ON KIRCHHOFF DOUBLE PHASE PROBLEMS WITH LOGARITHMIC PERTURBATION AND VARIABLE EXPONENTS

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Abstract. In this paper, we are interested in Kirchhoff problems driven by a double phase operator with a logarithmic perturbation and with variable exponents. Employing variational methods, we first establish the existence of at least one nontrivial weak solution for the problem under consideration, supposed that the nonlinearity satisfies very general assumptions. Moreover, via modifying slightly the hypotheses on the reaction term and making use of a variant of the symmetric mountain pass theorem, we also produce infinitely many solutions for our problem.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $C^{0,1}(\overline{\Omega})$ be the space of all Lipschitz continuous functions $u : \overline{\Omega} \to \mathbb{R}$. In the present article, the functions p, q and μ are supposed to satisfy the following assumptions:

(H1) $p, q \in C^{0,1}(\overline{\Omega})$ are such that $1 < p(x) < N$ and

$$p(x) < q(x) < 2\,q^+ < (p^*)^- \leq p^*(x) := \frac{Np(x)}{N-p(x)}$$

for all $x \in \overline{\Omega}$, where for $m \in C(\overline{\Omega})$ with $m(x) > 1$ for all $x \in \overline{\Omega}$ we put

$$m^- := \min_{x\in\overline{\Omega}} m(x) \quad \text{and} \quad m^+ := \max_{x\in\overline{\Omega}} m(x);$$

$\mu \in L^\infty(\Omega) \setminus \{0\}$ is such that

$$\mu(x) \geq 0 \quad \text{for a.a.}\, x \in \Omega.$$

Also, e stands for Euler's number and we consider $a > 0$ such that $ap^- \geq 1$. In this paper, we focus on problems driven by logarithmic operators with variable exponents and a-logarithmic perturbation given by

$$\begin{aligned} \operatorname{div}\mathcal{L}(u) := \operatorname{div}\Bigg[&\left(|\nabla u|^{p(x)-2} + \mu(x)|\nabla u|^{q(x)-2}\right)\nabla u \log(e + a|\nabla u|) + \\ &\left(\frac{1}{p(x)}|\nabla u|^{p(x)-2} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)-2}\right)\nabla u\,\frac{|\nabla u|}{e + a|\nabla u|}\Bigg], \end{aligned} \tag{1.1}$$

for any function u belonging to an appropriate Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}_L}(\Omega)$, which will be defined in Section 2. This operator and the related Musielak-Orlicz Sobolev space were recently introduced by Lu–Vetro–Zeng [24],

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who studied existence and uniqueness of equations involving such an operator in the context of a nonlinear problem with convection (that is, the nonlinearity depends on the weak gradient of the solution). We point out that the operator in (1.1) is connected to integral functionals of the form

$$u \mapsto \int_\Omega (|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}) \log(e + a|\nabla u|)\,\mathrm{d}x,$$

which originates from functionals with nearly linear growth of type

$$u \mapsto \int_\Omega |\nabla u| \log(1 + |\nabla u|)\,\mathrm{d}x. \tag{1.2}$$

Functionals as in (1.2) have been considered by Fuchs–Mingione [18] and Marcellini–Papi [25]. We stress that such type of functional appears in the context of plasticity with logarithmic hardening, as one can see for example by Seregin–Frehse [28].

In the present work, we focus on the following Kirchhoff type problem

$$\begin{aligned} -\mathcal{K}\left[\int_\Omega \Phi(u)\log(e+a|\nabla u|)\,\mathrm{d}x\right] \operatorname{div} \mathcal{L}(u) &= f(x,u) \qquad &&\text{in } \Omega,\\ u &= 0 &&\text{on } \partial\Omega \end{aligned} \tag{1.3}$$

with

$$\Phi(u) = \frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)},$$

where the operator $\operatorname{div}\mathcal{L}$ is as in (1.1), a is such that $ap^- \geq 1$, while $\mathcal{K} = K'$ with a function $K\colon [0,+\infty) \to [0,+\infty)$ satisfying the following assumptions:

(H2) $K \in C^1([0,+\infty))$ is such that
  (i) there exists $\vartheta \in \left[1, \frac{(p^*)^-}{2\,q^+}\right)$ so that the function
  $$\frac{K(t)}{t^\vartheta} \quad \text{is nonincreasing on } (0,+\infty);$$
  (ii) there exists $k > 0$ such that
  $$\mathcal{K}(t) \geq k \quad \text{for all } t \geq 1.$$

Example 1.1. *For $k > 0$ and $\vartheta \in \left[1, \frac{(p^*)^-}{2\,q^+}\right)$ fixed, we consider the function $K\colon [0,+\infty) \to [0,+\infty)$ defined by*

$$K(t) := kt + t^\vartheta \quad \text{for all } t \geq 0.$$

We point out that $K(\cdot)$ satisfies all the assumptions in hypotheses (H2).

We recall that in the case $p(x) = q(x) = 2$ for all $x \in \overline{\Omega}$ and $a = 0$, equation (1.3) can be directly linked to a stationary analogue (the corresponding elliptic equation) of the classical governing (parabolic) equation of nonlinear beam vibration posed by Kirchhoff [21], where a nonlocal term of type $\mathcal{K}(t) = kt + b$ is involved. There the real parameters k and b summarize physical features of the beam such as mass density, area of the cross-section, and length. Now, the idea of considering a general multiplicative function $\mathcal{K}$ in equation (1.3) is aimed to better model situations when for instance shape and area of the cross-section may change along beam's length, or when precise inertia effects are investigated (see, for example, Andrianov–Koblik [1] and the references cited therein). It is noted that such investigations are performed using experiments, and so the controllability of the physical setting is crucial for a successful process. From a mathematical perspective, this is realized by imposing a suitable set of assumptions on $\mathcal{K}$ to ensure that its influence is well-controlled throughout the analysis (see hypothesis (H2)). Finally, we remark that the attention of scholars was directed to nonlocal type problems by the Lions' approach (that is the Lions' representation theorem to generalize the Lax-Milgram theorem) with applications to physical systems (see Lions [22]). Furthermore, the hypothesis (H2) can be considered minimal, hence it is dictated by the specific needs of our proofs. In detail, (H2) gives well-posedness of the energy functional associated to problem (1.3) in order to fulfill the mountain-pass geometry (see Lemmas 3.2–3.4), and (H2)(ii) avoids the degeneracy of the main operator as well.

Our aim is to investigate the existence of nontrivial weak solutions (see (3.3)) of problem (1.3) by applying variational tools. To be more precise, we make use of the classical mountain pass theorem in order to derive the existence of at least one nontrivial weak solution for problem (1.3), supposed that the nonlinearity f satisfies very general conditions (see hypothesis (H3) and Theorem 3.5). Then, we produce infinitely many weak solutions for problem (1.3) (see Theorem 4.3) using a variant of the symmetric mountain pass theorem, namely, the Fountain theorem which can be found in Willem (see [33, Theorem 3.6]). Here, we obtain such multiplicity result under different hypotheses on the nonlinearity f, (see hypotheses (H3)(ii) and (H4)).

A special feature of the problem (1.3) is the fact that it combines the operator with variable exponents and a-logarithmic perturbation given in (1.1) along with a Kirchhoff term. As far as we know, this is the first paper that treats such a type of operator together with a Kirchhoff term. We point out that a Kirchhoff type problem driven by the classical double phase operator with variable exponents and with a nonlinearity which is just locally defined was considered recently by Ho–Winkert [20]. Differently from our study, the authors established the existence of infinitely many solutions via an abstract critical point result due to Kajikiya. For Kirchhoff problems in the double phase setting but with constant exponents we refer to the papers by Arora–Fiscella–Mukherjee–Winkert [4, 5], Borer–Pimenta–Winkert [7], Cen–Vetro–Zeng [13], Colasuonno–Perera [9], Crespo-Blanco–Gasiński–Winkert [14], Fiscella–Marino–Pinamonti–Verzellesi [16], Fiscella–Pinamonti [17], and Yang–Liu–Meng [32]. We also recall that nonlocal problems in the context of equations driven by the Laplacian or the p-Laplacian have been studied by a number of authors. We mention the papers by Alves–Figueiredo [2], Bueno–Ercole–Ferreira–Zumpano [8], Corrêa [10], and Corrêa–Figueiredo [11]. Further, existence results on degenerate and nondegenerate Kirchhoff problems can be found in the papers of Autuori–Pucci–Salvatori [6] and Xiang–Zhang–Rădulescu [34], see also Vetro [29] for $p(x)$-Kirchhoff type problems with convection.

Finally, we note that another logarithmic double phase operator different from the one in (1.1) has been recently introduced by Arora–Crespo-Blanco–Winkert [3] which has the form

$$\begin{aligned}\operatorname{div}\Bigg[&|\nabla u|^{p(x)-2}\nabla u + \\ &\mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right)|\nabla u|^{q(x)-2}\nabla u\Bigg]\end{aligned} \tag{1.4}$$

with $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$, where $\mathcal{H}_{\log}\colon \Omega \times [0,+\infty) \to [0,+\infty)$ is given by

$$\mathcal{H}_{\log}(x,t) = t^{p(x)} + \mu(x) t^{q(x)} \log(e+t).$$

The operator defined in (1.4) also appears in recent works by Carranza–Pimenta–Vetro–Winkert [12], Vetro [30] and Vetro–Winkert [31]. Note that in [12] the authors focused on a problem with a right-hand side consisting of a Carathéodory perturbation defined only locally and of a critical term. Therein, making use of appropriate truncation techniques and a suitable auxiliary problem, they produced a whole sequence of sign-changing solutions to the problem which converges to 0 in $L^\infty(\Omega)$ as well as in the logarithmic Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$. In [30], the author explores a similar problem to that under consideration here, namely, in the equation studied in [30] we have a multiplicative function $\mathcal{K}$ as well. But there the function $\mathcal{K}$ has to satisfy a condition which links it to its integral function, while here we claim a monotonicity condition on $\mathcal{K}$. Further, in [30] the growth condition on the nonlinearity f also depends on a constant η that we do not consider here. Lastly, in [31] the authors obtained existence and uniqueness results for a problem involving a nonlinearity which also depends on the gradient of the solution. Further, they proved the boundedness, closedness and compactness of the related solution set to the problem under consideration.

The paper is organized as follows. In Section 2 we recall the basic properties of Sobolev spaces with variable exponents and of logarithmic Musielak-Orlicz Sobolev spaces. Furthermore, we mention the properties of the operator (1.1) and fix some notation. Section 3 states and proves an existence results on very mild assumptions on the data (see Theorem 3.5) while Section 4 shows the existence of infinitely many solutions to problem (1.3) (see Theorem 4.3).

2. Preliminaries

In this section, we collect some facts on Lebesgue spaces and Musielak-Orlicz Sobolev spaces with variable exponents which will be needed later. One can find these topics in the books of Harjulehto–Hästö [19] and Musielak [26], see also the papers by Crespo-Blanco–Gasiński–Harjulehto–Winkert [15] and Lu–Vetro–Zeng [24] for more information.

To this end, let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $m \in C(\overline{\Omega})$ be such that $m(x) > 1$ for all $x \in \overline{\Omega}$. We write $m'(\cdot)$ to denote the conjugate variable exponent of $m(\cdot)$, which means that

$$\frac{1}{m(x)} + \frac{1}{m'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

By $L^{m(\cdot)}(\Omega)$ we denote the variable exponent Lebesgue space defined by

$$L^{m(\cdot)}(\Omega) = \left\{ u \in M(\Omega)\colon \rho_{m(\cdot)}(u) < +\infty \right\},$$

where $M(\Omega)$ stands for the set of all measurable functions $u\colon \Omega \to \mathbb{R}$ and the modular $\rho_{m(\cdot)}$ is given by

$$\rho_{m(\cdot)}(u) := \int_\Omega |u|^{m(x)} \, \mathrm{d}x.$$

As usual, we equip $L^{m(\cdot)}(\Omega)$ with the Luxemburg norm defined by

$$\|u\|_{m(\cdot)} = \inf \left\{ \alpha > 0\colon \rho_{m(\cdot)}\left(\frac{u}{\alpha}\right) \leq 1 \right\}$$

for all $u \in L^{m(\cdot)}(\Omega)$. With such norm the space $L^{m(\cdot)}(\Omega)$ becomes a separable, uniformly convex and hence reflexive Banach space whose dual space is given by $L^{m'(\cdot)}(\Omega)$. In addition, we know that the Hölder-type inequality

$$\int_\Omega |uv| \, \mathrm{d}x \leq \left[\frac{1}{r^-} + \frac{1}{(r')^-}\right] \|u\|_{m(\cdot)} \, \|v\|_{m'(\cdot)}$$

holds for all $u \in L^{m(\cdot)}(\Omega)$ and for all $v \in L^{m'(\cdot)}(\Omega)$. Also, if $m_1, m_2 \in C(\overline{\Omega})$ are such that $1 \leq m_1(x) \leq m_2(x)$ for all $x \in \overline{\Omega}$, then we have the continuous embedding

$$L^{m_2(\cdot)}(\Omega) \hookrightarrow L^{m_1(\cdot)}(\Omega).$$

Now, we focus on the nonlinear function $\mathcal{H}_L\colon \Omega \times [0, +\infty) \to [0, +\infty)$ defined by

$$\mathcal{H}_L(x,t) = \left(t^{p(x)} + \mu(x)t^{q(x)}\right) \log(e + at)$$

for all $x \in \Omega$ and for all $t \geq 0$, where $a \geq 0$ and the exponents as well as the weight function verify hypothesis (H1). We stress that $\mathcal{H}_L(\cdot, t)$ is a locally integrable, generalized N-function satisfying the Δ_2-condition (see Lu–Vetro–Zeng [24, Section 2]). Thus, the corresponding Musielak-Orlicz space $L^{\mathcal{H}_L}(\Omega)$ is given by

$$L^{\mathcal{H}_L}(\Omega) = \left\{u \in M(\Omega)\colon \rho_{\mathcal{H}_L}(u) < +\infty\right\},$$

with the modular $\rho_{\mathcal{H}_L}(\cdot)$ defined by

$$\rho_{\mathcal{H}_L}(u) := \int_\Omega \mathcal{H}_L(x, |u|) \, \mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right) \log(e + a|u|) \, \mathrm{d}x.$$

We consider on $L^{\mathcal{H}_L}(\Omega)$ the Luxemburg norm, that is,

$$\|u\|_{\mathcal{H}_L} := \inf\left\{\beta > 0\colon \rho_{\mathcal{H}_L}\left(\frac{u}{\beta}\right) \leq 1\right\},$$

for all $u \in L^{\mathcal{H}_L}(\Omega)$. This norm makes $L^{\mathcal{H}_L}(\Omega)$ to be a separable and reflexive Banach space (see Lu–Vetro–Zeng [24, Proposition 2.13]). Also, from Proposition 2.21 of [24] we know that the norm $\|\cdot\|_{\mathcal{H}_L}$ and the modular $\rho_{\mathcal{H}_L}$ are related to each other. In fact, let a_1 be the positive constant given by

$$a_1 := \frac{t_1}{\log(e + at_1)} \frac{\log(e + at_2)}{t_2}, \tag{2.1}$$

where t_1 and t_2 are a local maximum point and a local minimum point, respectively, for the function defined by $h(t) := \frac{t}{\log(e+at)}$ for all $t \geq 0$. Then, we have the following result.

Proposition 2.1. *Let hypothesis* (H1) *be satisfied. Then the following hold:*

(i) $\|u\|_{\mathcal{H}_L} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{\mathcal{H}_L}(u) < 1$ *(resp.* $> 1, = 1$*);*
(ii) $\min\{\|u\|_{\mathcal{H}_L}^{p^-}, a_1 \|u\|_{\mathcal{H}_L}^{q^+ + 1}\} \leq \rho_{\mathcal{H}_L}(u) \leq \max\{\|u\|_{\mathcal{H}_L}^{p^-}, a_1 \|u\|_{\mathcal{H}_L}^{q^+ + 1}\}$*, being* a_1 *as given in* (2.1)*;*
(iii) $\|u\|_{\mathcal{H}_L} \to 0$ *if and only if* $\rho_{\mathcal{H}_L}(u) \to 0$*;*
(v) $\|u\|_{\mathcal{H}_L} \to +\infty$ *if and only if* $\rho_{\mathcal{H}_L}(u) \to +\infty$*;*
(vi) $\|u\|_{\mathcal{H}_L} \to 1$ *if and only if* $\rho_{\mathcal{H}_L}(u) \to 1$*.*

Next, the Musielak-Orlicz Sobolev space corresponding to $L^{\mathcal{H}_L}(\Omega)$, denoted by $W^{1,\mathcal{H}_L}(\Omega)$, is given by

$$W^{1,\mathcal{H}_L}(\Omega) = \left\{u \in L^{\mathcal{H}_L}(\Omega)\colon |\nabla u| \in L^{\mathcal{H}_L}(\Omega)\right\}$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}_L} := \|u\|_{\mathcal{H}_L} + \|\nabla u\|_{\mathcal{H}_L},$$

where $\|\nabla u\|_{\mathcal{H}_L} := \| \, |\nabla u| \, \|_{\mathcal{H}_L}$. Then, we define $W_0^{1,\mathcal{H}_L}(\Omega)$ by the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}_L}(\Omega)$. From Propositions 2.13, 2.23 and 2.24 of [24] we see that $W^{1,\mathcal{H}_L}(\Omega)$ and $W_0^{1,\mathcal{H}_L}(\Omega)$ are separable, reflexive Banach spaces satisfying the following embeddings.

Proposition 2.2. *Let hypothesis (H1) be satisfied. Then the following hold:*

(i) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$ *are compact for* $m \in C(\overline{\Omega})$ *with* $1 \leq m(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$;

(ii) $W^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{\mathcal{H}_L}(\Omega)$ *is compact and there exists a constant* $b > 0$ *such that*

$$\|u\|_{\mathcal{H}_L} \leq b \, \|\nabla u\|_{\mathcal{H}_L} \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega).$$

As the Poincaré inequality holds, we can endow the space $W_0^{1,\mathcal{H}_L}(\Omega)$ with the equivalent norm given by

$$\|u\| := \|\nabla u\|_{\mathcal{H}_L} \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega).$$

Finally, we introduce the nonlinear operator $V_{\mathcal{L}} \colon W_0^{1,\mathcal{H}_L}(\Omega) \to W_0^{1,\mathcal{H}_L}(\Omega)^*$ defined by

$$\langle V_{\mathcal{L}}(u), w\rangle := \int_\Omega \mathcal{L}(u) \cdot \nabla w \, \mathrm{d}x \tag{2.2}$$

for all $u, w \in W_0^{1,\mathcal{H}_L}(\Omega)$, where $\mathcal{L}$ is as in (1.1) and $\langle \cdot, \cdot \rangle$ stands for the dual pairing between $W_0^{1,\mathcal{H}_L}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}_L}(\Omega)^*$. From Theorems 3.5 and 3.6 of [24] we know that such operator is characterized by several notable properties. In particular, we stress that the following results hold.

Proposition 2.3. *Let hypothesis (H1) be satisfied. Then, the operator* $V_{\mathcal{L}}$ *is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone, coercive and of* (S_+)*-type, that is,*

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty} \langle V_{\mathcal{L}}(u_n), u_n - u\rangle \leq 0$$

imply

$$u_n \to u \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega).$$

We conclude this section by recalling that a C^1-functional $\psi \colon W_0^{1,\mathcal{H}_L}(\Omega) \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_L}(\Omega)$ such that

$$\begin{aligned} &\{\psi(u_n)\}_{n\in\mathbb{N}} \subset \mathbb{R} \quad \text{is bounded and} \\ &\psi'(u_n) \to 0 \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega)^* \text{ as } n \to +\infty \end{aligned}$$

admits a convergent subsequence in $W_0^{1,\mathcal{H}_L}(\Omega)$. Also, we fix some notations which will be needed later. We denote by $|\Omega|$ the Lebesgue measure of Ω in $\mathbb{R}^N$ and for any $s \in \mathbb{R}$ we put $s_\pm := \max\{\pm s, 0\}$ which means that $s = s_+ - s_-$ and $|s| = s_+ + s_-$. For any function $u \colon \Omega \to \mathbb{R}$ we write $u_\pm(\cdot) := [u(\cdot)]_\pm$. With the purpose to lighten the notation, from now on we will use C as a generic constant, which may change from line to line, but does not depend on the crucial quantities.

3. An existence result

In this section, we suppose the following assumptions on the reaction term f while ϑ is as given in hypothesis (H2)(i):

(H3) $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the following conditions:

(i) there exists $r_1 \in ((q^+ + 1)\,\vartheta,\, (p^*)^-)$ such that for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$|f(x,t)| \le (q^+ + 1)\,\vartheta\,\varepsilon\,|t|^{(q^+ + 1)\,\vartheta - 1} + r_1\,\delta_\varepsilon\,|t|^{r_1 - 1}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(ii) there exist $r_2 \in (2\,q^+\,\vartheta,\, (p^*)^-)$ and $t_0 \ge 0$ such that

$$c \le r_2\,F(x,t) \le t f(x,t)$$

for some $c > 0$, for a.a. $x \in \Omega$ and for any $|t| \ge t_0$ with $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$.

For the sake of reader convenience, an example of the function verifying the aforementioned condition is provided next.

Example 3.1. *Let $\vartheta \in \left[1, \frac{(p^*)^-}{2\,q^+}\right)$ and $r_1 \in ((q^+ + 1)\,\vartheta,\, (p^*)^-)$ be fixed. The odd function $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ defined by*

$$f(x,t) := \frac{r_1}{2}\; t^{r_1 - 1} \quad \textit{for a.a. } x \in \Omega \textit{ and for all } t \ge 0$$

satisfies all the assumptions in (H3). In fact, we have that condition (H3)(i) holds for all $\varepsilon > 0$ if we take $\delta_\varepsilon = 1$, while condition (H3)(ii) holds for $t_0 = 1$ and $r_2 \in (2q^+\vartheta, r_1)$.

Our aim is to show that problem (1.3) admits at least one nontrivial weak solution in $W_0^{1,\mathcal{H}_L}(\Omega)$. In order to do this, the idea is to use the classical mountain pass theorem. For this purpose, we introduce the C^1-functional $\phi\colon W_0^{1,\mathcal{H}_L}(\Omega)\to\mathbb{R}$ defined by

$$\begin{aligned}\phi(u) := K\Bigg[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e + a|\nabla u|)\,\mathrm{d}x\Bigg] \\ - \int_\Omega F(x,u)\,\mathrm{d}x\end{aligned} \tag{3.1}$$

for all $u \in W_0^{1,\mathcal{H}_L}(\Omega)$. We point out that the derivative of ϕ is given by

$$\begin{aligned}\langle \phi'(u), w\rangle = \mathcal{K}\Bigg[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e + a|\nabla u|)\,\mathrm{d}x\Bigg] \\ \times \int_\Omega \mathcal{L}(u)\cdot\nabla w\,\mathrm{d}x - \int_\Omega f(x,u)\,w\,\mathrm{d}x\end{aligned} \tag{3.2}$$

for all $u, w \in W_0^{1,\mathcal{H}_L}(\Omega)$. Also, we recall that $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ is a weak solution of problem (1.3) if the following equality

$$\begin{aligned}\mathcal{K}\Bigg[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e + a|\nabla u|)\,\mathrm{d}x\Bigg] \\ \times \int_\Omega \mathcal{L}(u)\cdot\nabla w\,\mathrm{d}x \;=\; \int_\Omega f(x,u)\,w\,\mathrm{d}x\end{aligned} \tag{3.3}$$

holds for all $w \in W_0^{1,\mathcal{H}_L}(\Omega)$. Thus, according to (3.2) and (3.3) we have that the weak solutions of problem (1.3) coincide with the critical points of ϕ. Taking this into account, if we show that the functional ϕ satisfies the geometric features of the classical mountain pass theorem, then we are able to apply such theorem in order to obtain the existence of a nontrivial critical value of ϕ and consequently of a nontrivial weak solution of problem (1.3) in $W_0^{1,\mathcal{H}_L}(\Omega)$.

Therefore, we are going to prove that ϕ satisfies the geometry of the mountain pass theorem. We will do this in three steps. We start by the following result.

Lemma 3.2. *Let hypotheses (H1), (H2) and (H3)(i) be satisfied. Then, there exist $\nu \in (0,1]$ and $\zeta := \zeta(\nu) > 0$ such that*

$$\phi(u) \geq \zeta \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega) \quad \text{with } \|u\| = \nu.$$

Proof. First, we point out that hypothesis (H2) ensures that the following inequality

$$K(t) \geq K(1)\, t^{\vartheta}$$

holds for all $t \in [0,1]$. Moreover, from Proposition 2.1(ii) we see that for all $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ with $\|u\| \leq 1$, we have

$$a_1 \|u\|^{q^+ + 1} \leq \rho_{\mathcal{H}_L}(\nabla u) \leq \|u\|^{p^-} \leq 1$$

with $a_1 > 0$ as given in (2.1). This implies

$$\int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \right) \log(e + a|\nabla u|)\,\mathrm{d}x < 1,$$

according to the fact that

$$\int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \right) \log(e + a|\nabla u|)\,\mathrm{d}x \leq \frac{1}{p^-}\, \rho_{\mathcal{H}_L}(\nabla u).$$

Also, we point out that hypothesis (H3)(i) assures that, for any $\varepsilon > 0$, it is possible to find $\delta_\varepsilon > 0$ such that

$$|F(x,t)| \leq \varepsilon\, |t|^{(q^+ + 1)\,\vartheta} + \delta_\varepsilon\, |t|^{r_1}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$. Keeping this in mind, we are able to affirm that

$$\begin{aligned}
\phi(u) &\geq K(1) \left[\int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \right) \log(e + a|\nabla u|)\,\mathrm{d}x \right]^{\vartheta} \\
&\quad - \int_\Omega \varepsilon |u|^{(q^+ + 1)\,\vartheta}\,\mathrm{d}x - \int_\Omega \delta_\varepsilon |u|^{r_1}\,\mathrm{d}x \\
&\geq \frac{a_1^{\vartheta}\, K(1)}{(q^+)^{\vartheta}}\, \|u\|^{(q^+ + 1)\,\vartheta} - \varepsilon\, \|u\|_{(q^+ + 1)\,\vartheta}^{(q^+ + 1)\,\vartheta} - \delta_\varepsilon\, \|u\|_{r_1}^{r_1}.
\end{aligned}$$

Note that the embeddings

$$W_0^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{(q^+ + 1)\,\vartheta}(\Omega) \quad \text{and} \quad W_0^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{r_1}(\Omega)$$

are both compact due to the fact that $(q^+ + 1)\,\theta < 2\,q^+\,\theta < (p^*)^-$ from hypothesis (H2)(i) and $r_1 < (p^*)^-$ from hypothesis (H3)(i) (see Proposition 2.2(i)). Taking this into account, we denote by ℓ and $\tilde{\ell}$ the best positive constants such that

$$\|u\|_{(q^+ + 1)\,\vartheta}^{(q^+ + 1)\,\vartheta} \leq \ell \|u\|^{(q^+ + 1)\,\vartheta} \quad \text{and} \quad \|u\|_{r_1}^{r_1} \leq \tilde{\ell} \|u\|^{r_1},$$

for all $u \in W_0^{1,\mathcal{H}_L}(\Omega)$, respectively. Then, we have that

$$\begin{aligned}\phi(u) &\geq \frac{a_1^{\vartheta}\,K(1)}{(q^+)^{\vartheta}}\,\|u\|^{(q^++1)\,\vartheta} - \varepsilon\,\ell\,\|u\|^{(q^++1)\vartheta} - \delta_\varepsilon\,\tilde{\ell}\,\|u\|^{r_1}\\ &= \left(\frac{a_1^{\vartheta}K(1)}{(q^+)^{\vartheta}} - \varepsilon\,\ell\right)\|u\|^{(q^++1)\,\vartheta} - \delta_\varepsilon\,\tilde{\ell}\,\|u\|^{r_1}\\ &= \left[\left(\frac{a_1^{\vartheta}\,K(1)}{(q^+)^{\vartheta}} - \varepsilon\,\ell\right) - \delta_\varepsilon\,\tilde{\ell}\,\|u\|^{r_1-(q^++1)\,\vartheta}\right]\|u\|^{(q^++1)\,\vartheta},\end{aligned}$$

where $r_1 - (q^+ + 1)\,\vartheta > 0$ according to hypothesis (H3)(i). From here, if we choose $\varepsilon > 0$ small enough such that

$$\frac{a_1^{\vartheta}\,K(1)}{(q^+)^{\vartheta}} - \varepsilon\,\ell > 0,$$

then for any $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ with

$$\|u\| = \nu \in \left(0,\, \min\left\{1, \left[\frac{1}{\delta_\varepsilon\,\tilde{\ell}}\left(\frac{a_1^{\vartheta}\,K(1)}{(q^+)^{\vartheta}} - \varepsilon\,\ell\right)\right]^{\frac{1}{r_1-(q^++1)\,\vartheta}}\right\}\right)$$

we have that

$$\phi(u) \geq \left[\left(\frac{a_1^{\vartheta}\,K(1)}{(q^+)^{\vartheta}} - \varepsilon\,\ell\right) - \delta_\varepsilon\,\tilde{\ell}\,\nu^{r_1-(q^++1)\,\vartheta}\right]\nu^{(q^++1)\,\vartheta} := \zeta > 0.$$

This shows the assertion of the lemma. □

Now, we give the following result.

Lemma 3.3. *Let hypotheses (H1), (H2) and (H3) be satisfied. Then, there exists $v \in W_0^{1,\mathcal{H}_L}(\Omega)$ such that*

$$\phi(v) < 0 \quad \textit{and} \quad \|v\| > 1.$$

Proof. From hypothesis (H2)(i) we have for all $\varepsilon > 0$ the inequality

$$K(t) \leq \frac{K(\varepsilon)}{\varepsilon^{\vartheta}}\,t^{\vartheta} \tag{3.4}$$

whenever $t \geq \varepsilon$. Also, we point out that Proposition 2.1(ii) guarantees that for all $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ with $\|u\| \geq (q^+)^{\frac{1}{p^-}} > 1$, it leads to

$$1 < \|u\|^{p^-} \leq \rho_{\mathcal{H}_L}(\nabla u).$$

Hence, we deduce that

$$\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e + a|\nabla u|)\,\mathrm{d}x \geq 1 \tag{3.5}$$

taking into account that

$$\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e + a|\nabla u|)\,\mathrm{d}x \geq \frac{1}{q^+}\rho_{\mathcal{H}_L}(\nabla u).$$

Next, due to hypotheses (H3), we know that it is possible to find $c_1 > 0$ and $c_2 \geq 0$ so that the inequality

$$F(x,t) \geq c_1|t|^{r_2} - c_2 \tag{3.6}$$

is satisfied for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$.

Based on this, for $t \geq (q^+)^{\frac{1}{p^-}}$ and $w \in W_0^{1,\mathcal{H}_L}(\Omega)$ such that $\|w\| = 1$ we have that

$$\begin{aligned}
\phi(tw) &\leq K(1) \left[\int_\Omega \left(\frac{1}{p(x)} |\nabla(tw)|^{p(x)} + \frac{\mu(x)}{q(x)} |\nabla(tw)|^{q(x)} \right) \log(e + a|\nabla(tw)|) \, \mathrm{d}x \right]^\vartheta \\
&\quad - c_1 \int_\Omega |t\, w|^{r_2} \, \mathrm{d}x + c_2 \int_\Omega \mathrm{d}x \\
&\leq \frac{K(1)}{p^-} t^{(q^+ + 1)\, \vartheta} \left[\int_\Omega \left(|\nabla w|^{p(x)} + \mu(x)\, |\nabla w|^{q(x)} \right) \log(e + a|\nabla w|) \, \mathrm{d}x \right]^\vartheta \\
&\quad - c_1\, t^{r_2}\, \|w\|_{r_2}^{r_2} + c_2\, |\Omega| \\
&= \frac{K(1)}{p^-} t^{(q^+ + 1)\, \vartheta} [\rho_{\mathcal{H}_L}(\nabla\, w)]^\vartheta - c_1\, t^{r_2}\, \|w\|_{r_2}^{r_2} + c_2\, |\Omega|.
\end{aligned}$$

Recall that $\|w\| := \|\nabla\, w\|_{\mathcal{H}_L} = 1$. This, according to Proposition 2.1(i), implies that $\rho_{\mathcal{H}_L}(\nabla\, w) = 1$ as well. Moreover, as $r_2 < (p^*)^-$ due to hypothesis (H3)(ii), Proposition 2.2(i) guarantees that the embedding $W_0^{1,\mathcal{H}_L}(\Omega) \hookrightarrow L^{r_2}(\Omega)$ is compact. This leads to

$$\phi(tw) \;\leq\; \frac{K(1)}{p^-} t^{(q^+ + 1)\, \vartheta} - c_1\, C\, t^{r_2} + c_2\, |\Omega|$$

for some $C > 0$. Taking into account that $r_2 > (q^+ + 1)\, \vartheta$ from hypothesis (H3)(ii), we conclude that

$$\phi(t\, w) \to -\infty \quad \text{as } t \to +\infty.$$

This shows that if $\bar{t} > 0$ is large enough, then $v = \bar{t}\, w \in W_0^{1,\mathcal{H}_L}(\Omega)$ is such that

$$\|v\| > 1 \quad \text{and further} \quad \phi(v) < 0.$$

This finishes the proof. □

Next, we show that the functional ϕ verifies the Palais-Smale condition.

Lemma 3.4. *Let hypotheses (H1), (H2) and (H3) be satisfied. Then, the functional ϕ satisfies the Palais-Smale condition.*

Proof. In order to prove the claim, we consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_L}(\Omega)$ satisfying the following conditions:

$$\begin{aligned}
&\{\phi(u_n)\}_{n\in\mathbb{N}} \subset \mathbb{R} \text{ is bounded}, \\
&\phi'(u_n) \to 0 \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega)^* \text{ as } n \to +\infty.
\end{aligned} \tag{3.7}$$

First, we are going to prove that the sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_L}(\Omega)$ is bounded. We prove this via contradiction. For this purpose, assume that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is unbounded. This means that it is possible to find a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, such that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty \quad \text{and} \quad \|u_n\| \geq (q^+)^{\frac{1}{p^-}} \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

Note that hypothesis (H2)(i) implies that

$$t\, \mathcal{K}(t) \leq \vartheta K(t) \tag{3.9}$$

for all $t \geq 0$. Then, recalling that $ap^- \geq 1$, using (3.9) we see that

$$\phi(u_n) - \frac{1}{r_2} \langle \phi'(u_n), u_n \rangle$$

$$\begin{aligned}
&= K\left[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\right] - \int_\Omega F(x,u_n)\,\mathrm{d}x\\
&\quad - \frac{1}{r_2}\,\mathcal{K}\left[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\right]\\
&\quad \times \int_\Omega \left[\left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\right.\\
&\qquad \left. + \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\frac{|\nabla u|}{(e+a|\nabla u|)}\right]\mathrm{d}x + \frac{1}{r_2}\int_\Omega f(x,u_n)\,u_n\,\mathrm{d}x\\
&\geq \frac{1}{q^+\vartheta}\,\mathcal{K}\left[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\right]\\
&\quad \times \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\\
&\quad - \frac{1}{r_2}\,\mathcal{K}\left[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\right]\\
&\quad \times \int_\Omega \left[\left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\right.\\
&\qquad \left. + \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\frac{|\nabla u|}{(e+a|\nabla u|)}\right]\mathrm{d}x\\
&\quad - \int_\Omega \left(F(x,u_n) - \frac{1}{r_2}\,f(x,u_n)u_n\right)\mathrm{d}x\\
&\geq \left(\frac{1}{q^+\vartheta} - \frac{1}{r_2}\right)\mathcal{K}\left[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\right]\\
&\quad \times \rho_{\mathcal{H}_L}(\nabla u_n) - \int_\Omega \left(F(x,u_n) - \frac{1}{r_2}\,f(x,u_n)u_n\right)\mathrm{d}x\\
&\quad - \frac{1}{r_2}\,\mathcal{K}\left[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\right]\\
&\qquad \times \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\frac{|\nabla u|}{(e+a|\nabla u|)}\,\mathrm{d}x\\
&\geq \left(\frac{1}{q^+\vartheta} - \frac{2}{r_2}\right)\mathcal{K}\left[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\right]\\
&\quad \times \rho_{\mathcal{H}_L}(\nabla u_n) - \int_{\Omega_n}\left[F(x,u_n) - \frac{1}{r_2}\,f(x,u_n)u_n\right]_+\mathrm{d}x,
\end{aligned} \tag{3.10}$$

where we have used hypothesis (H3)(ii) to get

$$\frac{1}{q^+\vartheta} - \frac{2}{r_2} > 0$$

and we put

$$\Omega_n := \{x \in \Omega\colon |u_n(x)| \leq t_0\}$$

with t_0 as given in hypothesis (H3)(ii).

Now, from hypothesis (H2)(ii) we know that there exists $k > 0$ such that

$$\mathcal{K}(t) \geq k \quad \text{whenever } t \geq 1. \tag{3.11}$$

Also, from (3.5) and (3.8) we see that

$$\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x \geq 1.$$

Putting

$$M := |\Omega_n| \sup_{x\in\Omega_n,\,|t|\leq t_0}\left[F(x,t) - \frac{1}{r_2} f(x,t)t\right]_+ < +\infty$$

(note that $M < +\infty$ due to hypothesis (H3)(i)) and using (3.10) along with (3.11) and Proposition 2.1(ii), we derive that

$$\phi(u_n) - \frac{1}{r_2}\langle \phi'(u_n), u_n\rangle \geq \left(\frac{1}{q^+\vartheta} - \frac{2}{r_2}\right) k\,\|u_n\|^{p^-} - M.$$

For this reason and since (3.7) holds, we can affirm that there exist $c_3, c_4 > 0$ such that

$$c_3 + c_4\,\|u_n\| \geq \left(\frac{1}{q^+\vartheta} - \frac{2}{r_2}\right) k\,\|u_n\|^{p^-} - M \quad \text{for all } n\in\mathbb{N}.$$

Since this contradicts to (3.8) due to $p^- > 1$, we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, we may suppose (for a subsequence if necessary, not relabeled) that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega),\\ &u_n \to u \quad \text{in } \ L^{(q^++1)\,\vartheta}(\Omega) \text{ and in } L^{r_1}(\Omega). \end{aligned} \tag{3.12}$$

We emphasize at this point that if we take $w = u_n - u$ in (3.2) and use (3.7), we obtain

$$\begin{aligned} \lim_{n\to+\infty} \mathcal{K}\Bigg[&\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\Bigg]\\ &\times \int_\Omega \mathcal{L}(u_n)\cdot\nabla(u_n-u)\,\mathrm{d}x - \lim_{n\to+\infty}\int_\Omega f(x,u_n)\,(u_n-u)\,\mathrm{d}x = 0. \end{aligned} \tag{3.13}$$

Next, using hypothesis (H3)(i) with $\varepsilon = 1$, Hölder's inequality and (3.12) we can see that

$$\begin{aligned} &\left|\int_\Omega f(x,u_n)\,(u_n-u)\,\mathrm{d}x\right|\\ &\leq \int_\Omega \left((q^++1)\,\vartheta\,|u_n|^{(q^++1)\,\vartheta-1} + r_1\,\delta_1\,|u_n|^{r_1-1}\right)|u_n-u|\,\mathrm{d}x\\ &\leq (q^++1)\,\vartheta\,\|u_n\|_{(q^++1)\,\vartheta}^{(q^++1)\,\vartheta-1}\|u_n-u\|_{(q^++1)\,\vartheta}\\ &\quad + r_1\,\delta_1\,\|u_n\|_{r_1}^{r_1-1}\,\|u_n-u\|_{r_1}\\ &\to 0 \qquad \text{as } n\to+\infty. \end{aligned} \tag{3.14}$$

Recall that the operator $V_{\mathcal{L}}$ defined in (2.2) is continuous and bounded (see Proposition 2.3). Further, we know that (3.5) holds and $\mathcal{K}(t) > k$ for all $t \geq 1$ (see (3.11)). Keeping all this in mind, from (3.13) using (3.14) we deduce that

$$\int_\Omega \mathcal{L}(u_n)\cdot\nabla(u_n-u)\,\mathrm{d}x \to 0 \quad \text{as } n\to+\infty$$

and hence

$$\limsup_{n\to+\infty} \langle V_{\mathcal{L}}(u_n), u_n - u\rangle \le 0.$$

Now, as the operator $V_{\mathcal{L}}$ is of (S_+)-type (see again Proposition 2.3), we conclude that

$$u_n \to u \quad \text{in } W_0^{1,\mathcal{H}_L}(\Omega).$$

□

Finally, we can state and prove the existence result in this section.

Theorem 3.5. *Let hypotheses (H1), (H2) and (H3) be satisfied. Then, problem (1.3) admits at least one nontrivial weak solution in $W_0^{1,\mathcal{H}_L}(\Omega)$.*

Proof. Let ϕ be the C^1-functional introduced in (3.1). We recall that from (3.2) and (3.3) we know that the critical points of ϕ are the weak solutions of problem (1.3). For way of this, in order to obtain the claim, it is sufficient to show that ϕ has at least one nontrivial critical point. Now, as $\phi(0) = 0$ and since Lemmas 3.2, 3.3 and 3.4 hold, we are in the position to apply the mountain pass theorem to derive that there exists a nontrivial critical value of ϕ, which is a nontrivial weak solution of problem (1.3) in $W_0^{1,\mathcal{H}_L}(\Omega)$. □

4. Infinitely many solutions

In this section, we present our second existence result. To be more precise, we are going to show that problem (1.3) has infinitely many weak solutions in $W_0^{1,\mathcal{H}_L}(\Omega)$. In order to do this, we will make use of a variant of the symmetric mountain pass theorem, namely, the Fountain theorem which can be found in the monograph by Willem [33, Theorem 3.6]. First, we need new assumptions on the nonlinearity f. Precisely, we now suppose that the Carathéodory function $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies hypothesis (H3)(ii) and in addition we assume that:

(H4) f is odd with respect to the second variable and there exists $r \in (p^-, (p^*)^-)$ such that

$$|f(x,t)| \le d\,(1 + |t|^{r-1})$$

for some $d > 0$, for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$.

Remark 4.1. *We point out that if we replace hypothesis (H3)(i) with hypothesis (H4), then Lemma 3.4 is still true. We can easily deduce this from the proof of Lemma 3.4 recalling that since $r < (p^*)^-$ we have the compact embedding of $W_0^{1,\mathcal{H}_L}(\Omega)$ into $L^r(\Omega)$.*

Before formulating the main result of this section, we note that $W_0^{1,\mathcal{H}_L}(\Omega)$ is a separable and reflexive Banach space. Therefore, we can find sequences

$$\{v_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_L}(\Omega) \quad \text{and} \quad \{g_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_L}(\Omega)^*$$

such that

$$W_0^{1,\mathcal{H}_L}(\Omega) := \overline{\operatorname{span}\{v_n : n\in\mathbb{N}\}}, \quad W_0^{1,\mathcal{H}_L}(\Omega)^* := \overline{\operatorname{span}\{g_n : n\in\mathbb{N}\}}$$

$$\text{and further} \quad \langle g_j,\, v_n\rangle := \begin{cases} 1 & \text{if } n = j,\\ 0 & \text{if } n \ne j.\end{cases}$$

Next, we put

$$Z_n := \operatorname{span}\{v_n\}, \quad \widetilde{Z}_n := \bigoplus_{j=1}^{n} Z_j, \quad \widehat{Z}_n := \bigoplus_{j=n}^{+\infty} \widetilde{Z}_j$$

and

$$\xi_n := \sup_{u \in \widehat{Z}_n,\, \|u\|=1} \|u\|_r,$$

where r is from hypothesis (H4).

Remark 4.2. *We point out that*

$$\xi_n \to 0 \quad \text{as } n \to +\infty,$$

see Lemma 7.1 by Liu–Dai [23].

Now, we are ready to state our multiplicity result.

Theorem 4.3. *Let hypotheses* (H1)*,* (H2)*,* (H3)(ii) *and* (H4) *be satisfied. Then, problem* (1.3) *admits infinitely many weak solutions in* $W_0^{1,\mathcal{H}_L}(\Omega)$*.*

Proof. Let ϕ be the C^1-functional introduced in (3.1). From hypothesis (H4) we know that f is odd with respect to the second variable. This in particular guarantees that ϕ is an even functional. Also, according to Remark 4.1 we know that ϕ verifies the Palais-Smale condition. Then, in order to use the Fountain theorem, we only need to show that for all $n \geq 1$ there exist $\sigma_n > \gamma_n > 0$ such that

$$l_n := \inf\{\phi(u) \colon u \in \widehat{Z}_n, \|u\| = \gamma_n\} \to +\infty \quad \text{as } n \to +\infty \tag{4.1}$$

and

$$\max\{\phi(u) \colon u \in \widetilde{Z}_n, \|u\| = \sigma_n\} \leq 0. \tag{4.2}$$

As first step, we have to determinate $\gamma_n > 0$ such that (4.1) holds. To this end, taking hypothesis (H2)(ii) into account there exists $k > 0$ such that for all $u \in \widehat{Z}_n$ with $\|u\| \geq (q^+)^{\frac{1}{p^-}} > 1$ it results

$$\mathcal{K}\left[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right) \log(e + a|\nabla u|)\,\mathrm{d}x\right] \geq k, \tag{4.3}$$

(see (3.11) and (3.5)). Also, from hypothesis (H4) we have that

$$|F(x,t)| \leq C(|t| + |t|^r) \tag{4.4}$$

for some $C > 0$, for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$.

Then, using (3.9), (4.3), (4.4), Proposition 2.1(ii) and Hölder's inequality, for all $u \in \widehat{Z}_n$ such that $\|u\| \geq (q^+)^{\frac{1}{p^-}} > 1$ we get that

$$\begin{aligned}
\phi(u) \geq\ & \frac{1}{q^+\,\vartheta}\,\mathcal{K}\bigg[\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e + a|\nabla u|)\,\mathrm{d}x\bigg] \\
& \times \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\right)\log(e + |\nabla u|)\,\mathrm{d}x \\
& - C\int_\Omega |u|\,\mathrm{d}x - C\int_\Omega |u|^r\,\mathrm{d}x \\
\geq\ & \frac{k}{q^+\,\vartheta}\|u\|^{p^-} - C\,|\Omega|^{\frac{r-1}{r}}\,\|u\|_r - C\|u\|_r^r \\
\geq\ & \frac{k}{q^+\,\vartheta}\|u\|^{p^-} - C\,\xi_n\,|\Omega|^{\frac{r-1}{r}}\,\|u\| - C\,\xi_n^r\,\|u\|^r \\
\geq\ & \frac{k}{q^+\,\vartheta}\|u\|^{p^-} - C\,\xi_n\,|\Omega|^{\frac{r-1}{r}}\,\|u\|^r - C\,\xi_n^r\,\|u\|^r \\
=\ & \left[\frac{k}{q^+\,\vartheta} - C\left(\xi_n\,|\Omega|^{\frac{r-1}{r}} + \xi_n^r\right)\|u\|^{r-p^-}\right]\|u\|^{p^-},
\end{aligned} \tag{4.5}$$

where we recall that $C > 0$ may change from line to line. Taking Remark 4.2 into account and recalling that $r > p^-$ from hypothesis (H4), by setting

$$\gamma_n := \left[\frac{k}{2\,q^+\,\vartheta}\ \frac{1}{C\,(\xi_n\,|\Omega|^{\frac{r-1}{r}} + \xi_n^r)}\right]^{\frac{1}{r-p^-}},$$

we have that

$$\gamma_n \to +\infty \quad \text{as } n \to +\infty. \tag{4.6}$$

This yields

$$\gamma_n > (q^+)^{\frac{1}{p^-}} \quad \text{for } n \text{ large enough.}$$

Thus, (4.5) and (4.6) permit us to affirm that for all $u \in \widehat{Z}_n$ with $\|u\| = \gamma_n$ and n large enough it holds

$$l_n \geq \frac{k}{2\,q^+\,\vartheta}\,\gamma_n^{p^-} \to +\infty \quad \text{as } n \to +\infty.$$

Consequently, we conclude that (4.1) holds.

Our goal is now to prove that (4.2) also holds. We recall that $\widetilde{Z}_n$ has finite dimension and hence all the norms on $\widetilde{Z}_n$ are equivalent (see, for example, [27, Proposition 3.1.17, p.183]). This means that there exists $d_{\widetilde{Z}_n} > 0$, independent of $u \in \widetilde{Z}_n$, such that

$$d_{\widetilde{Z}_n}\,\|u\|^{r_2} \leq \|u\|_{r_2}^{r_2},$$

where r_2 is from hypothesis (H3)(ii). Moreover, from Proposition 2.1(ii) we know that

$$\rho_{\mathcal{H}_L}(\nabla u) \leq a_1\,\|u\|^{q^+ + 1} \quad \text{whenever} \quad \|u\| := \|\nabla u\|_{\mathcal{H}_L} > 1,$$

with $a_1 > 0$ as given in (2.1). Then, using the previous facts along with (3.6), (3.4) (where we choose $\varepsilon = 1$) and (3.5), we have, for $u \in \widetilde{Z}_n$ such that $\|u\| \geq (q^+)^{\frac{1}{p^-}}$, that

$$
\begin{aligned}
\phi(u) &\le K(1)\left[\int_\Omega\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right)\log(e+a|\nabla u|)\,\mathrm{d}x\right]^\vartheta\\
&\quad - c_1\int_\Omega |u|^{r_2}\,\mathrm{d}x + c_2\int_\Omega \mathrm{d}x\\
&\le \frac{K(1)}{p^-}\left[\rho_{\mathcal{H}_L}(\nabla u)\right]^\vartheta - c_1\,\|u\|_{r_2}^{r_2} + c_2\,|\Omega|\\
&\le \frac{a_1^\vartheta\,K(1)}{p^-}\,\|u\|^{(q^++1)\,\vartheta} - c_1\,d_{\widetilde{Z}_n}\,\|u\|^{r_2} + c_2\,|\Omega|
\end{aligned}
\tag{4.7}
$$

since $(q^+)^{\frac{1}{p^-}} > 1$ according to hypothesis (H1). Now, we point out that hypothesis (H3)(ii) gives that $r_2 > (q^+ + 1)\,\vartheta$. So, if we take

$$\sigma_n > \max\left\{(q^+)^{\frac{1}{p^-}},\ \gamma_n\right\} \quad \text{large enough,}$$

with a view to (4.7), then we conclude that (4.2) holds.

Thus, the functional ϕ satisfies all the assumptions of the Fountain theorem. Consequently, we can apply such result which guarantees the existence of an unbounded sequence of critical points of ϕ. Recalling that the critical points of ϕ are weak solutions for problem (1.3) (see (3.3) and (3.2)), the theorem is proved. □

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