SINGULAR DIRICHLET (p,q)-EQUATIONS

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Abstract. We consider a nonlinear Dirichlet problem driven by the (p,q)-Laplacian and with a reaction having the combined effects of a singular term and of a parametric $(p-1)$-superlinear perturbation. We prove a bifurcation-type result describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. Moreover, we prove the existence of a minimal positive solution u_λ^* and study the monotonicity and continuity properties of the map $\lambda \to u_\lambda^*$.

1. Introduction

In a recent paper, the authors [15] studied the following singular parametric p-Laplacian Dirichlet problem

$$\begin{aligned} -\Delta_p u &= u^{-\eta} + \lambda f(x,u) \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega,\\ u > 0, \quad \lambda &> 0, \quad 0 < \eta < 1, \quad 1 < p. \end{aligned}$$

They proved a result describing the dependence of the set of positive solutions as the parameter $\lambda > 0$ varies, assuming that $f(x,\cdot)$ is $(p-1)$-superlinear.

In the present paper, we consider a singular parametric Dirichlet problem driven by the (p,q)-Laplacian, that is, the sum of a p-Laplacian and of a q-Laplacian with $1 < q < p$. To be more precise, the problem under consideration is the following

$$\begin{aligned} -\Delta_p u - \Delta_q u &= u^{-\eta} + \lambda f(x,u) \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \qquad (\mathrm{P}_\lambda)\\ u > 0, \quad \lambda &> 0, \quad 0 < \eta < 1, \quad 1 < q < p, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. In this problem, the differential operator is not homogeneous and so many of the techniques used in Papageorgiou-Winkert [15] are not applicable here. More precisely, in the proof of Proposition 3.1 in [15], the homogeneity of the p-Laplacian is crucial in the argument. It provides naturally an upper solution $\overline{u}$ which is an appropriate multiple of the unique solution $e \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ of problem (3.2) in [15] (see also the argument in (3.7)). In our setting, this is no longer possible since the differential operator, so the (p,q)-Laplacian, is not homogeneous. This makes our proof here of the fact that $\mathcal{L} \neq \emptyset$ (existence of admissible parameters, see Proposition 3.1) more involved and requires some preparation which involves Propositions 2.3 and 2.4. Moreover, the proof that the critical parameter $\lambda^* > 0$ is finite differs for the same reason and

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here is more involved and requires the use of a different strong comparison principle. In [15] (see Proposition 3.6) this is done easily since we can use the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$ and in particular the principal eigenvalue $\hat{\lambda}_1 > 0$ thanks to the homogeneity of the differential operator (see (3.25) in [15]). This reasoning fails in our setting and leads to a different geometry near zero (compare hypothesis H(iv) in [15] with hypothesis H(iv) in this paper). Furthermore, we now need to employ a different comparison argument based on a recent strong comparison principle due to Papageorgiou-Rădulescu-Repovš [12]. In addition, the proof of Proposition 3.7 in [15] cannot be extended to our problem (see the part from (3.42) and below). The presence of the q-Laplacian leads to difficulties. For this reason, our superlinearity condition (see hypothesis H(iii)) differs from the one used in [15]. However, we stress that both go beyond the classical Ambrosetti-Rabinowitz condition.

For the parametric perturbation of the singular term, $\lambda f(\cdot,\cdot)$ with $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$, we assume that f is a Carathéodory function, that is, $x \mapsto f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \mapsto f(x,s)$ is continuous for almost all (a. a.) $x \in \Omega$. Moreover we assume that $f(x,\cdot)$ exhibits $(p-1)$-superlinear growth as $s \to +\infty$ but it need not satisfy the usual Ambrosetti-Rabinowitz condition (the AR-condition for short) in such cases. Applying variational tools from critical point theory along with suitable truncation and comparison techniques, we prove a bifurcation-type result as in [15], which describes in a precise way the dependence of the set of positive solutions as the parameter $\lambda > 0$ changes.

In this direction we mention the recent works of Papageorgiou-Rădulescu-Repovš [12] and Papageorgiou-Vetro-Vetro [14] which also deal with nonlinear singular parametric Dirichlet problems. In theses works the parameter multiplies the singular term. Indeed, in Papageorgiou-Rădulescu-Repovš [12] the equation is driven by a nonhomogeneous differential operator and in the reaction we have the competing effects of a parametric singular term and of a $(p-1)$-superlinear perturbation. In Papageorgiou-Vetro-Vetro [14] the equation is driven by the $(p,2)$-Laplacian and in the reaction we have the competing effects of a parametric singular term and of a $(p-1)$-linear, resonant perturbation. The work of Papageorgiou-Vetro-Vetro [14] was continued by Bai-Motreanu-Zeng [2] where the authors examine the continuity properties with respect to the parameter of the solution multifunction.

Boundary value problems monitored by a combination of differential operators of different nature (such as (p,q)-equations), arise in many mathematical processes. We refer, for example, to the works of Bahrouni-Rădulescu-Repovš [1] (transonic flows), Benci-D'Avenia-Fortunato-Pisani [3] (quantum physics), Cherfils-Il′yasov [4] (reaction diffusion systems) and Zhikov [19] (elasticity theory). We also mention the survey paper of Rădulescu [18] on anistropic (p,q)-equations.

2. Preliminaries and Hypotheses

The main spaces which we will be using in the study of problem (P_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega})$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$ and because of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $\|\cdot\|_p$ denotes norm in $L^p(\Omega)$ and also in $L^p(\Omega;\mathbb{R}^N)$. From the context it will be clear which one is used.

The Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) \,:\, u\big|_{\partial\Omega} = 0\right\}$$

is an ordered Banach space with positive cone

$$C_0^1(\overline{\Omega})_+ = \left\{u \in C_0^1(\overline{\Omega}) \,:\, u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) = \left\{u \in C_0^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega\right\},$$

where $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$.

For every $r \in (1,\infty)$, let $A_r\colon W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ with $\frac{1}{r}+\frac{1}{r'}=1$ be the nonlinear map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}\nabla u \cdot \nabla h \, dx \quad \text{for all } u, h \in W_0^{1,r}(\Omega). \tag{2.1}$$

From Gasiński-Papageorgiou [5, Problem 2.192, p. 279] we have the following properties of A_r.

Proposition 2.1. *The map $A_r\colon W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ defined in (2.1) is bounded, that is, it maps bounded sets to bounded sets, continuous, strictly monotone, hence maximal monotone and it is of type $(\mathrm{S})_+$, that is,*

$$u_n \overset{\mathrm{w}}{\to} u \text{ in } W_0^{1,r}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A_r(u_n), u_n - u\rangle \leq 0,$$

imply $u_n \to u$ in $W_0^{1,r}(\Omega)$.

For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is well known that

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

For $u, v \in W_0^{1,p}(\Omega)$ with $u(x) \leq v(x)$ for a. a. $x \in \Omega$ we define

$$[u,v] = \left\{h \in W_0^{1,p}(\Omega) : u(x) \leq h(x) \leq v(x) \text{ for a. a. } x \in \Omega\right\},$$
$$[u) = \left\{h \in W_0^{1,p}(\Omega) : u(x) \leq h(x) \text{ for a. a. } x \in \Omega\right\}.$$

Given a set $S \subseteq W^{1,p}(\Omega)$ we say that it is "downward directed", if for any given $u_1, u_2 \in S$ we can find $u \in S$ such that $u \leq u_1$ and $u \leq u_2$.

If $h_1, h_2\colon \Omega \to \mathbb{R}$ are two measurable functions, then we write $h_1 \prec h_2$ if and only if for every compact $K \subseteq \Omega$ we have $0 < c_K \leq h_2(x) - h_1(x)$ for a. a. $x \in K$.

If X is a Banach space and $\varphi \in C^1(X,\mathbb{R})$, then we define

$$K_\varphi = \{u \in X \,:\, \varphi'(u) = 0\}$$

being the critical set of φ. Furthermore, we say that φ satisfies the Cerami condition (C-condition for short), if every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq1} \subseteq \mathbb{R}$ is bounded and such that $(1+\|u_n\|_X)\,\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

Our Hypotheses on the perturbation $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ are the following:

H: $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(x,0)=0$ for a. a. $x \in \Omega$ and

(i)

$$f(x,s) \leq a(x)\left(1+s^{r-1}\right)$$

for a.a. $x \in \Omega$, for all $s \geq 0$, with $a \in L^\infty(\Omega)$ and $p<r<p^*$, where p^* denotes the critical Sobolev exponent with respect to p given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p<N, \\ +\infty & \text{if } N \leq p; \end{cases}$$

(ii) if $F(x,s)=\int_0^s f(x,t)dt$, then

$$\lim_{s\to+\infty} \frac{F(x,s)}{s^p} = +\infty \quad \text{uniformly for a. a. } x \in \Omega;$$

(iii) there exists $\tau \in \left((r-p)\max\left\{\frac{N}{p},1\right\},p^*\right)$ with $\tau>q$ such that

$$0<c_0 \leq \liminf_{s\to+\infty} \frac{f(x,s)s-pF(x,s)}{s^\tau} \quad \text{uniformly for a. a. } x \in \Omega;$$

(iv)

$$\lim_{s\to 0^+} \frac{f(x,s)}{s^{q-1}} = 0 \quad \text{uniformly for a. a. } x \in \Omega$$

and there exists $\tau \in (q,p)$ such that

$$\liminf_{s\to 0^+} \frac{f(x,s)}{s^{\tau-1}} \geq \hat{\eta} > 0 \quad \text{uniformly for a. a. } x \in \Omega;$$

(v) for every $\hat{s}>0$ we have

$$f(x,s) \geq m_{\hat{s}} > 0$$

for a.a. $x \in \Omega$ and for all $s \geq \hat{s}$ and for every $\rho>0$ there exists $\hat{\xi}_\rho>0$ such that the function

$$s \to f(x,s)+\hat{\xi}_\rho s^{p-1}$$

is nondecreasing on $[0,\rho]$ for a.a. $x \in \Omega$.

Remark 2.2. *Since we are looking for positive solutions and the hypotheses above concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss generality, we may assume that*

$$f(x,s)=0 \quad \textit{for a.a. } x \in \Omega \textit{ and for all } s \leq 0. \tag{2.2}$$

Hypotheses H(ii), H(iii) imply that

$$\lim_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} = +\infty \quad \textit{uniformly for a.a. } x \in \Omega.$$

Hence, the perturbation $f(x,\cdot)$ is $(p-1)$-superlinear. In the literature, superlinear equations are usually treated by using the AR-condition. In our case, taking (2.2) into account, we refer to a unilateral version of this condition which says that there exist $M>0$ and $\mu>p$ such that

$$0<\mu F(x,s) \leq f(x,s)s \quad \textit{for a. a. } x \in \Omega \textit{ and for all } s \geq M, \tag{2.3}$$

$$0<\operatorname*{ess\,inf}_\Omega F(\cdot,M). \tag{2.4}$$

If we integrate (2.3) and use (2.4), we obtain the weaker condition

$$c_1 s^\mu \leq F(x,s) \quad \text{for a. a. } x \in \Omega, \text{ for all } s \geq M \text{ and for some } c_1 > 0.$$

This implies, due to (2.3), that

$$c_1 s^{\mu-1} \leq f(x,s) \quad \text{for a. a. } x \in \Omega \text{ and for all } s \geq M.$$

We see that the AR-condition is dictating that $f(x,\cdot)$ eventually has $(\mu-1)$-polynomial growth. Here, instead of the AR-condition, see (2.3), (2.4), we employ a less restrictive behavior near $+\infty$, see hypothesis H(iii). This way we are able to incorporate in our framework superlinear nonlinearities with "slower" growth near $+\infty$. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ (for the sake of simplicity we drop the x-dependence) defined by

$$f(x) = \begin{cases} s^{\mu-1} & \text{if } 0 \leq s \leq 1, \\ s^{p-1}\ln(x) + s^{\tilde{s}-1} & \text{if } 1 < s \end{cases}$$

with $q < \mu < p$ and $\tilde{s} < p$, see (2.2). This function satisfies hypotheses H, but fails to satisfy the AR-condition.

By a solution of (P_λ) we mean a function $u \in W_0^{1,p}(\Omega)$, $u \geq 0$, $u \neq 0$, such that $uh \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$ and

$$\langle A_p(u), h\rangle + \langle A_q(u), h\rangle = \int_\Omega u^{-\eta} h\, dx + \lambda \int_\Omega f(x,u) h\, dx \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

The energy functional $\varphi_\lambda\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ of the problem (P_λ) is given by

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{1-\eta}\int_\Omega \left(u^+\right)^{1-\eta} dx - \lambda \int_\Omega F\left(x,u^+\right) dx$$

for all $h \in W_0^{1,p}(\Omega)$.

We can find solutions of (P_λ) among the critical points of φ_λ. The problem that we face is that because of the third term, so the singular one, the energy functional φ_λ is not C^1. So, we cannot apply directly the minimax theorems of the critical point theory on φ_λ. Solving related auxiliary Dirichlet problems and then using suitable truncation and comparison techniques, we are able to overcome this difficulty, isolate the singularity and deal with C^1-functionals on which the classical critical point theory can be used.

To this end, first we consider the following purely singular Dirichlet problem

$$\begin{aligned} -\Delta_p u - \Delta_q u &= u^{-\eta} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ u > 0, \quad 0 < \eta &< 1, && 1 < q < p. \end{aligned} \tag{2.5}$$

From Proposition 10 of Papageorgiou-Rădulescu-Repovš [12] we have the following result concerning problem (2.5).

Proposition 2.3. *Problem (2.5) admits a unique solution $\underline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

From the Lemma in Lazer-McKenna [9] we know that

$$\underline{u}^{-\eta} \in L^1(\Omega).$$

Moreover, from Hardy's inequality we have

$$\underline{u}^{-\eta}h \in L^1(\Omega) \quad \text{and} \quad \int_\Omega \left|\underline{u}^{-\eta}h\right| dx \le \hat{c}\|h\|$$

for all $h \in W_0^{1,p}(\Omega)$. It follows that $\underline{u}^{-\eta}+1 \in W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$.

So, we can consider a second auxiliary Dirichlet problem

$$\begin{aligned}
-\Delta_p u - \Delta_q u &= \underline{u}^{-\eta}+1 \qquad && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,\\
0<\eta<1&, \quad 1<q<p.
\end{aligned} \tag{2.6}$$

We show that (2.6) has a unique solution.

Proposition 2.4. *Problem (2.6) admits a unique solution* $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Proof. Consider the operator $L\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ with $\frac{1}{p}+\frac{1}{p'}=1$ defined by

$$L(u) = A_p(u) + A_q(u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

This operator is continuous, strictly monotone, hence maximal monotone and coercive. Since $\underline{u}^{-\eta}+1 \in W^{-1,p'(\Omega)}$ (see the comments after Proposition 2.3), we can find $\overline{u} \in W_0^{1,p}(\Omega), \overline{u} \neq 0$ such that

$$L\left(\overline{u}\right) = \underline{u}^{-\eta}+1.$$

The strict monotonicity of L implies the uniqueness of $\overline{u}$ while Theorem B.1 of Giacomoni-Schindler-Takáč [7] implies that $\overline{u} \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Furthermore, we have

$$\Delta_p\overline{u}(x) + \Delta_q\overline{u}(x) \le 0 \quad \text{for a. a. } x \in \Omega.$$

Hence, from the nonlinear maximum principle, see Pucci-Serrin [17, pp. 111 and 120], we conclude that $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

3. Positive solutions

We introduce the following two sets

$$\begin{aligned}
\mathcal{L} &= \left\{\lambda>0 : \text{problem } (\mathrm{P}_\lambda) \text{ has a positive solution}\right\},\\
\mathcal{S}_\lambda &= \left\{u : u \text{ is a positive solution of problem } (\mathrm{P}_\lambda)\right\}.
\end{aligned}$$

Proposition 3.1. *If hypotheses H hold, then* $\mathcal{L} \neq \emptyset$.

Proof. Let $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ be as in Proposition 2.4. Hypothesis H(i) implies that $f(\cdot,\overline{u}(\cdot)) \in L^\infty(\Omega)$. So, we can find $\lambda_0>0$ such that

$$0 \le \lambda_0 f\left(x,\overline{u}(x)\right) \le 1 \quad \text{for a. a. } x \in \Omega. \tag{3.1}$$

From the weak comparison principle (see Pucci-Serrin [17, Theorem 3.4.1, p. 61]), we have $\underline{u} \le \overline{u}$. So, for given $\lambda \in (0,\lambda_0]$, we can define the following truncation of the reaction of problem (P_λ)

$$g_\lambda(x,s) = \begin{cases}
\underline{u}(x)^{-\eta} + \lambda f(x,\underline{u}(x)) & \text{if } s < \underline{u}(x),\\
s^{-\eta} + \lambda f(x,s) & \text{if } \underline{u}(x) \le s \le \overline{u}(x),\\
\overline{u}(x)^{-\eta} + \lambda f(x,\overline{u}(x)) & \text{if } \overline{u}(x) < s.
\end{cases} \tag{3.2}$$

This is a Carathéodory function. We set $G_\lambda(x,s)=\int_0^s g_\lambda(x,t)\,dt$ and consider the C^1-functional $\psi_\lambda\colon W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\psi_\lambda(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega G_\lambda(x,u)\,dx \quad \text{for all } u\in W_0^{1,p}(\Omega),$$

see also Papageorgiou-Smyrlis [13, Proposition 3]. From (3.2) we see that ψ_λ is coercive. Also, using the Sobolev embedding theorem, we see that ψ_λ is sequentially weakly lower semicontinuous. So, by the Weierstraß-Tonelli theorem, we can find $u_\lambda\in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_\lambda)=\min\left[\psi_\lambda(u)\,:\,u\in W_0^{1,p}(\Omega)\right].$$

This means, in particular, that $\psi_\lambda'(u_\lambda)=0$, which gives

$$\langle A_p(u_\lambda),h\rangle+\langle A_q(u_\lambda),h\rangle=\int_\Omega g_\lambda(x,u_\lambda)h\,dx \quad \text{for all } h\in W_0^{1,p}(\Omega). \tag{3.3}$$

First, we choose $h=(\underline{u}-u_\lambda)^+\in W_0^{1,p}(\Omega)$ in (3.3). This yields, because of (3.2), $f\geq 0$ and Proposition 2.3 that

$$\begin{aligned}
&\left\langle A_p(u_\lambda),(\underline{u}-u_\lambda)^+\right\rangle+\left\langle A_q(u_\lambda),(\underline{u}-u_\lambda)^+\right\rangle\\
&=\int_\Omega\left[\underline{u}^{-\eta}+\lambda f(x,\underline{u})\right](\underline{u}-u_\lambda)^+\,dx\\
&\geq\int_\Omega \underline{u}^{-\eta}(\underline{u}-u_\lambda)^+\,dx\\
&=\left\langle A_p(\underline{u}),(\underline{u}-u_\lambda)^+\right\rangle+\left\langle A_q(\underline{u}),(\underline{u}-u_\lambda)^+\right\rangle.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{\{\underline{u}>u_\lambda\}}\left(|\nabla\underline{u}|^{p-2}\nabla\underline{u}-|\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\cdot(\nabla\underline{u}-\nabla u_\lambda)\,dx\\
&+\int_{\{\underline{u}>u_\lambda\}}\left(|\nabla\underline{u}|^{q-2}\nabla\underline{u}-|\nabla u_\lambda|^{q-2}\nabla u_\lambda\right)\cdot(\nabla\underline{u}-\nabla u_\lambda)\,dx\\
&\leq 0,
\end{aligned}$$

which means $|\{\underline{u}>u_\lambda\}|_N=0$ with $|\cdot|_N$ being the Lebesgue measure of $\mathbb{R}^N$. Hence,

$$\underline{u}\leq u_\lambda. \tag{3.4}$$

Next, we choose $h=(u_\lambda-\overline{u})^+\in W_0^{1,p}(\Omega)$ in (3.3). Applying (3.2), (3.4), (3.1) and recall that $0<\lambda\leq\lambda_0$, we obtain

$$\begin{aligned}
&\left\langle A_p(u_\lambda),(u_\lambda-\overline{u})^+\right\rangle+\left\langle A_q(u_\lambda),(u_\lambda-\overline{u})^+\right\rangle\\
&=\int_\Omega\left[\overline{u}^{-\eta}+\lambda f(x,\overline{u})\right](u_\lambda-\overline{u})^+\,dx\\
&\leq\int_\Omega\left[\underline{u}^{-\eta}+1\right](u_\lambda-\overline{u})^+\,dx\\
&=\left\langle A_p(\overline{u}),(u_\lambda-\overline{u})^+\right\rangle+\left\langle A_q(\overline{u}),(u_\lambda-\overline{u})^+\right\rangle
\end{aligned}$$

From this we see that

$$\begin{aligned}&\int_{\{u_\lambda>\overline{u}\}}\left(|\nabla u_\lambda|^{p-2}\nabla u_\lambda-|\nabla\overline{u}|^{p-2}\nabla\overline{u}\right)\cdot\left(\nabla u_\lambda-\nabla\overline{u}\right)dx\\&+\int_{\{u_\lambda>\overline{u}\}}\left(|\nabla u_\lambda|^{q-2}\nabla u_\lambda-|\nabla\overline{u}|^{q-2}\nabla\overline{u}\right)\cdot\left(\nabla u_\lambda-\nabla\overline{u}\right)dx\\&\leq 0\end{aligned}$$

and so $|\{u_\lambda>\overline{u}\}|_N=0$. Thus, $u_\lambda\leq\overline{u}$. So, we have proved that

$$u_\lambda\in[\underline{u},\overline{u}]. \tag{3.5}$$

Then, (3.5), (3.2) and (3.3) imply that $u_\lambda\in\mathcal{S}_\lambda$ and so $(0,\lambda_0]\subseteq\mathcal{L}\neq\emptyset$. ☐

Proposition 3.2. *If hypotheses H hold and $\lambda\in\mathcal{L}$, then $\underline{u}\leq u$ for all $u\in\mathcal{S}_\lambda$.*

Proof. Let $u\in\mathcal{S}_\lambda$. On $\Omega\times(0,+\infty)$ we introduce the Carathéodory function $k(\cdot,\cdot)$ defined by

$$k(x,s)=\begin{cases}s^{-\eta}&\text{if } 0<s\leq u(x),\\u(x)^{-\eta}&\text{if } u(x)<s\end{cases} \tag{3.6}$$

for all $(x,s)\in\Omega\times(0,+\infty)$. Then we consider the following Dirichlet (p,q)-problem

$$\begin{aligned}-\Delta_p u-\Delta_q u&=k(x,u)&&\text{in }\Omega,\\u&=0&&\text{on }\partial\Omega,\\u&>0,\quad 1<q<p.\end{aligned}$$

Proposition 10 of Papageorgiou-Rădulescu-Repovš [12] implies that this problem admits a solution

$$\underline{\tilde{u}}\in\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.7}$$

This means

$$\langle A_p(\underline{\tilde{u}}),h\rangle+\langle A_q(\underline{\tilde{u}}),h\rangle=\int_\Omega k(x,\underline{\tilde{u}})\,h\,dx\quad\text{for all } h\in W_0^{1,p}(\Omega). \tag{3.8}$$

Choosing $h=(\underline{\tilde{u}}-u)^+\in W_0^{1,p}(\Omega)$ in (3.8) and applying (3.6), $f\geq 0$ and $u\in\mathcal{S}_\lambda$ gives

$$\begin{aligned}&\left\langle A_p(\underline{\tilde{u}}),(\underline{\tilde{u}}-u)^+\right\rangle+\left\langle A_q(\underline{\tilde{u}}),(\underline{\tilde{u}}-u)^+\right\rangle\\&=\int_\Omega u^{-\eta}\left(\underline{\tilde{u}}-u\right)^+dx\\&\leq\int_\Omega\left[u^{-\eta}+\lambda f(x,u)\right]\left(\underline{\tilde{u}}-u\right)^+dx\\&=\left\langle A_p(u),(\underline{\tilde{u}}-u)^+\right\rangle+\left\langle A_q(u),(\underline{\tilde{u}}-u)^+\right\rangle.\end{aligned}$$

This implies

$$\begin{aligned}&\int_{\{\underline{\tilde{u}}>u\}}\left(|\nabla\underline{\tilde{u}}|^{p-2}\nabla\underline{\tilde{u}}-|\nabla u|^{p-2}\nabla u\right)\cdot\left(\nabla\underline{\tilde{u}}-\nabla u\right)dx\\&+\int_{\{\underline{\tilde{u}}>u\}}\left(|\nabla\underline{\tilde{u}}|^{q-2}\nabla\underline{\tilde{u}}-|\nabla u|^{q-2}\nabla u\right)\cdot\left(\nabla\underline{\tilde{u}}-\nabla u\right)dx\\&\leq 0,\end{aligned}$$

which means $|\{\tilde{\underline{u}} > u\}|_N = 0$. Thus,

$$\tilde{\underline{u}} \le u. \tag{3.9}$$

From (3.9), (3.7), (3.6), (3.8) and Proposition 2.3 it follows that $\tilde{\underline{u}} = u$. Therefore, $\underline{u} \le u$ for all $u \in \mathcal{S}_\lambda$. □

As before, using Theorem B.1 of Giacomoni-Schindler-Takáč [7], we have the following result about the solution set S_λ.

Proposition 3.3. *If hypotheses H hold and $\lambda \in \mathcal{L}$, then $S_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 3.4. *If hypotheses H hold, then $\lambda^* < \infty$.*

Proof. Hypotheses H(ii), (iii) imply that we can find $M > 0$ such that

$$f(x,s) \ge s^{p-1} \quad \text{for a. a. } x \in \Omega \text{ and for all } s \ge M.$$

Moreover, hypothesis H(iv) implies that there exist $\delta \in (0,1)$ and $\hat{\eta}_1 \in (0,\hat{\eta})$ such that

$$f(x,s) \ge \hat{\eta}_1 s^{\tau-1} \ge \hat{\eta}_1 s^{p-1}$$

for a. a. $x \in \Omega$ and for all $0 \le s \le \delta$ since $\tau < p$ and $\delta < 1$. This yields

$$\frac{1}{\hat{\eta}_1} f(x,s) \ge s^{p-1} \quad \text{for a. a. } x \in \Omega \text{ and for all } 0 \le s \le \delta.$$

In addition, on account of hypothesis H(v) we can find $\tilde{\lambda} > 0$ large enough such that

$$\tilde{\lambda} f(x,s) \ge M^{p-1} \quad \text{for a. a. } x \in \Omega \text{ and for all } \delta \le s \le M.$$

Therefore, taking into account the calculations above, there exists $\hat{\lambda} > 0$ large enough such that

$$s^{p-1} \le \hat{\lambda} f(x,s) \quad \text{for a. a. } x \in \Omega \text{ and for all } s \ge 0. \tag{3.10}$$

Let $\lambda > \hat{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Proposition 3.3. Let $\Omega' \subset\subset \Omega$ with C^2-boundary $\partial\Omega'$. Then $m_0 = \min_{\overline{\Omega'}} u_\lambda > 0$ since $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(v). For $\delta > 0$, we set $m_0^\delta = m_0 + \delta$. Applying (3.10), hypothesis H(v) and $u_\lambda \in \mathcal{S}_\lambda$, we have for a. a. $x \in \Omega'$

$$\begin{aligned}
& -\Delta_p m_0^\delta - \Delta_q m_0^\delta + \lambda \hat{\xi}_\rho \left(m_0^\delta\right)^{p-1} - \lambda \left(m_0^\delta\right)^{-\eta} \\
& \le \lambda \hat{\xi}_\rho m_0^{p-1} + \chi(\delta) \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
& \le \left[\lambda \hat{\xi}_\rho + 1\right] m_0^{p-1} + \chi(\delta) \\
& \le \hat{\lambda} f(x,m_0) + \lambda \hat{\xi}_\rho m_0^{p-1} + \chi(\delta) \\
& = \lambda \left[f(x,m_0) + \hat{\xi}_\rho m_0^{p-1}\right] - \left(\lambda - \hat{\lambda}\right) f(x,m_0) + \chi(\delta) \\
& \le \lambda \left[f\left(x,u_\lambda(x)\right) + \hat{\xi}_\rho u_\lambda(x)^{p-1}\right] \quad \text{for } \delta > 0 \text{ small enough} \\
& = -\Delta_p u_\lambda(x) - \Delta_q u_\lambda(x) + \lambda \hat{\xi}_\rho u_\lambda(x)^{p-1} - \lambda u_\lambda(x)^{-\eta}.
\end{aligned}$$

Note that for $\delta > 0$ small enough, we will have

$$0 < \hat{\eta} \leq \left[\lambda - \hat{\lambda}\right] f(x, m_0) - \chi(\delta) \quad \text{for a. a. } x \in \Omega',$$

see hypothesis H(v). Then, invoking Proposition 6 of Papageorgiou-Rădulescu-Repovš [12], it follows that

$$m_0^\delta < u_\lambda(x) \quad \text{for a. a. } x \in \Omega' \text{ and for } \delta > 0 \text{ small enough,}$$

which contradicts the definition of m_0. Therefore, $\lambda \notin \mathcal{L}$ and so we conclude that $\lambda^* \leq \hat{\lambda} < \infty$. □

Next, we are going to show that $\mathcal{L}$ is an interval. So, we have

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*] .$$

Proposition 3.5. *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. We know that $\underline{u} \leq u_\lambda$, see Proposition 3.2. So, we can define the following truncation $e_\mu \colon \Omega \times \mathbb{R} \to \mathbb{R}$ of the reaction for problem (P_λ)

$$e_\mu(x,s) = \begin{cases} \underline{u}(x)^{-\eta} + \mu f(x, \underline{u}(x)) & \text{if } s < \underline{u}(x), \\ s^{-\eta} + \mu f(x,s) & \text{if } \underline{u}(x) \leq s \leq u_\lambda(x), \\ u_\lambda(x)^{-\eta} + \mu f\left(x, u_\lambda(x)\right) & \text{if } u_\lambda(x) < s, \end{cases} \tag{3.11}$$

which is a Carathéodory function. We set $E_\mu(x,s) = \int_0^s e_\mu(x,t)\, dt$ and consider the C^1-functional $\hat{\varphi}_\mu \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\mu(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega E_\mu(x,u)\, dx \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

see Papageorgiou-Vetro-Vetro [14]. From (3.11) it is clear that $\hat{\varphi}_\mu$ is coercive. Moreover, it is sequentially weakly lower semicontinuous. Therefore, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\mu\left(u_\mu\right) = \min\left[\hat{\varphi}_\mu(u) \,:\, u \in W_0^{1,p}(\Omega)\right].$$

In particular, we have $\hat{\varphi}_\mu'\left(u_\mu\right) = 0$ which means

$$\langle A_p\left(u_\mu\right), h\rangle + \langle A_q\left(u_\mu\right), h\rangle = \int_\Omega e_\mu(x,u) h\, dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.12}$$

Choosing $h = (\underline{u} - u_\mu)^+ \in W_0^{1,p}(\Omega)$ in (3.12) and applying (3.11), $f \geq 0$ and Proposition 2.3 yields

$$\begin{aligned} &\left\langle A_p\left(u_\mu\right), \left(\underline{u} - u_\mu\right)^+\right\rangle + \left\langle A_q\left(u_\mu\right), \left(\underline{u} - u_\mu\right)^+\right\rangle \\ &= \int_\Omega \left[\underline{u}^{-\eta} + \mu f(x,\underline{u})\right]\left(\underline{u} - u_\mu\right)^+ dx \\ &\geq \int_\Omega \underline{u}^{-\eta}\left(\underline{u} - u_\mu\right)^+ dx \\ &= \left\langle A_p\left(\underline{u}\right), \left(\underline{u} - u_\mu\right)^+\right\rangle + \left\langle A_q\left(\underline{u}\right), \left(\underline{u} - u_\mu\right)^+\right\rangle. \end{aligned}$$

We obtain $\underline{u} \leq u_\mu$. Furthermore, choosing $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.12) and applying (3.11), $\mu < \lambda$ and $u_\lambda \in \mathcal{S}_\lambda$, we get

$$\begin{aligned}
&\left\langle A_p(u_\mu), (u_\mu - u_\lambda)^+ \right\rangle + \left\langle A_q(u_\mu), (u_\mu - u_\lambda)^+ \right\rangle \\
&= \int_\Omega \left[u_\lambda^{-\eta} + \mu f(x, u_\lambda)\right] (u_\mu - u_\lambda)^+ \, dx \\
&\leq \int_\Omega \left[u^{-\eta} + \lambda f(x, u_\lambda)\right] (u_\mu - u_\lambda)^+ \, dx \\
&= \left\langle A_p(u_\lambda), (u_\mu - u_\lambda)^+ \right\rangle + \left\langle A_q(u_\lambda), (u_\mu - u_\lambda)^+ \right\rangle .
\end{aligned}$$

Hence, $u_\mu \leq u_\lambda$ and so we have proved that

$$u_\mu \in [\underline{u}, u_\lambda]. \tag{3.13}$$

From (3.13), (3.11) and (3.12) we infer that

$$u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right).$$

Thus, $\mu \in \mathcal{L}$. □

A byproduct of the proof above is the following corollary.

Corollary 3.6. *If hypotheses H hold, $\lambda \in \mathcal{L}$, $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\mu \in (0, \lambda)$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_\mu \leq u_\lambda$.*

Using the strong comparison principle of Papageorgiou-Rădulescu-Repovš [12] we can improve the conclusion of this corollary as follows.

Proposition 3.7. *If hypotheses H hold, $\lambda \in \mathcal{L}$, $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\mu \in (0, \lambda)$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that*

$$u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right).$$

Proof. From Corollary 3.6 we already have that $\mu \in \mathcal{L}$ and we also know that there exists $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that

$$u_\mu \leq u_\lambda. \tag{3.14}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(v). Applying $u_\mu \in \mathcal{S}_\mu$, (3.14), hypothesis H(v) and $\mu < \lambda$, we obtain

$$\begin{aligned}
&-\Delta_p u_\mu(x) - \Delta_q u_\mu(x) + \lambda \hat{\xi}_\rho u_\mu(x)^{p-1} - u_\mu(x)^{-\eta} \\
&= \mu f(x, u_\mu(x)) + \lambda \hat{\xi}_\rho u_\mu(x)^{p-1} \\
&= \lambda \left[f(x, u_\mu(x)) + \hat{\xi}_\rho u_\mu(x)^{p-1}\right] - (\lambda - \mu) f(x, u_\mu(x)) \\
&\leq \lambda \left[f(x, u_\lambda(x)) + \hat{\xi}_\rho u_\lambda(x)^{p-1}\right] \\
&= -\Delta_p u_\lambda(x) - \Delta_q u_\lambda(x) + \lambda \hat{\xi}_\rho u_\lambda(x)^{p-1} - u_\lambda(x)^{-\eta} \quad \text{for a. a. } x \in \Omega.
\end{aligned} \tag{3.15}$$

Since $u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, because of hypothesis H(v), we have

$$0 \prec (\lambda - \mu) f(\cdot, u_\mu(\cdot)).$$

Then, from (3.15) and Proposition 7 of Papageorgiou-Rădulescu-Repovš [12] we conclude that $u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

Proposition 3.8. *If hypotheses H hold and $\lambda \in (0,\lambda^*)$, then problem* ($P_\lambda$) *has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \quad u_0 \leq \hat{u}, \quad u_0 \neq \hat{u}.$$

Proof. Let $\lambda < \vartheta < \lambda^*$. Due to Proposition 3.7, we can find $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $u_0 \in \mathcal{S}_\lambda$ such that

$$u_\vartheta - u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.16}$$

From Proposition 3.2 we know that $\underline{u} \leq u_0$. Therefore, $u_0^{-\eta} \in L^1(\Omega)$. So, we can define the following truncation $w_\lambda \colon \Omega \times \mathbb{R} \to \mathbb{R}$ of the reaction in problem (P_λ)

$$w_\lambda(x,s) = \begin{cases} u_0(x)^{-\eta} + \lambda f(x,u_0(x)) & \text{if } s \leq u_0(x),\\ s^{-\eta} + \lambda f(x,s) & \text{if } u_0(x) < s. \end{cases} \tag{3.17}$$

Also, using (3.16), we can consider the truncation $\hat{w}_\lambda \colon \Omega\times\mathbb{R} \to \mathbb{R}$ of $w_\lambda(x,\cdot)$ defined by

$$\hat{w}_\lambda(x,s) = \begin{cases} w_\lambda(x,s) & \text{if } s \leq u_\vartheta(x),\\ w_\lambda(x,u_\vartheta(x)) & \text{if } u_\vartheta(x) < s. \end{cases} \tag{3.18}$$

It is clear that both are Carathéodory function. We set

$$W_\lambda(x,s) = \int_0^s w_\lambda(x,t)\,dt \quad \text{and} \quad \hat{W}_\lambda(x,s) = \int_0^s \hat{w}_\lambda(x,t)\,dt$$

and consider the C^1-functionals $\sigma_\lambda, \hat{\sigma}_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \sigma_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega W_\lambda(x,u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega),\\ \hat{\sigma}_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \hat{W}_\lambda(x,u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega). \end{aligned}$$

From (3.17) and (3.18) it is clear that

$$\sigma_\lambda\big|_{[0,u_\vartheta]} = \hat{\sigma}_\lambda\big|_{[0,u_\vartheta]} \quad \text{and} \quad \sigma_\lambda'\big|_{[0,u_\vartheta]} = \hat{\sigma}_\lambda'\big|_{[0,u_\vartheta]}. \tag{3.19}$$

Using (3.17), (3.18) and the nonlinear regularity theory of Lieberman [10] we obtain that

$$K_{\sigma_\lambda} \subseteq [u_0) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad K_{\hat{\sigma}_\lambda} \subseteq [u_0,u_\vartheta] \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.20}$$

From (3.20) we see that we may assume that

$$K_{\sigma_\lambda} \text{ is finite and } K_{\sigma_\lambda} \cap [u_0,u_\vartheta] = \{u_0\}. \tag{3.21}$$

Otherwise we already have a second positive smooth solution larger that u_0 and so we are done.

From (3.18) and since $u_0^{-\eta} \in L^1(\Omega)$, it is clear that $\hat{\sigma}_\lambda$ is coercive and it is also sequentially weakly lower semicontinuous. Hence, we find its global minimizer $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma}_\lambda(\tilde{u}_0) = \min\left[\hat{\sigma}_\lambda(u) \,:\, u \in W_0^{1,p}(\Omega)\right].$$

By (3.20) we see that $\tilde{u}_0 \in K_{\hat{\sigma}_\lambda} \subseteq [u_0, u_\vartheta] \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Then, (3.19) and (3.21) imply $\tilde{u}_0 = u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Finally, from (3.16) we obtain that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of σ_λ and then by Gasiński-Papageorgiou [6] we have that

$$u_0 \text{ is also a local } W_0^{1,p}(\Omega)\text{-minimizer of } \sigma_\lambda. \tag{3.22}$$

From (3.22), (3.21) and Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [11, p. 449] we know that we can find $\rho \in (0,1)$ small enough such that

$$\sigma_\lambda(u_0) < \inf\left[\sigma_\lambda(u) : \|u - u_0\| = \rho\right] = m_\lambda. \tag{3.23}$$

Hypothesis H(ii) implies that if $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, then

$$\sigma_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.24}$$

Claim: The functional σ_λ satisfies the C-condition.
Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\sigma_\lambda(u_n)| \leq c_6 \quad \text{for some } c_6 > 0 \text{ and for all } n \in \mathbb{N}, \tag{3.25}$$

$$(1 + \|u_n\|)\sigma_\lambda'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.26}$$

From (3.26) we have

$$\left|\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle - \int_\Omega w_\lambda(x, u_n) h\, dx\right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{3.27}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. We choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ in (3.27) and obtain, by applying (3.17), that

$$\left\|u_n^-\right\|^p \leq c_7 \quad \text{for some } c_7 > 0 \text{ and for all } n \in \mathbb{N}.$$

This shows that

$$\left\{u_n^-\right\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.28}$$

From (3.25) and (3.28) it follows that

$$\left\|\nabla u_n^+\right\|_p^p + \frac{p}{q}\left\|\nabla u_n^+\right\|_q^q - \int_\Omega pF\left(x, u_n^+\right)\, dx \leq c_8\left[1 + \left\|u_n^+\right\|_\tau\right] \tag{3.29}$$

for some $c_8 > 0$ and for all $n \in \mathbb{N}$, see (3.17). Moreover, choosing $h = u_n^+ \in W_0^{1,p}(\Omega)$ in (3.27), we obtain by using (3.17)

$$-\left\|\nabla u_n^+\right\|_p^p - \left\|\nabla u_n^+\right\|_q^q + \int_\Omega f\left(x, u_n^+\right) u_n^+\, dx \leq c_9 \tag{3.30}$$

for some $c_9 > 0$ and for all $n \in \mathbb{N}$. Adding (3.29) and (3.30) and recall that $q < p$, gives

$$\int_\Omega \left[f\left(x, u_n^+\right) u_n^+ - pF\left(x, u_n^+\right)\right] dx \leq c_{10}\left[1 + \left\|u_n^+\right\|_\tau\right] \tag{3.31}$$

for some $c_{10} > 0$ and for all $n \in \mathbb{N}$.

Taking hypotheses H(i), (iii) into account, we see that we can find constants $c_{11}, c_{12} > 0$ such that

$$c_{11}s^\tau - c_{12} \leq f(x,s)s - pF(x,s) \quad \text{for a. a. } x \in \Omega \text{ and for all } s \geq 0. \tag{3.32}$$

Applying (3.32) in (3.31), we infer that

$$\left\|u_n^+\right\|_\tau^{\tau-1} \leq c_{13}$$

for some $c_{13} > 0$ and for all $n \in \mathbb{N}$. Therefore,

$$\left\{u_n^+\right\}_{n\geq 1} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{3.33}$$

First assume that $p \neq N$. From hypothesis H(iii), we see that we can always assume that $\tau < r < p^*$. So, we can find $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}. \tag{3.34}$$

Invoking the interpolation inequality, see Papageorgiou-Winkert [16, Proposition 2.3.17, p. 116], we have

$$\left\|u_n^+\right\|_r \leq \left\|u_n^+\right\|_\tau^{1-r} \left\|u_n^+\right\|_{p^*}^t.$$

Hence, by (3.33),

$$\left\|u_n^+\right\|_r^r \leq c_{14} \left\|u_n^+\right\|^{tr} \tag{3.35}$$

for some $c_{14} > 0$ and for all $n \in \mathbb{N}$. We choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ in (3.27) to get

$$\left\|u_n^+\right\|^p \leq \int_\Omega w_\lambda\left(x, u_n^+\right) u_n^+ \, dx.$$

Then, from (3.17) and hypothesis H(i), it follows that

$$\left\|u_n^+\right\|^p \leq \int_\Omega c_{15}\left[1 + \left(u_n^+\right)^r\right] \, dx$$

for some $c_{15} > 0$ and for all $n \in \mathbb{N}$. This implies

$$\left\|u_n^+\right\|^p \leq c_{16}\left[1 + \left\|u_n^+\right\|_r^r\right]$$

for some $c_{16} > 0$ and for all $n \in \mathbb{N}$. Finally, from (3.35), we then obtain

$$\left\|u_n^+\right\|^p \leq c_{17}\left[1 + \left\|u_n^+\right\|^{tr}\right] \tag{3.36}$$

for some $c_{17} > 0$ and for all $n \in \mathbb{N}$.

If $N < p$, then $p^* = \infty$ and so from (3.34) we have $tr = r - \tau$, which by hypothesis H(iii) leads to $tr < p$.

If $N > p$, then $p^* = \frac{Np}{N-p}$. From (3.34) it follows

$$tr = \frac{(r-\tau)p^*}{p^* - \tau},$$

which implies

$$tr = \frac{(r-\tau)Np}{N(p-\tau) + \tau p} < p.$$

Therefore, from (3.36) we infer that

$$\left\{u_n^+\right\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.37}$$

If $N = p$, then by the Sobolev embedding theorem, we know that $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ continuously for all $1 \leq s < \infty$. So, for the argument above to work, we need to replace p^* by $s > r > \tau$ in (3.34) which yields

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{s}.$$

Then, by hypothesis H(iii), we obtain

$$tr = \frac{(r-\tau)s}{s-\tau} \to r - \tau < p \quad \text{as } s \to +\infty.$$

We choose $s > r$ large enough so that $tr < p$. Then, we reach again (3.37).

From (3.37) and (3.28) it follows that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^r(\Omega). \tag{3.38}$$

In (3.27) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.38). This gives

$$\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle\right] = 0.$$

The monotonicity of A_q implies

$$\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A_q(u), u_n - u\rangle\right] \leq 0$$

and from (3.38) one has

$$\limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \leq 0.$$

Hence, by Proposition 2.1, it follows

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega).$$

Therefore, σ_λ satisfies the C-condition and this proves the Claim.

Then, (3.23), (3.24) and the Claim permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\sigma_\lambda} \subseteq [u_0) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad \sigma_\lambda(u_0) < m_\lambda \leq \sigma_\lambda\left(\hat{u}\right), \tag{3.39}$$

see (3.20) and (3.23), respectively.

From (3.39), (3.17) and (3.27), we conclude that

$$\hat{u} \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \quad u_0 \leq \hat{u}, \quad u_0 \neq \hat{u}.$$

□

Proposition 3.9. *If hypotheses H hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $0 < \lambda_n < \lambda^*$ with $n \in \mathbb{N}$ and assume that $\lambda_n \nearrow \lambda^*$. By Proposition 3.2 we can find $u_n \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that

$$\underline{u} \leq u_n \quad \text{for all } n \in \mathbb{N}$$

and

$$\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle = \int_\Omega \left[u_n^{-\eta} + \lambda_n f(x, u_n)\right] h \, dx \tag{3.40}$$

for all $h \in W_0^{1,p}(\Omega)$ and for all $n \in \mathbb{N}$. From hypothesis H(iii), we have

$$\varphi_\lambda(u_n) \leq c_{18} \tag{3.41}$$

for some $c_{18} > 0$ and for all $n \in \mathbb{N}$, where φ_λ is the energy functional of problem (P_λ).

From (3.40), (3.41) and reasoning as in the Claim in the proof of Proposition 3.8, we obtain that

$$u_n \to u_* \quad \text{in } W_0^{1,p}(\Omega). \tag{3.42}$$

So, if in (3.40) we pass to the limit as $n \to \infty$ and use (3.42), then

$$\langle A_p(u_*), h\rangle + \langle A_q(u_*), h\rangle = \int_\Omega \left[u_*^{-\eta} + \lambda^* f(x, u_*)\right] h\,dx$$

for all $h \in W_0^{1,p}(\Omega)$ and $\underline{u} \leq u_*$. It follows that $u_* \in \mathcal{S}_{\lambda^*} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so $\lambda^* \in \mathcal{L}$. □

Therefore, we have

$$\mathcal{L} = (0, \lambda^*]\,.$$

We can state the following bifurcation-type theorem describing the variations in the set of positive solutions as the parameter λ moves in $(0, +\infty)$.

Theorem 3.10. *If hypotheses H hold, then there exist $\lambda^* > 0$ such that*

(a) for every $0 < \lambda < \lambda^*$, problem (P_λ) has at least two positive solutions

$$u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \quad u_0 \leq \hat{u}, \quad u_0 \neq \hat{u};$$

(b) for $\lambda = \lambda^*$, problem (P_λ) has at least one positive solution

$$u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right);$$

(c) for every $\lambda > \lambda^*$, problem (P_λ) has no positive solutions.

4. Minimal positive solutions

In this section we show that for every $\lambda \in \mathcal{L} = (0, \lambda^*]$, problem ($P_\lambda$) has a smallest positive solutions $u^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and we investigate the monotonicity and continuity properties of the map $\lambda \to u_\lambda^*$.

Proposition 4.1. *If hypotheses H hold and $\lambda \in \mathcal{L}$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, that is, $u_\lambda^* \leq u$ for all $u \in \mathcal{S}_\lambda$.*

Proof. From Proposition 18 of Papageorgiou-Rădulescu-Repovš [12] we know that the set $\mathcal{S}_\lambda \subseteq W_0^{1,p}(\Omega)$ is downward directed. So, invoking Lemma 3.10 of Hu-Papageorgiou [8, p. 178], we can find a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq \mathcal{S}_\lambda$ such that

$$\underline{u} \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}, \quad \inf_{n\geq 1} u_n = \inf \mathcal{S}_\lambda, \tag{4.1}$$

see Proposition 3.2. From (4.1) we see that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. From this, as in the proof of Proposition 3.8, using Proposition 2.1, we obtain

$$u_n \to u_\lambda^* \quad \text{in } W_0^{1,p}(\Omega), \quad \underline{u} \leq u_\lambda^*.$$

From (4.1) it follows

$$u_\lambda^* \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad u_\lambda^* = \inf \mathcal{S}_\lambda.$$

□

In the next proposition we examine the monotonicity and continuity properties of the map $\lambda \to u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$.

Proposition 4.2. *If hypotheses H hold, then the minimal solution map $\lambda \to u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is*

(a) strictly increasing in the sense that

$$0 < \mu < \lambda \leq \lambda^* \quad \textit{implies} \quad u_\lambda^* - u_\mu^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right);$$

(b) left continuous.

Proof. (a) Let $0 < \mu < \lambda \leq \lambda^*$. According to Proposition 3.2 we can find $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_\lambda^* - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Since $u_\lambda^* \leq u_\mu$ we obtain the desired conclusion.

(b) Suppose that $\lambda_n \to \lambda^- \leq \lambda^*$. Then $\{u_n^*\}_{n\geq 1} := \{u_{\lambda_n}^*\}_{n\geq 1} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is increasing and

$$\underline{u} \leq u_n^* \leq u_{\lambda^*}^* \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

From (4.2) and the nonlinear regularity theory of Lieberman [10] we have that $\{u_n^*\}_{n\geq 1} \subseteq C_0^1(\overline{\Omega})$ is relatively compact and so

$$u_n^* \to \tilde{u}_\lambda^* \quad \text{in } C_0^1(\overline{\Omega}). \tag{4.3}$$

If $\tilde{u}_\lambda^* \neq u_\lambda^*$, then we can find $z_0 \in \Omega$ such that

$$u_\lambda^*(z_0) < \tilde{u}_\lambda^*(z_0).$$

From (4.3) we then derive

$$u_\lambda^*(z_0) < u_n^*(z_0) \quad \text{for all } n \geq n_0,$$

which contradicts (a). So, $\tilde{u}_\lambda^* = u_\lambda^*$ and we conclude the left continuity of $\lambda \to u_\lambda^*$. □

Summarizing our findings in this section, we can state the following theorem.

Theorem 4.3. *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem ($P_\lambda$) admits a smallest positive solution $u_\lambda^* \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and the map $\lambda \to u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is*

(a) strictly increasing;
(b) left continuous.

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