \|q\|_\infty \hat{s} \geq h \frac{2m}{(b-a)K} \frac{d^2}{F(d)} \quad \text{for all } s \in U.$$

Therefore,

$$\lambda f(s) - q(x)\hat{s} \geq h \frac{2m}{(b-a)K} \frac{d^2}{F(d)} \quad \text{for all } \lambda > \frac{2m}{(b-a)K} \frac{d^2}{F(d)},$$

for all $s \in U$, and for a.a. $x \in [a, b]$. This gives

$$\lambda f^-(\hat{s}) - q(x)\hat{s} > 0 \quad \text{for a.a. } x \in [a, b] \text{ and for all } \lambda > \frac{2m}{(b-a)K} \frac{d^2}{F(d)}.$$

The same can be done assuming $\hat{s} \leq 0$ by arguing with q_0 instead of $\|q\|_\infty$. Hence, (H2) of Theorem 3.1 is fulfilled which implies the existence of a nontrivial weak solution u_λ of (3.1) having the required properties. □

Based on the strong maximum principle, we have the following special case of our result.

Corollary 3.5. *If $f : \mathbb{R} \to \mathbb{R}$ is locally essentially bounded, almost everywhere continuous and nonnegative, satisfying*

(K1) $\displaystyle\lim_{s \to 0^+} \frac{f(s)}{s} = +\infty,$
and

(K2') *for all $s \in D_f$ such that $f^-(s) = 0$ one has $f(s) = 0$.*

Then, for each $c > 0$, there exists $\lambda_c^* = \frac{c^2}{F(c)}$ such that, for each $\lambda \in]0, \lambda_c^*[$, problem (1.2) admits at least one positive weak solution u_λ such that $|u_\lambda(x)| < c$ for all $x \in [a, b]$.

Proof. Owing to (K1), we see that we can find $\delta > 0$ such that $f(t) > 0$ for all $t \in]0, \delta[$. Thus, $\frac{c^2}{F(c)} > 0$ for all $c \in]0, \delta[$ and $\lambda_c^* > 0$. Fixing $\lambda \in]0, \lambda_c^*[$ implies the existence of $c \in]0, \delta[$ such that

$$\frac{1}{\lambda} > \frac{F(c)}{c^2}. \tag{3.22}$$

On the other hand, thanks again to (K1), we have

$$\lim_{s \to 0^+} \frac{F(s)}{s^2} = +\infty.$$

Then we find a number $d > 0$ such that $d < c$ and

$$\frac{K}{2} \frac{F(d)}{d^2} > \frac{1}{\lambda}. \tag{3.23}$$

Combining (3.22) and (3.23) yields

$$\frac{F(c)}{c^2} < \frac{K}{2} \frac{F(d)}{d^2}$$

with $d < c$. Hence, taking the nonnegativity of f into account, assumption (H1') is satisfied (see Remark 3.3). Moreover, as a simple computation shows, (K2') implies (H2).

As the assumptions of Theorem 3.1 are satisfied, we obtain the existence of a nontrivial weak solution u_λ of (1.2) satisfying $|u_\lambda(x)| < c$ for all $x \in [a, b]$. Taking $v = u^- = \max(-u, 0) \in W_0^{1,2}([a, b])$ as test function in the weak formulation of (3.1) gives $u \geq 0$ since f is positive on $\mathbb{R}$. Moreover, we have

$$u''(x) - [-\lambda f(u(x))] = 0 \quad \text{in }]a, b[. \tag{3.24}$$

Owing to (3.24) and due to the nonnegativity of f we see that the assumptions of Pucci-Serrin [20, Theorem 11.1] are satisfied. This gives $u \equiv 0$ in $]a, b[$ or $u > 0$ in $]a, b[$. Since u is nontrivial it cannot be identically zero. Hence, $u > 0$ in $]a, b[$. □

Remark 3.6. Theorem 1.1 is a special case of Corollary 3.5. Indeed, since $\inf_{\mathbb{R}} f > 0$, there holds

$$\frac{f(s)}{s} \geq \frac{\inf_{s \in \mathbb{R}} f(s)}{s} \to +\infty \quad \text{as } s \to 0^+.$$

Therefore, (K1) is satisfied. Finally, because of $\inf_{\mathbb{R}} f > 0$ it is easy to see that (K2') is fulfilled as well.

Let us give some examples of functions satisfying the assumptions of Corollary 3.5.

Example 3.7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \in C, \\ 2 & \text{if } x \notin C, \end{cases}$$

where C is the Cantor set. One easily verifies that f is continuous in every $x \notin C$ and since the Lebesgue measure of C is zero we conclude that f is almost everywhere continuous. Of course, Corollary 3.5 is satisfied. We note that in this case the set of discontinuity points of f is uncountable.

Example 3.8. Given $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} |x|^{\frac{1}{2}} & \text{if } x < 1, \\ e^x & \text{if } x \geq 1, \end{cases}$$

it is easy to see that the hypotheses of Corollary 3.5 are fulfilled.

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