SOLUTIONS WITH SIGN INFORMATION FOR NONLINEAR NONHOMOGENEOUS PROBLEMS

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Abstract. We consider a parametric elliptic equation driven by a nonlinear nonhomogeneous differential operator and with a Robin boundary condition. In the first part we prove the existence of positive solutions and state a bifurcation-type result describing how the set of positive solutions changes as the parameter $\lambda > 0$ varies. In the second part we show that problem admits nodal (sign-changing) solutions provided the parameter $\lambda > 0$ is sufficiently large.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear nonhomogeneous parametric Robin problem

$$\begin{aligned} -\operatorname{div} a(x, \nabla u) &= f(x, u, \lambda) && \text{in } \Omega,\\ \frac{\partial u}{\partial n_a} + \beta(x)|u|^{p-2}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{P$_\lambda$}$$

where $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map which is strictly monotone in the second variable and satisfies appropriate regularity and growth conditions listed in hypotheses H(a) below. These hypotheses are general enough to incorporate many differential operators of interest in our framework such as the weighted p-Laplacian ($1 < p < \infty$) and the weighted (p, q)-differential operator ($1 < q < p < \infty$), that is, a sum of weighted p- and q-Laplacians. In the reaction term $f : \Omega \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ on the right-hand side, $\lambda > 0$ is a parameter and $(x, s) \to f(x, s, \lambda)$ is a Carathéodory function for every $\lambda > 0$, that is, $x \to f(x, s, \lambda)$ is measurable for all $x \in \mathbb{R}$ and $s \to f(x, s, \lambda)$ is continuous for a.a. (almost all) $x \in \Omega$. In the boundary condition, the term $\frac{\partial u}{\partial n_a}$ denotes the generalized normal derivative defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to (a(x, \nabla u), n)_{\mathbb{R}^N}$$

with n being the outward unit normal on $\partial\Omega$ according to Green's theorem.

In the first part of the paper, we are interested in the existence of positive solutions and examine how the set of positive solutions varies with $\lambda > 0$ which is known as a bifurcation-type result. We also prove the existence of a smallest positive solution u_λ^* and study the monotonicity as well as the continuity properties of the map $\lambda \to u_\lambda^*$. In the second part of the paper we are interested in finding so called nodal (sign-changing) solutions. Indeed, we can prove that such solutions exist provided the parameter $\lambda > 0$ is sufficiently large.

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In the past such problems were investigated primarily in the context of Dirichlet problems and usually for equations with competing nonlinearities which are called "concave-convex problems". We mention the works of Ambrosetti-Brezis-Cerami [2], Brock-Iturriaga-Ubilla [5], García Azorero-Peral Alonso-Manfredi [9], Gasiński-Papageorgiou [11], Guo [13], Guo-Zhang [14], Takeuchi [22]. For the Neumann problem there is the work of Cardinali-Papageorgiou-Rubbioni [6] for logistic equations while there is a recent work of Fragnelli-Mugnai-Papageorgiou [8] for a class of nonlinear parametric Robin problems driven by the p-Laplacian.

Our approach is variational based on the critical point theory. We also make use of critical groups in order to distinguish between solutions. In the next section we recall the main analytical tools which we will use in the sequel.

2. Preliminaries

Let X be a Banach space and let X^* be its topological dual while $\langle \cdot,\cdot\rangle$ denotes the duality brackets to the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$ we say that φ satisfies the Palais-Smale condition, the PS-condition for short, if every sequence $(u_n)_{n\geq 1} \subseteq X$ such that $(\varphi(u_n))_{n\geq 1} \subseteq \mathbb{R}$ is bounded and such that $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This compactness-type condition on the functional φ leads to a deformation theorem from which one can derive the minimax theory for the critical values of φ. A central result of this theory is the so-called mountain pass theorem due to Ambrosetti-Rabinowitz [3] which we recall next.

Theorem 2.1. *Let $\varphi \in C^1(X,\mathbb{R})$ be a functional satisfying the PS-condition and let $u_1, u_2 \in X$, $\|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$. Then $c \geq m_\rho$ with c being a critical value of φ.

We denote by $W^{1,p}(\Omega)$ for $1 < p < \infty$ the usual Sobolev space equipped with the norm

$$\|u\|_{1,p} = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{\frac{1}{p}} \quad \text{for } u \in W^{1,p}(\Omega)$$

where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$, resp. $L^p(\Omega,\mathbb{R}^N)$. It is well-known that $W^{1,p}(\Omega)$ is a separable, reflexive Banach space. The boundary Lebesgue space is denoted by $L^q(\partial\Omega)$ for $1 \leq q \leq \infty$.

In addition to the Sobolev space $W^{1,p}(\Omega)$ we will also use the ordered Banach space $C^1(\overline{\Omega})$ and its positive cone

$$C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior containing the open set

$$D_+ = \left\{u \in C^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}$$

and

$$\operatorname{int}\left(C^1(\overline{\Omega})_+\right) = \hat{D}_+ = \Big\{u \in C^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \Omega,$$
$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega \cap u^{-1}(0)} < 0 \text{ if } \partial\Omega \cap u^{-1}(0) \neq \emptyset \Big\}.$$

Evidently, we have $D_+ \subseteq \hat{D}_+$.

The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is well known that

$$u^\pm \in W^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

For $u, v \in W^{1,p}(\Omega)$ with $u \leq v$ we denote by $[u,v]$ the order interval defined by the two Sobolev functions, that is,

$$[u,v] = \left\{h \in W^{1,p}(\Omega) : u(x) \leq h(x) \leq v(x) \text{ for a.a. } x \in \Omega\right\}.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. On the boundary $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ. Having this measure, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$ for $1 \leq q \leq \infty$. From the theory of Sobolev spaces we know that there exists a unique linear, continuous map $\gamma_0 : W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$ with

$$p_* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ \text{any } q \in [1,\infty) & \text{if } p \geq N. \end{cases}$$

called the trace map such that

$$\gamma_0(u) = u\big|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

The trace map gives meaning to the notion of boundary values for an arbitrary Sobolev function. Furthermore, the trace map γ_0 is compact into $L^q(\partial\Omega)$ for all $q < p_*$. Moreover it holds

$$\ker \gamma_0 = W_0^{1,p}(\Omega) \quad \text{and} \quad \operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \text{for } \frac{1}{p'} + \frac{1}{p} = 1.$$

In what follows, for the sake of notational simplicity we drop the usage of the trace operator γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Now let us introduce the hypotheses on the map $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ involved in the definition of the differential operator. So, let $\vartheta \in C^1(0,\infty)$ be a function such that

$$0 < \hat{c}_0 \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2\left(t^{\tau-1} + t^{p-1}\right) \tag{2.1}$$

for all $t > 0$, with some constants $\hat{c}_0, c_0, c_1, c_2 > 0$ and for $1 \leq \tau < p < \infty$. The hypotheses on $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ read as follows.

H(a): $a(x,\xi) = a_0\left(x, \|\xi\|_{\mathbb{R}^N}\right)\xi$ with $a_0 \in C(\overline{\Omega} \times \mathbb{R}_+)$ for all $\xi \in \mathbb{R}^N$ where $\mathbb{R}_+ = [0,+\infty)$ and with $a_0(x,t) > 0$ for all $x \in \overline{\Omega}$, for all $t > 0$ and

(i) $a_0 \in C^1(\overline{\Omega} \times (0,\infty))$, $t \to ta_0(x,t)$ is strictly increasing in $(0,\infty)$, $\lim\limits_{t\to 0^+} ta_0(x,t) = 0$ for all $x \in \overline{\Omega}$ and

$$\lim_{t\to 0^+} \frac{ta_0'(x,t)}{a_0(x,t)} = c > -1 \quad \text{for all } x \in \overline{\Omega};$$

(ii) $\|\nabla_\xi a(x,\xi)\|_{\mathbb{R}^N} \leq c_3 \dfrac{\vartheta\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}}$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and for some $c_3 > 0$;

(iii) $(\nabla_\xi a(x,\xi)y,y)_{\mathbb{R}^N} \geq \dfrac{\vartheta\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}} \|y\|^2_{\mathbb{R}^N}$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and for all $y \in \mathbb{R}^N$;

(iv) there exists $\delta \in (0,1)$ such that

$$\|\nabla_x a_0(x,t)\|_{\mathbb{R}^N} \leq c_4 \left(1 + |\ln \delta|\right) a_0(x,t)$$

for all $x \in \overline{\Omega}$, for all $t \in [\delta, 1]$ and for some $c_4 > 0$.

Remark 2.2. *These conditions on the map $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ are designed in order to use the nonlinear regularity theory of Lieberman [16] and the nonlinear maximum principle of Zhang [25]. If we set*

$$G_0(x,t) = \int_0^t a_0(x,s)s ds,$$

then $G_0 \in C^1(\overline{\Omega} \times \mathbb{R}_+)$ and the function $G_0(x,\cdot)$ is increasing and strictly convex for all $x \in \overline{\Omega}$. We set $G(x,\xi) = G_0(x, \|\xi\|_{\mathbb{R}^N})$ for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$ and obtain that $G \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ and that the function $\xi \to G(x,\xi)$ is convex. Moreover, we easily derive that

$$\nabla_\xi G(x,\xi) = (G_0)_t'(x, \|\xi\|_{\mathbb{R}^N}) \frac{\xi}{\|\xi\|_{\mathbb{R}^N}} = a_0(x, \|\xi\|_{\mathbb{R}^N})\xi = a(x,\xi)$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\nabla_\xi G(x,0) = 0$. So, $G(x,\cdot)$ is the primitive of $a(x,\cdot)$. This fact, the convexity of $G(x,\cdot)$ and since $G(x,0) = 0$ for all $x \in \overline{\Omega}$ imply that

$$G(x,\xi) \leq (a(x,\xi),\xi)_{\mathbb{R}^N} \quad \text{for all } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{2.2}$$

The next lemma summarizes the main properties of $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$. The result is an easy consequence of (2.1) and the hypotheses H(a)(i), (ii), (iii).

Lemma 2.3. *If hypotheses H(a)(i)–(iii) are satisfied, the the following hold:*

(i) $a \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$ and the map $\xi \to a(x,\xi)$ is continuous and strictly monotone and so maximal monotone as well for all $x \in \overline{\Omega}$;

(ii) $\|a(x,\xi)\|_{\mathbb{R}^N} \leq c_5 \left(1 + \|\xi\|^{p-1}_{\mathbb{R}^N}\right)$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$ and for some $c_5 > 0$;

(iii) $(a(x,\xi),\xi)_{\mathbb{R}^N} \geq \frac{c_1}{p-1} \|\xi\|^p_{\mathbb{R}^N}$ for all $x \in \overline{\Omega}$ and for all $\xi \in \mathbb{R}^N$.

From this lemma along with (2.2) we easily deduce the following growth estimates for the primitive $G(x,\cdot)$.

Corollary 2.4. *If hypotheses H(a)(i)–(iii) hold, then*

$$\frac{c_1}{p(p-1)}\|\xi\|^p_{\mathbb{R}^N} \leq G(x,\xi) \leq c_6\left(1+\|\xi\|^p_{\mathbb{R}^N}\right)$$

for all $x \in \overline{\Omega}$, *for all* $\xi \in \mathbb{R}^N$ *and for some* $c_6 > 0$.

Example 2.5. *Let* $\hat{a} \in C^1(\overline{\Omega})$ *be such that*

$$0 < \eta_0 \leq \hat{a}(x) \leq \eta_1 \quad \text{and} \quad 0 < \eta_0 \leq \|\nabla\hat{a}(x)\|_{\mathbb{R}^N} \leq \eta_1 \quad \text{for all } x \in \overline{\Omega}.$$

We consider the following maps:

$$\begin{aligned}
&a_1(x,y) = \hat{a}(x)\|y\|^{p-2}_{\mathbb{R}^N}y \quad \text{with } 1 < p < \infty,\\
&a_2(x,y) = \hat{a}(x)\|y\|^{p-2}_{\mathbb{R}^N}y + \|y\|^{q-2}_{\mathbb{R}^N}y \quad \text{with } 1 < q < p < \infty,\\
&a_3(x,y) = \hat{a}(x)\left(1+\|y\|^2_{\mathbb{R}^N}\right)^{\frac{p-2}{2}} y \quad \text{with } 1 < p < \infty.
\end{aligned}$$

These maps satisfy hypotheses H(a). The map a_1 *corresponds to a weighted version of the p-Laplacian*

$$\operatorname{div}\left(\hat{a}(x)\|\nabla u\|^{p-2}_{\mathbb{R}^N}\nabla u\right) \quad \text{for all } u \in W^{1,p}(\Omega)$$

while the map a_2 *corresponds to a weighted* (p,q)*-Laplacian given by*

$$\operatorname{div}\left(\hat{a}(x)\|\nabla u\|^{p-2}_{\mathbb{R}^N}\nabla u\right) + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Now, let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(x,\nabla u), \nabla h)_{\mathbb{R}^N}\,dx \quad \text{for all } u,h \in W^{1,p}_0(\Omega).$$

Hypotheses H(a) imply that A is continuous, monotone, hence maximal monotone as well.

Consider now a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$|f_0(x,s)| \leq a_0(x)\left(1+|s|^{r-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and for all } x \in \mathbb{R}$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 \leq r \leq p^*$, where p^* denotes the critical Sobolev exponent given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N,\\ +\infty & \text{if } N \leq p.\end{cases}$$

We set $F_0(x,s) = \int_0^s f_0(x,t)dt$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(x,\nabla u)dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)|u|^p d\sigma - \int_\Omega F_0(x,u)dx.$$

From Papageorgiou-Rădulescu [19] we have the following result.

Proposition 2.6. *Let the assumptions in H(a)(i)–(iii) be satisfied. If* $u_0 \in W^{1,p}(\Omega)$ *is a local* $C^1(\overline{\Omega})$*-minimizer of* φ_0, *that is, there exists* $\rho_0 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0+h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$ *and* u_0 *is a local* $W^{1,p}(\Omega)$*-minimizer of* φ_0, *that is, there exists* $\rho_1 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0+h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{1,p} \leq \rho_1.$$

The hypotheses on the boundary coefficient β are the following ones.

H(β): $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$ and $\beta(x) > 0$ for all $x \in \partial\Omega$.

Remark 2.7. *This hypothesis excludes the Neumann problem, that is, $\beta \equiv 0$, from our consideration. Indeed, as we will see in Section 3, the Neumann problem does not have positive solutions for any $\lambda > 0$. The requirement that $\beta(x) > 0$ for all $x \in \partial\Omega$ is in order to use the strong comparison principles, see Propositions 2.9 and 2.10.*

The next result will be useful in obtaining a priori estimates. It extends a corresponding result of Zeidler [24, p. 1033].

Proposition 2.8. *If $\beta \in L^\infty(\partial\Omega)$, $\beta(x) \geq 0$ σ-a.e. on $\partial\Omega$ and $\beta \not\equiv 0$, then*

$$u \to |u|_{1,p} = \|\nabla u\|_p + \left(\int_{\partial\Omega} \beta(x)|u|^q d\sigma\right)^{\frac{1}{q}}$$

with $1 \leq q \leq \frac{(N-1)p}{N-p}$ if $N > p$ and $1 \leq q < \infty$ if $p \geq N$, is an equivalent norm on $W^{1,p}(\Omega)$.

Proof. Taking the continuity of the trace map into account, we have for every $u \in W^{1,p}(\Omega)$

$$|u|_{1,p} \leq \|\nabla u\|_p + \|\beta\|_{L^\infty(\partial\Omega)}\|\gamma_0(u)\|_{L^q(\partial\Omega)} \leq \|\nabla u\|_p + \|\beta\|_{L^\infty(\partial\Omega)}\|\gamma_0\|_L c_7\|u\|_{1,p}$$

for some $c_7 > 0$ where $\|\gamma_0\|_L$ denotes the operator norm of γ_0. This gives

$$|u|_{1,p} \leq c_8\|u\|_{1,p} \quad \text{for some } c_8 > 0. \tag{2.3}$$

Next we show that

$$\|u\|_q \leq c_9|u|_{1,p} \quad \text{for some } c_9 > 0. \tag{2.4}$$

Arguing by contradiction, suppose that (2.4) is not true. Then there exists a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that $\|u_n\|_q > n|u_n|$ for all $n \in \mathbb{N}$. Let $y_n = \frac{u_n}{\|u_n\|_q}$ for all $n \in \mathbb{N}$. Then $\|y_n\|_q = 1$ for all $n \geq 1$ and we have

$$\frac{1}{n} > |y_n|_{1,p}, \tag{2.5}$$

which shows that $|y_n|_{1,p} \to 0$ as $n \to \infty$. Recall that $u \to \|\nabla u\|_p + \|u\|_q$ is an equivalent norm on $W^{1,p}(\Omega)$, see, for example, Gasiński-Papageorgiou [10, Theorem 2.5.24, p. 227]. So, it follows that $\{y_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. Hence we may assume that

$$y_n \overset{w}{\to} y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^q(\Omega) \text{ and in } L^q(\partial\Omega). \tag{2.6}$$

Here we have used the Rellich-Kondrachov theorem and the compactness of the trace map. From (2.5) and (2.6) we obtain

$$\|\nabla y\|_p + \left(\int_{\partial\Omega} \beta(x)|y|^q d\sigma\right)^{\frac{1}{q}} \leq 0, \tag{2.7}$$

which implies, due to H(β), that $y \equiv \xi \in \mathbb{R}$. From (2.7) and hypothesis H(β) it follows that

$$0 < |\xi| \left(\int_\Omega \beta(x) d\sigma\right)^{\frac{1}{q}} \leq 0,$$

a contradiction. Therefore, (2.4) is true and we have

$$\|u\|_{1,p} \leq c_{10}|u|_{1,p} \quad \text{for some } c_{10} > 0 \text{ and for all } u \in W^{1,p}(\Omega). \tag{2.8}$$

From (2.3) and (2.8) we conclude that $|\cdot|_{1,p}$ is an equivalent norm on $W^{1,p}(\Omega)$. $\square$

It is well known that it is difficult to have strong comparison principles for nonlinear equations. We refer to the works of Arcoya-Ruiz [4] and Cuesta-Takáč [7] who proved such results for Dirichlet problems. The result of Arcoya-Ruiz [4] was extended to Robin problems by Fragnelli-Mugnai-Papageorgiou [8]. We recall their result for future use.

Let $h_1, h_2 \in L^\infty(\Omega)$. We write $h_1 \prec h_2$ if for every compact $K \subseteq \Omega$ there exists $\varepsilon = \varepsilon(K) > 0$ such that

$$h_1(x) + \varepsilon \leq h_2(x) \quad \text{for a.a. } x \in K.$$

Evidently, if $h_1, h_2 \in C(\Omega)$ and $h_1(x) < h_2(x)$ for all $x \in \Omega$, then $h_1 \prec h_2$. The strong comparison result of Fragnelli-Mugnai-Papageorgiou [8] extends Proposition 6 of Arcoya-Ruiz [4].

Proposition 2.9. *If hypotheses $H(a)(i)$, (ii), (iii) hold, $\xi \in L^\infty(\Omega)_+, h_1, h_2 \in L^\infty(\Omega)$ with $h_1 \prec h_2$, $u \in C^1(\overline{\Omega}), u \neq 0, v \in D_+, u \leq v$ and they satisfy*

$$\begin{aligned} -\operatorname{div} a(x, \nabla u(x)) + \xi(x)|u(x)|^{p-2}u(x) &= h_1(x) \quad \textit{for a.a. } x \in \Omega, \\ -\operatorname{div} a(x, \nabla v(x)) + \xi(x)v(x)^{p-1} &= h_2(x) \quad \textit{for a.a. } x \in \Omega \end{aligned}$$

and $\frac{\partial v}{\partial n}\big|_{\partial\Omega} < 0$, *then* $v - u \in \hat{D}_+$.

The next strong comparison principle extends Theorem 2.1 of Cuesta-Takáč [7].

Proposition 2.10. *If hypotheses $H(a)(i)$, (ii), (iii) hold, $h_1, h_2 \in L^\infty(\Omega)$, $h_1(x) \leq h_2(x)$ for a.a. $x \in \Omega$ and the inequality is strict on a set of positive measure and $u, v \in C^1(\overline{\Omega})$ satisfy $u \leq v$ on Ω and*

$$-\operatorname{div} a(x, \nabla u(x)) = h_1(x) \quad \textit{for a.a. } x \in \Omega, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0,$$

$$-\operatorname{div} a(x, \nabla v(x)) = h_2(x) \quad \textit{for a.a. } x \in \Omega, \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} < 0,$$

then $v - u \in \hat{D}_+$.

Proof. From the hypotheses on the normal derivatives of u and v we see that for $\delta > 0$ small enough we have

$$\left|\nabla\left[(1-t)u(x) + tv(x)\right]\right| \geq \varepsilon > 0 \quad \text{for all } t \in [0,1] \text{ and for all } x \in \overline{\Omega}_\delta, \tag{2.9}$$

where $\overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}$. Then from the hypotheses of the proposition we get

$$-\operatorname{div}\left[a(x, \nabla v(x)) - a(x, \nabla u(x)\right] \geq 0 \quad \text{for a.a. } x \in \Omega. \tag{2.10}$$

Let $a = (a_k)_{k=1}^N$. Then for $k \in \{1, \ldots, N\}$, by the mean value theorem, we obtain

$$a_k(x, \xi) - a_k(x, \xi') = \sum_{k=1}^N \int_0^1 \frac{\partial a_k}{\partial y_m}\left(x, \xi' + t(\xi - \xi')\right)(\xi_m - \xi'_m)\, dt$$

for all $\xi=(\xi_m)_{m=1}^N$ and $\xi'=(\xi'_m)_{m=1}^N$. On Ω_δ we define the following coefficients

$$c_{k,m}(x)=\int_0^1 \frac{\partial a_k}{\partial y_m}(\nabla u(x)+t(\nabla v(x)-\nabla u(x)))dt.$$

Using these coefficients we introduce the following second order differential operator

$$L(w)=-\operatorname{div}\left(\sum_{m=1}^N c_{k,m}(x)\frac{\partial w}{\partial z_m}\right)\quad \text{for all } w\in W^{1,p}(\Omega_\delta).$$

From (2.9) we see that the operator L is strictly elliptic and due to (2.10) one has

$$L(v-u)(x)\geq 0 \quad \text{for a.a. } x\in\Omega_\delta. \tag{2.11}$$

We will show that $u\neq v$ on Ω_δ. Arguing by contradiction, suppose that $u=v$ on Ω_δ. Since by hypotheses $h_1\neq h_2$, we obtain $h_1\neq h_2$ on $\Omega\setminus\Omega_\delta$. Then we choose $\vartheta\in W^{1,p}(\Omega)$ such that

$$\vartheta>0 \text{ on } \Omega \text{ and } \vartheta\big|_{\Omega\setminus\Omega_\delta}\equiv 1. \tag{2.12}$$

In what follows we denote by $\langle\cdot,\cdot\rangle_{\partial\Omega}$ the duality brackets for the pair

$$\left(W^{-\frac{1}{p'},p'}(\partial\Omega),W^{\frac{1}{p'},p}(\partial\Omega)\right)\quad \text{with}\quad \frac{1}{p'}+\frac{1}{p}=1.$$

Applying the nonlinear Green's identity, see, for example Gasiński-Papageorgiou [10, Theorem 2.4.53, p. 210], (2.12) and the fact that $u=v$ on Ω_δ give

$$\begin{aligned}
\int_\Omega h_1\vartheta dx &= \int_\Omega (a(x,\nabla u),\nabla\vartheta)_{\mathbb{R}^N}dx-\left\langle\frac{\partial u}{\partial n_a},\vartheta\right\rangle_{\partial\Omega}\\
&=\int_{\Omega_\delta}(a(x,\nabla u),\nabla\vartheta)_{\mathbb{R}^N}-\left\langle\frac{\partial u}{\partial n_a},\vartheta\right\rangle_{\partial\Omega}\\
&=\int_{\Omega_\delta}(a(x,\nabla v),\nabla\vartheta)_{\mathbb{R}^N}-\left\langle\frac{\partial v}{\partial n_a},\vartheta\right\rangle_{\partial\Omega}\\
&=\int_\Omega h_2\vartheta dx.
\end{aligned}\tag{2.13}$$

But from (2.12) and since $h_1\neq h_2$ we see that

$$\int_\Omega (h_2-h_1)\vartheta dx>0. \tag{2.14}$$

Comparing (2.13) and (2.14) we reach a contradiction. Hence, $u\neq v$ on Ω_δ. Then from (2.11) and the strong maximum principle, see, for example Motreanu-Motreanu-Papageorgiou [17, Theorem 8.27, p. 217], we obtain

$$(v-u)(x)>0 \quad \text{for all } x\in\Omega, \quad \left.\frac{\partial(v-u)}{\partial n}\right|_{\partial\Omega\cap(v-u)^{-1}(0)}<0.$$

Therefore, $v-u\in\hat{D}_+$. □

Next, let us recall some basic definitions and facts about critical groups which will be used in the sequel. Let X be a Banach space and let (Y_1,Y_2) be a topological pair such that $Y_2\subseteq Y_1\subseteq X$. For every integer $k\geq 0$ the term $H_k(Y_1,Y_2)$ stands for the $k\,\underline{\underline{th}}$-relative singular homology group with integer coefficients. Recall that

$$H_k(Y_1,Y_2)={Z_k(Y_1,Y_2)}\Big/{B_k(Y_1,Y_2)}\quad \text{for all } k\in\mathbb{N}_0,$$

where $Z_k(Y_1, Y_2)$ is the group of relative singular k-cycles of Y_1 mod Y_2 (that is, $Z_k(Y_1, Y_2) = \ker \partial_k$ with ∂_k being the boundary homomorphism) and $B_k(Y_1, Y_2)$ is the group of relative singular k-boundaries of Y_1 mod Y_2 (that is, $B_k(Y_1, Y_2) = \operatorname{im} \partial_{k+1}$). We know that $\partial_{k-1} \circ \partial_k = 0$ for all $k \in \mathbb{N}$, hence $B_k(Y_1, Y_2) \subseteq Z_k(Y_1, Y_2)$ and so the quotient

$$Z_k(Y_1, Y_2) \Big/ B_k(Y_1, Y_2)$$

makes sense.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\begin{aligned}
&\varphi^c = \{u \in X : \varphi(u) \leq c\} && \text{(the sublevel set of } \varphi \text{ at } c\text{)},\\
&K_\varphi = \{u \in X : \varphi'(u) = 0\} && \text{(the critical set of } \varphi\text{)},\\
&K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\} && \text{(the critical set of } \varphi \text{ at the level } c\text{)}.
\end{aligned}$$

For every isolated critical point $u \in K_\varphi^c$ the critical groups of φ at $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}\big) \quad \text{for all } k \geq 0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology theory implies that the definition of critical groups above is independent of the particular choice of the neighborhood U.

3. Positive Solutions

In this section we study the existence and multiplicity of the positive solutions as the parameter $\lambda > 0$ varies.

The hypotheses on the reaction term $f : \Omega \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ are the following ones.

H(f): $f : \Omega \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is a function such that $(x, s) \to f(x, s, \lambda)$ is a Carathéodory function for every $\lambda > 0$, $f(x, 0, \lambda) = 0$ for a.a. $x \in \Omega$, for all $\lambda > 0$ and

(i) for every $\rho > 0$ and every $\lambda_0 > 0$ there exists $a_\rho^{\lambda_0} \in L^\infty(\Omega)_+$ such that

$$0 \leq f(x, s, \lambda) \leq a_\rho^{\lambda_0}(x)$$

for a.a. $x \in \Omega$, for all $s \in [0, \rho]$ and for all $0 < \lambda \leq \lambda_0$;

(ii) for every $\lambda > 0$ there holds

$$\lim_{s \to +\infty} \frac{f(x, s, \lambda)}{s^{p-1}} = 0 \quad \text{uniformly for a.a. } x \in \Omega;$$

(iii) for every $\lambda > 0$ there holds

$$\lim_{s \to 0^+} \frac{f(x, s, \lambda)}{s^{p-1}} = 0 \quad \text{uniformly for a.a. } x \in \Omega;$$

(iv)
- $s \to f(x, s, \lambda)$ is nondecreasing on $\mathbb{R}_+$ for a.a. $x \in \Omega$ and for every $\lambda > 0$;
- $\lambda \to f(x, s, \lambda)$ is strictly increasing on $(0, +\infty)$ for a.a. $x \in \Omega$ and for all $s > 0$;

- $f(x,s,\lambda) \to 0^+$ as $\lambda \to 0^+$ uniformly for a.a. $x \in \Omega$ and for all $x \in K \subseteq \mathbb{R}_+$ with K being compact; moreover $f(x,s,\lambda) \to +\infty$ as $\lambda \to +\infty$ for a.a. $x \in \Omega$ and for all $s > 0$.

Remark 3.1. *Since we are interested in the existence of positive solutions and all hypotheses above concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality, we may assume that $f(x,\cdot,\lambda)\big|_{(-\infty,0]} \equiv 0$ for a.a. $x \in \Omega$ and for all $\lambda > 0$.*

Example 3.2. *For the sake of simplicity we drop the x-dependence. Let $f_1 : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ be defined by*

$$f_1(s,\lambda) = \begin{cases} \lambda s^{\tau-1} & \text{if } s \in [0,1], \\ s^{q-1}\ln(s) + \lambda s^{\eta-1} & \text{if } 1 < s, \end{cases}$$

with $1 < q, \eta < p < \tau < +\infty$. Then f_1 satisfies hypotheses H(f). Let $f_2 : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ be defined by

$$f_2(s,\lambda) = \begin{cases} \lambda s^{\tau-1} & \text{if } s \in [0,r(\lambda)], \\ \lambda s^{q-1} + \mu(\lambda) & \text{if } r(\lambda) < s, \end{cases}$$

with a strictly increasing differentiable function $r : (0,+\infty) \to (1,+\infty)$ such that $r(\lambda) \to 1^+$ as $\lambda \to 0^+$, $\mu(\lambda) = \lambda\left[r(\lambda)^{\tau-1} - r(\lambda)^{q-1}\right]$ and $1 < q < p < \tau < +\infty$. Then f_2 satisfies hypotheses H(f).

We introduce the following two sets:

$$\begin{aligned} \mathcal{L} &= \{\lambda > 0 : \text{problem } (\mathrm{P}_\lambda) \text{ admits a positive solution}\}, \\ \mathcal{S}(\lambda) &= \{u : u \text{ is a positive solution of problem } (\mathrm{P}_\lambda)\}. \end{aligned}$$

Moreover, we set $\lambda^* = \inf \mathcal{L} \geq 0$.

Proposition 3.3. *If hypotheses H(a), H(β) and H(f) hold, then $\mathcal{S}(\lambda) \subseteq D_+$ for every $\lambda > 0$ and $\lambda^* > 0$.*

Proof. Let $u \in \mathcal{S}(\lambda)$. From Papageorgiou-Rădulescu [18] we have

$$\begin{aligned} -\operatorname{div} a(x,\nabla u(x)) &= f(x,u(x),\lambda) && \text{for a.a. } x \in \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(x)u^{p-1} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Moreover, from Winkert [23] we obtain that $u \in L^\infty(\Omega)$ and the nonlinear regularity theory of Lieberman [16] gives $u \in C^1(\overline{\Omega})_+ \setminus \{0\}$. Note that $f(x,s,\lambda) \geq 0$ for a.a. $x \in \Omega$, for all $s \geq 0$ and for all $\lambda > 0$. Then, (3.1) implies $\operatorname{div} a(x,\nabla u(x)) \leq 0$ for a.a. $x \in \Omega$. Applying Theorem 1.2 of Zhang [25] gives $u \in D_+$.

We have proved that $\mathcal{S}(\lambda) \subseteq D_+$ for all $\lambda > 0$. Using Proposition 2.8 there exists $c_{11} > 0$ such that

$$c_{11}\|u\|_{1,p}^p \leq \frac{c_1}{p-1}\|\nabla u\|_p^p + \int_{\partial\Omega} \beta(x)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega). \tag{3.2}$$

Hypotheses H(f)(iii) imply that for a given $\varepsilon \in (0,c_{11})$ there exists $\tilde{\lambda} > 0$ such that

$$0 \leq f\left(x,s,\tilde{\lambda}\right) \leq \varepsilon s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq 0. \tag{3.3}$$

Let $\lambda \in (0, \tilde{\lambda}]$ and suppose that $\lambda \in \mathcal{L}$. Then we find $u_\lambda \in \mathcal{S}(\lambda) \subseteq D_+$ which means

$$\langle A(u_\lambda), h\rangle + \int_{\partial\Omega} \beta(x) u_\lambda^{p-1} h d\sigma = \int_\Omega f(x, u_\lambda, \lambda) h dx \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.4}$$

Choosing $h = u_\lambda \in D_+$ in (3.4) and using Lemma 2.3 gives

$$\frac{c_1}{p-1} \|\nabla u_\lambda\|_p^p + \int_{\partial\Omega} \beta(x) u_\lambda^p d\sigma \le \int_\Omega f(x, u_\lambda, \lambda) u_\lambda dx.$$

Applying (3.2), (3.3) and hypothesis H(f)(iv) implies $c_{11}\|u_\lambda\|_{1,p}^p \le \varepsilon \|u_\lambda\|_{1,p}^p$ and so $c_{11} \le \varepsilon$, a contradiction. Therefore, $\lambda \notin \mathcal{L}$, hence $0 < \tilde{\lambda} \le \lambda^*$. □

Proposition 3.4. *If hypotheses H(a), H(β) and H(f) hold, then $\mathcal{L} \neq \emptyset$.*

Proof. Using hypotheses H(f)(i), (ii) we see that for given $\varepsilon > 0$ and $\lambda > 0$ there exists $c_{12} = c_{12}(\varepsilon, \lambda) > 0$ such that

$$F(x, s, \lambda) \le \frac{\varepsilon}{p} s^{p-1} + c_{12} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \ge 0. \tag{3.5}$$

We consider the C^1-functional $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \int_\Omega G(x, \nabla u) dx + \frac{1}{p} \int_{\partial\Omega} \beta(x) |u|^p d\sigma - \int_\Omega F(x, s, \lambda) dx.$$

Using Corollary 2.4, (3.5) and Proposition 2.8 results in

$$\begin{aligned}\varphi_\lambda(u) &= \frac{c_1}{p(p-1)} \|\nabla\|_p^p + \frac{1}{p} \int_{\partial\Omega} \beta(x) |u|^p d\sigma - \frac{\varepsilon}{p} \|u^+\|_p^p - c_{12} |\Omega|_N \\ &\ge c_{13} \|u\|_{1,p}^p - c_{12} |\Omega|_N\end{aligned}$$

for some $c_{13} > 0$. Hence, φ_λ is coercive. Applying the Rellich-Kondrachov theorem and the compactness of the trace map, we easily see that φ_λ is sequentially weakly lower semicontinuous. So, by the Weierstraß-Tonelli theorem there exists $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \inf \left[\varphi_\lambda(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.6}$$

Hypothesis H(f)(iv) implies that

$$f(x, s, \lambda) > 0 \quad \text{for a.a. } x \in \Omega, \text{ for all } s > 0 \text{ and for all } \lambda > 0.$$

This gives

$$F(x, s, \lambda) > 0 \quad \text{for a.a. } x \in \Omega, \text{ for all } s > 0 \text{ and for all } \lambda > 0.$$

So, if $\tilde{u} \in D_+$, then

$$\int_\Omega F(x, \tilde{u}, \lambda)\, dx > 0$$

and by Fatou's lemma we have

$$\int_\Omega F(x, \tilde{u}, \lambda)\, dx \to +\infty \quad \text{as } \lambda \to +\infty.$$

Therefore, we can find $\tilde{\lambda} > 0$ such that

$$\int_\Omega G(x, \nabla \tilde{u}) dx + \frac{1}{p} \int_{\partial\Omega} \beta(x) \tilde{u}^p d\sigma < \int_\Omega F(x, \tilde{u}, \lambda)\, dx \quad \text{for all } \lambda > \tilde{\lambda}.$$

Hence $\varphi_\lambda(\tilde{u}) < 0 = \varphi_\lambda(0)$ for all $\lambda > \tilde{\lambda}$ and so, due to (3.6), we have $\varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0)$ for all $\lambda > \tilde{\lambda}$. This shows that $u_\lambda \neq 0$ for all $\lambda > \tilde{\lambda}$.

Since u_λ is a global minimizer of φ_λ we have $\varphi'_\lambda(u_\lambda)=0$ which is equivalent to

$$\langle A(u_\lambda),h\rangle+\int_{\partial\Omega}\beta(x)|u_\lambda|^{p-2}u_\lambda h d\sigma=\int_\Omega f(x,u_\lambda,\lambda)hdx \tag{3.7}$$

for all $h\in W^{1,p}(\Omega)$. We choose $h=-u_\lambda^-\in W^{1,p}(\Omega)$ in (3.7) and use Lemma 2.3 to obtain

$$\frac{c_1}{p-1}\|\nabla u_\lambda^-\|_p^p+\int_{\partial\Omega}\beta(x)(u_\lambda^-)^p d\sigma\leq 0.$$

Proposition 2.8 then implies that

$$c_{14}\|u_\lambda^-\|_{1,p}^p\leq 0 \quad \text{for some } c_{14}>0.$$

Hence, $u_\lambda\geq 0$ and $u_\lambda\neq 0$ for all $\lambda>\tilde{\lambda}$. Therefore, $u_\lambda\in\mathcal{S}(\lambda)\subseteq D_+$ for all $\lambda>\tilde{\lambda}$ and so $\mathcal{L}\neq\emptyset$. □

Proposition 3.5. *If hypotheses $H(a)$, $H(\beta)$, $H(f)$ hold and $\lambda\in\mathcal{L}$, then $(\lambda,+\infty)\subseteq\mathcal{L}$.*

Proof. Let $\mu>\lambda$ and let $u_\lambda\in\mathcal{S}(\lambda)\subseteq D_+$. We introduce the following truncation perturbation of the right-hand side nonlinearity in (P_λ)

$$k_\mu(x,s)=\begin{cases} f(x,u_\lambda(x),\mu)+(u_\lambda(x))^{p-1} & \text{if } s<u_\lambda(x),\\ f(x,s,\mu)+s^{p-1} & \text{if } u_\lambda(x)\leq s.\end{cases} \tag{3.8}$$

It is easy to see that this is a Carathéodory function. Furthermore, we set $K_\mu(x,s)=\int_0^s k_\mu(x,t)dt$ and consider the C^1-functional $\hat{\varphi}_\mu: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\varphi}_\mu(u)=\int_\Omega G(x,\nabla u)dx+\frac{1}{p}\|u\|_p^p+\frac{1}{p}\int_{\partial\Omega}\beta(x)|u|^p d\sigma-\int_\Omega k_\mu(x,u_\mu)dx.$$

As we did in Proposition 3.4 for the functional φ_λ, we can show that $\hat{\varphi}_\mu$ is coercive and sequentially weakly lower semicontinuous. Therefore, there exists a global minimizer $u_\mu\in W^{1,p}(\Omega)$ of $\hat{\varphi}_\mu$, that is,

$$\hat{\varphi}_\mu(u_\mu)=\inf\left[\hat{\varphi}_\mu(u): u\in W^{1,p}(\Omega)\right].$$

Hence, $\hat{\varphi}'_\mu(u_\mu)=0$ which means

$$\langle A(u_\mu),h\rangle+\int_\Omega|u_\mu|^{p-2}u_\mu hdx+\int_{\partial\Omega}\beta(x)|u_\mu|^{p-2}u_\mu hd\sigma=\int_\Omega k_\mu(x,u_\mu)hdx \tag{3.9}$$

for all $h \in W^{1,p}(\Omega)$. We choose $h = (u_\lambda - u_\mu)^+ \in W^{1,p}(\Omega)$ in (3.9). This along with (3.8), hypothesis H(f)(iv) and the fact that $u_\lambda \in \mathcal{S}(\lambda)$ yields

$$\begin{aligned}
&\langle A(u_\mu), (u_\lambda - u_\mu)^+\rangle + \int_\Omega |u_\mu|^{p-2}u_\mu(u_\lambda - u_\mu)^+ dx \\
&\quad + \int_{\partial\Omega} \beta(x)|u_\mu|^{p-2}u_\mu(u_\lambda - u_\mu)^+ d\sigma \\
&= \int_\Omega [f(x, u_\lambda, \mu) + u_\lambda^{p-1}](u_\lambda - u_\mu)^+ dx \\
&\geq \int_\Omega [f(x, u_\lambda, \lambda) + u_\lambda^{p-1}](u_\lambda - u_\mu)^+ dx \\
&= \langle A(u_\lambda), (u_\lambda - u_\mu)^+\rangle + \int_\Omega u_\lambda^{p-1}(u_\lambda - u_\mu)^+ dx \\
&\quad + \int_{\partial\Omega} \beta(x)u_\lambda^{p-1}(u_\lambda - u_\mu)^+ d\sigma.
\end{aligned}$$

This implies

$$\begin{aligned}
&\langle A(u_\lambda) - A(u_\mu), (u_\lambda - u_\mu)^+\rangle + \int_\Omega \left[u_\lambda^{p-1} - |u_\mu|^{p-2}u_\mu\right](u_\lambda - u_\mu)^+ dx \\
&+ \int_{\partial\Omega} \beta(x)\big[u_\lambda^{p-1} - |u_\mu|^{p-2}u_\mu\big](u_\lambda - u_\mu)^+ d\sigma \leq 0.
\end{aligned}$$

Hence, due to Lemma 2.3 and hypotheses H(β), $u_\lambda \leq u_\mu$. Then from (3.8) and (3.9) it follows that $u_\mu \in \mathcal{S}(\mu) \subseteq D_+$ which says that $\mu \in \mathcal{L}$. So we have proved that $[\lambda, +\infty) \subseteq \mathcal{L}$. □

An interesting byproduct of the proof above is the following monotonicity result.

Corollary 3.6. *If hypotheses H(a), H(β), H(f) hold, $\lambda \in \mathcal{L}, u_\lambda \in \mathcal{S}(\lambda)$ and $\mu > \lambda$, then there exists $u_\mu \in \mathcal{S}(\mu)$ such that $u_\lambda \leq u_\mu$.*

Proposition 3.5 implies that

$$(\lambda^*, +\infty) \subseteq \mathcal{L} \subseteq [\lambda^*, +\infty)\,. \tag{3.10}$$

Next we show that problem (P_λ) has multiple solutions for all $\lambda > \lambda^*$.

Proposition 3.7. *If hypotheses H(a), H(β), H(f) hold and $\lambda > \lambda^*$, then problem (P_λ) has at least two positive solutions $u_\lambda, \hat{u}_\lambda \in D_+$.*

Proof. Let $\lambda^* < \tau < \lambda < \mu$. From (3.10) we know that $\tau, \mu \in \mathcal{L}$ and applying Corollary 3.6 there exist $u_\tau \in \mathcal{S}(\tau) \subseteq D_+$ and $u_\mu \in \mathcal{S}(\mu) \subseteq D_+$ such that

$$u_\tau \leq u_\mu, \quad u_\tau \neq u_\mu.$$

We introduce the following truncation perturbation of the right-hand side nonlinearity of problem (P_λ)

$$e_\lambda(x, s) = \begin{cases} f(x, u_\tau(x), \lambda) + (u_\tau(x))^{p-1} & \text{if } s < u_\tau(x), \\ f(x, s, \lambda) + s^{p-1} & \text{if } u_\tau(x) \leq s \leq u_\mu(x), \\ f(x, u_\mu(x), \lambda) + (u_\mu(x))^{p-1} & \text{if } u_\mu(x) < s, \end{cases} \tag{3.11}$$

which is a Carathéodory function. Setting $E_\lambda(x,s)=\int_0^s e_\lambda(x,t)dt$ we then introduce the C^1-functional $\psi_\lambda : W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\psi_\lambda(u)=\int_\Omega G(x,\nabla u)dx+\frac{1}{p}\|u\|_p^p+\frac{1}{p}\int_{\partial\Omega}\beta(x)|u|^p d\sigma-\int_\Omega E_\lambda(x,u)dx.$$

From (3.11) and hypotheses H(β) we see that ψ_λ is coercive. Moreover, the Rellich-Kondrachov theorem and the compactness of the trace operator imply that ψ_λ is sequentially weakly lower semicontinuous. Hence, we find $u_\lambda\in W^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_\lambda)=\inf\left[\psi_\lambda(u) : u\in W^{1,p}(\Omega)\right]. \tag{3.12}$$

From (3.12) we have $\psi_\lambda'(u_\lambda)=0$ which gives

$$\langle A(u_\lambda),h\rangle+\int_\Omega |u_\lambda|^{p-2}u_\lambda h dx+\int_{\partial\Omega}\beta(x)|u_\lambda|^{p-2}u_\lambda h d\sigma=\int_\Omega e_\lambda(x,u_\lambda)h dx. \tag{3.13}$$

We choose $h=(u_\lambda-u_\mu)^+\in W^{1,p}(\Omega)$ in (3.13). Then using (3.11), hypothesis H(f)(iv) and $u_\mu\in\mathcal{S}(\mu)$ we obtain

$$\begin{aligned}
&\langle A(u_\lambda),(u_\lambda-u_\mu)^+\rangle+\int_\Omega u_\lambda^{p-1}(u_\lambda-u_\mu)^+dx\\
&\quad+\int_{\partial\Omega}\beta(x)u_\lambda^{p-1}(u_\lambda-u_\mu)^+d\sigma\\
&=\int_\Omega[f(x,u_\mu,\lambda)+u_\mu^{p-1}](u_\lambda-u_\mu)^+dx\\
&\leq\int_\Omega[f(x,u_\mu,\mu)+u_\mu^{p-1}](u_\lambda-u_\mu)^+dx\\
&=\langle A(u_\mu),(u_\lambda-u_\mu)^+\rangle+\int_\Omega u_\mu^{p-1}(u_\lambda-u_\mu)^+dx\\
&\quad+\int_{\partial\Omega}\beta(x)u_\mu^{p-1}(u_\lambda-u_\mu)^+d\sigma.
\end{aligned}$$

This gives

$$\begin{aligned}
&\langle A(u_\lambda)-A(u_\mu),(u_\lambda-u_\mu)^+\rangle+\int_\Omega\left[u_\lambda^{p-1}-u_\mu^{p-1}\right](u_\lambda-u_\mu)^+dx\\
&+\int_{\partial\Omega}\beta(x)\left[u_\lambda^{p-1}-u_\mu^{p-1}\right](u_\lambda-u_\mu)^+d\sigma\leq 0.
\end{aligned}$$

Then Lemma 2.3 and hypotheses H(β) imply that $u_\lambda\leq u_\mu$. If we choose $h=(u_\tau-u_\lambda)^+\in W^{1,p}(\Omega)$ and reason as above, we obtain $u_\tau\leq u_\lambda$. So we have proved that

$$u_\lambda\in[u_\tau,u_\mu],\quad u_\lambda\notin\{u_\tau,u_\mu\}, \tag{3.14}$$

see hypothesis H(f)(iv). Then from (3.13), (3.14) and (3.11) it follows that $u_\lambda\in\mathcal{S}(\lambda)\subseteq D_+$. We have

$$\begin{aligned}
-\operatorname{div} a(x,\nabla u_\tau(x))&=f(x,u_\tau(x),\tau)=:h_1(x)\quad\text{for a.a. }x\in\Omega,\\
-\operatorname{div} a(x,\nabla u_\lambda(x))&=f(x,u_\lambda(x),\lambda)=:h_2(x)\quad\text{for a.a. }x\in\Omega,\\
-\operatorname{div} a(x,\nabla u_\mu(x))&=f(x,u_\mu(x),\mu)=:h_3(x)\quad\text{for a.a. }x\in\Omega.
\end{aligned}$$

Note that (3.14) and hypothesis H(f)(iv) imply that

$$h_1\leq h_2\leq h_3,\quad h_1\neq h_2,\quad h_2\neq h_3.$$

Furthermore, we have

$$\left.\frac{\partial u_\tau}{\partial n_a}\right|_{\partial\Omega} < 0, \quad \left.\frac{\partial u_\lambda}{\partial n_a}\right|_{\partial\Omega} < 0, \quad \left.\frac{\partial u_\mu}{\partial n_a}\right|_{\partial\Omega} < 0$$

since $u_\tau, u_\lambda, u_\mu \in D_+$. Thus, we can apply Proposition 2.10 and obtain

$$u_\lambda - u_\tau \in \hat{D}_+ \quad \text{and} \quad u_\mu - u_\lambda \in \hat{D}_+. \tag{3.15}$$

Consider now the C^1-functional φ_λ introduced in the proof of Proposition 3.4. From (3.11) we see that

$$\psi_\lambda = \varphi_\lambda + \hat{\xi}_\lambda \text{ on } [u_\tau, u_\mu] \text{ with } \hat{\xi}_\lambda \in \mathbb{R}.$$

Then, due to (3.15), we see that u_λ is a local $C^1(\overline{\Omega})$-minimizer of φ_λ and so,

$$u_\lambda \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda, \tag{3.16}$$

see Proposition 2.6. Hypothesis H(f)(iii) implies that for a given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 \leq F(x,s,\lambda) \leq \frac{\varepsilon}{p} s^p \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \delta. \tag{3.17}$$

Let $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \leq \delta$. Then from Corollary 2.4, (3.17) and Proposition 2.8 we see that

$$\begin{aligned}\varphi_\lambda(u) &\geq \frac{c_1}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(x)|u|^p d\sigma - \frac{\varepsilon}{p}\|u\|_p^p\\ &\geq c_{15}\|u\|_{1,p}^p - \frac{\varepsilon}{p}\|u\|_{1,p}^p\end{aligned}$$

for some $c_{15} > 0$. Let $\varepsilon \in (0, pc_{15})$. Then we see that $u = 0$ is a local $C^1(\overline{\Omega})$-minimizer of φ_λ and so, again due to Proposition 2.6, $u = 0$ is a local $W^{1,p}(\Omega)$-minimizer of φ_λ.

We may assume, without any loss of generality, that $0 = \varphi_\lambda(0) \leq \varphi_\lambda(u_\lambda)$. The treatment is similar if the opposite inequality holds. Moreover, we assume that K_{φ_λ} is finite, otherwise we already have an infinite number of positive solutions of problem (P_λ). From (3.16) it follows that there exists $\rho \in (0,1)$ small enough such that

$$0 = \varphi_\lambda(0) \leq \varphi_\lambda(u_\lambda) < \inf\left[\varphi_\lambda(u) : \|u - u_\lambda\| = \rho\right] = m_\lambda, \quad \|u_\lambda\| > \rho, \tag{3.18}$$

see Aizicovici-Papageorgiou-Staicu [1, Proof of Proposition 29]. Recall that φ_λ is coercive, see the proof of Proposition 3.4. Hence

$$\varphi_\lambda \text{ satisfies the PS-condition.} \tag{3.19}$$

From (3.18) and (3.19) we see that we can apply the mountain pass theorem stated as Theorem 2.1. This gives $\hat{u}_\lambda \in W^{1,p}(\Omega)$ such that $\hat{u}_\lambda \in K_{\varphi_\lambda}$ and $m_\lambda \leq \varphi_\lambda(\hat{u}_\lambda)$. Hence, $\hat{u}_\lambda \in \mathcal{S}(\lambda) \subseteq D_+$ and due to (3.18) we obtain that $\hat{u}_\lambda \notin \{0, u_\lambda\}$. ☐

It is natural to ask whether the critical parameter $\lambda^* > 0$ is admissible. The next proposition shows that λ^* is indeed admissible, that is, $\lambda^* \in \mathcal{L}$.

Proposition 3.8. *If hypotheses H(a), H(β) and H(f) hold, then $\lambda^* \in \mathcal{L}$, that is, $\mathcal{L} = [\lambda^*, +\infty)$, see (3.10).*

Proof. Let $\{\lambda_n\}_{n\geq 1} \subseteq (\lambda^*, +\infty)$ be a sequence such that $\lambda_n \searrow \lambda^*$. From the proof of Proposition 3.7 we see that we can find $u_n \in \mathcal{S}(\lambda_n) \subseteq D_+$ for $n \in \mathbb{N}$ such that $\{u_n\}_{n\geq 1}$ is decreasing. Since $u_n \in \mathcal{S}(\lambda_n)$ we have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(x) u_n^{p-1} d\sigma = \int_\Omega f(x, u_n, \lambda_n) h dx \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.20}$$

We choose $h = u_n \in W^{1,p}(\Omega)$ in (3.20). Then, from Lemma 2.3, hypothesis H(f)(i) and since $0 \leq u_n \leq u_1$ for all $n \in \mathbb{N}$, we obtain

$$\frac{c_1}{p-1} \|\nabla u_n\|_p^p + \int_{\partial\Omega} \beta(x) u_n^p d\sigma \leq c_{16} \quad \text{for some } c_{16} > 0 \text{ and for all } n \in \mathbb{N}.$$

Hence

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded,} \tag{3.21}$$

see Proposition 2.8. So, we may assume that

$$u_n \xrightarrow{\text{w}} u_{\lambda^*} \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_{\lambda^*} \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \tag{3.22}$$

From (3.21) and Winkert [23] we see that there exists $c_{17} > 0$ such that $\|u_n\|_\infty \leq c_{17}$ for all $n \in \mathbb{N}$. Then from Lieberman [16] we know that we can find $\alpha' \in (0,1)$ and $c_{18} > 0$ such that

$$u_n \in C^{1,\alpha'}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha'}(\overline{\Omega})} \leq c_{18} \quad \text{for all } n \in \mathbb{N}. \tag{3.23}$$

From the compact embedding $C^{1,\alpha'}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and from (3.22) as well as (3.23) it follows that

$$u_n \to u_{\lambda^*} \text{ in } C^1(\overline{\Omega}). \tag{3.24}$$

Hypothesis H(f)(iii) implies that for a given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 \leq f(x, s, \lambda_1) s \leq \varepsilon s^p \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \delta.$$

Then, by hypothesis H(f)(iv) we obtain

$$0 \leq f(x, s, \lambda_n) s \leq \varepsilon s^p \quad \text{for a.a. } x \in \Omega, \text{ for all } 0 \leq s \leq \delta \text{ and for all } n \in \mathbb{N}. \tag{3.25}$$

Suppose that $u_{\lambda^*} = 0$. From (3.24) we see that there exists $n_0 \in \mathbb{N}$ such that

$$u_n(x) \in (0, \delta] \quad \text{for all } x \in \overline{\Omega} \text{ and for all } n \geq n_0. \tag{3.26}$$

So, if we choose $h = u_n \in D_+$ in (3.20), we obtain, due to (3.25) and (3.26), that

$$\frac{c_1}{p-1} \|\nabla u_n\|_p^p + \int_\Omega \beta(x) u_n^p d\sigma \leq \frac{\varepsilon}{p} \|u_n\|_p^p \quad \text{for all } n \geq n_0.$$

Hence $c_{19}\|u_n\|_{1,p}^p \leq \varepsilon \|u_n\|_{1,p}^p$ for all $n \geq n_0$ and for some $c_{19} > 0$. Therefore, $c_{19} \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \searrow 0$, we reach a contradiction. Thus, $u_{\lambda^*} \neq 0$. If we pass to the limit in (3.20) as $n \to \infty$ and use (3.24), then we conclude that $u_{\lambda^*} \in \mathcal{S}(\lambda^*) \subseteq D_+$ and so $\lambda^* \in \mathcal{L}$. □

So we can state our first theorem. This is a bifurcation-type result describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies.

Theorem 3.9. *If hypotheses H(a), H(β) and H(f) hold, then there exists $\lambda^* > 0$ such that the following is satisfied:*

(a) problem (P_λ) *has at least two positive solutions $u_\lambda, \hat{u}_\lambda \in D_+$ for all $\lambda > \lambda^*$;*
(b) problem (P_λ) *has at least one positive solution $u_{\lambda^*} \in D_+$ for $\lambda = \lambda^*$;*

(c) problem (P_λ) *has no positive solution for all* $\lambda \in (0, \lambda^*)$*;*

Remark 3.10. *Hypothesis H(β) leaves out the Neumann problem of our considerations, that is, the case* $\beta \equiv 0$*. Indeed, under hypotheses H(f), problem* (P_λ) *with* $\beta \equiv 0$ *and* $\lambda > 0$ *cannot have positive solutions. In order to see this, suppose we could find a positive solution* u_λ*. Then, as before, using the nonlinear regularity theory and the nonlinear maximum principle, we can show that* $u_\lambda \in D_+$*. Moreover*

$$\langle A(u_\lambda), h\rangle = \int_\Omega f(x, u_\lambda, \lambda) h dx \quad \text{for all } h \in W^{1,p}(\Omega).$$

Choosing $h \equiv 1 \in W^{1,p}(\Omega)$ *gives*

$$\int_\Omega f(x, u_\lambda, \lambda) dx = 0.$$

Since $u_\lambda \in D_+$ *we get* $0 < m_\lambda = \min_{\overline{\Omega}} u_\lambda$ *and then for* $\tau < \lambda$ *we have*

$$\int_\Omega f(x, m_\lambda, \tau) dx < 0,$$

see hypothesis H(f)(iv), a contradiction. So, the Neumann problem cannot have positive solutions.

In the last part of this section we show that problem (P_λ) has a smallest positive solution $u^*_\lambda \in D_+$ for every $\lambda \in \mathcal{L}$ and we investigate the monotonicity and continuity properties of the map $\lambda \to u^*_\lambda$.

First we prove the existence of a smallest positive solution $u^*_\lambda \in D_+$ for every $\lambda \in \mathcal{L}$.

Proposition 3.11. *If hypotheses H(a), H(β) and H(f) hold, then problem* (P_λ) *admits a smallest positive solution* $u^*_\lambda \in D_+$ *for every* $\lambda \in \mathcal{L}$*.*

Proof. From Papageorgiou-Rădulescu-Repovš [20, see the proof of Proposition 7] we know that $\mathcal{S}(\lambda)$ is downward directed, that is, if $u, \hat{u} \in \mathcal{S}(\lambda)$, then there exists $\tilde{u} \in \mathcal{S}(\lambda)$ such that $\tilde{u} \le u$ and $\tilde{u} \le \hat{u}$. Invoking Lemma 3.10 of Hu-Papageorgiou [15] there exists a decreasing sequence $\{u_n\}_{n\ge1} \subseteq \mathcal{S}(\lambda)$ such that $\inf \mathcal{S}(\lambda) = \inf_{n\ge1} u_n$. We have $0 \le u_n \le u_1 \in D_+$ for all $n \in \mathbb{N}$ and

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(x) u_n^{p-1} h dx = \int_\Omega f(x, u_n, \lambda) h dx \tag{3.27}$$

for all $h \in W^{1,p}(\Omega)$ and for all $n \in \mathbb{N}$. Choosing $h = u_n \in D_+$ and using Lemma 2.3, Proposition 2.8 as well as hypothesis H(f)(i) we easily see that

$$\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{3.28}$$

As in the proof Proposition 3.8, using (3.28) and the nonlinear regularity theory, we obtain, at least for a subsequence, that

$$u_n \to u^*_\lambda \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{3.29}$$

Again, as in the proof of Proposition 3.8, using hypothesis H(f)(iii) and (3.29), we show that $u^*_\lambda \ne 0$. Then, if we pass to the limit in (3.27) as $n \to \infty$ and use (3.29), we infer that $u^*_\lambda \in \mathcal{S}(\lambda) \subseteq D_+$. Therefore, $u^*_\lambda = \inf \mathcal{S}(\lambda)$. □

Next we consider the map $\gamma : \mathcal{L} \to C^1(\overline{\Omega})$ defined by

$$\gamma(\lambda) = u^*_\lambda \quad \text{for all } \lambda \in \mathcal{L} = [\lambda^*, +\infty).$$

Proposition 3.12. *If hypotheses H(a), H(β) and H(f) hold, then the map γ is strictly increasing in the sense that $\lambda < \mu$ implies $u_\mu^* - u_\lambda^* \in \hat{D}_+$. Moreover, γ is left continuous on $\mathcal{L}_0 = (\lambda^*, +\infty)$.*

Proof. Let $\lambda \in \mathcal{L}$ and let $\mu > \lambda$. Then $\mu \in \mathcal{L}$. We consider $u_\mu^* \in \mathcal{S}(\mu) \subseteq D_+$ which is the minimal positive solution of problem (P_μ). According to Corollary 3.6 there exists $u_\lambda \in \mathcal{S} \subseteq D_+$ such that $u_\lambda \leq u_\mu^*$. Hence, $u_\lambda^* \leq u_\mu^*$. In fact, as in the proof of Proposition 3.7, using Proposition 2.10, we can show that $u_\mu^* - u_\lambda^* \in \hat{D}_+$.

Now we prove the left continuity of γ on $\mathcal{L}_0 = (\lambda^*, +\infty)$. So, suppose that $\lambda_n \to \lambda^-$ with $\lambda_n > \lambda^*$ for all $n \in \mathbb{N}$. We have $u_{\lambda_n}^* \leq u_\lambda^*$ for all $n \in \mathbb{N}$. As in the proof of Proposition 3.8, we can show that

$$u_{\lambda_n}^* \to \tilde{u}_\lambda \quad \text{in } C^1(\overline{\Omega}). \tag{3.30}$$

If $\tilde{u}_\lambda \neq u_\lambda^*$, then we can find $z_0 \in \overline{\Omega}$ such that $u_\lambda^*(z_0) < \tilde{u}_\lambda(z_0)$. From (3.30) we see that $u_\lambda^*(z_0) < u_{\lambda_n}^*(z_0)$ for all $n \geq n_0$ which contradicts the monotonicity of γ. $\square$

4. Nodal Solutions

In this section we are interested in the existence of nodal solutions for problem (P_λ). In order to do this, we need to impose the conditions on $f(x, \cdot)$ on all of $\mathbb{R}$. So, we introduce the following bilateral version of hypothesis H(f).

H$(f)_1$: $f : \Omega \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is a function such that $(x, s) \to f(x, s, \lambda)$ is a Carathéodory function for every $\lambda > 0$, $f(x, 0, \lambda) = 0$ for a.a. $x \in \Omega$, for all $\lambda > 0$ and

(i) for every $\rho > 0$ and every $\lambda_0 > 0$ there exists $a_\rho^{\lambda_0} \in L^\infty(\Omega)_+$ such that

$$0 \leq f(x, s, \lambda) \leq a_\rho^{\lambda_0}(x)$$

for a.a. $x \in \Omega$, for all $|s| \leq \rho$ and for all $0 < \lambda \leq \lambda_0$;

(ii) for every $\lambda > 0$ there holds

$$\lim_{s \to \pm\infty} \frac{f(x, s, \lambda)}{|s|^{p-2}s} = 0 \quad \text{uniformly for a.a. } x \in \Omega;$$

(iii) for every $\lambda > 0$ there holds

$$\lim_{s \to 0} \frac{f(x, s, \lambda)}{|s|^{p-2}s} = 0 \quad \text{uniformly for a.a. } x \in \Omega;$$

(iv)
- $s \to f(x, s, \lambda)$ is nondecreasing on $\mathbb{R}$ for a.a. $x \in \Omega$ and for every $\lambda > 0$;
- $\lambda \to f(x, s, \lambda)$ is strictly increasing (resp. strictly decreasing) for a.a. $x \in \Omega$ and for all $s > 0$ (resp. $s < 0$);
- $f(x, s, \lambda) \to 0$ as $\lambda \to 0^+$ uniformly for a.a. $x \in \Omega$ and for all $x \in K \subseteq \mathbb{R}$ with K being compact; moreover $f(x, s, \lambda) \to +\infty$ (resp. $-\infty$) as $\lambda \to +\infty$ for a.a. $x \in \Omega$ and for all $s > 0$ (resp. $s < 0$).

The new conditions also apply on the negative semiaxis $(-\infty, 0]$. So, working as in the first part of this section, we can have a bifurcation-type result for negative solutions of (P_λ). More precisely, let

$$\tilde{\mathcal{L}} = \{\lambda > 0: \text{ problem } (P_\lambda) \text{ has a negative solution}\},$$
$$\tilde{\mathcal{S}}(\lambda) = \{u : u \text{ is a negative solution of problem } (P_\lambda)\}.$$

Then there exists $\tilde{\lambda}^* > 0$ such that

$$\tilde{\mathcal{L}} = \left[\tilde{\lambda}^*, +\infty\right), \quad \tilde{\mathcal{S}}(\lambda) \subseteq -D_+ \quad \text{for all } \lambda \in \tilde{\mathcal{L}},$$

problem (P_λ) has two negative solutions for all $\lambda > \tilde{\lambda}^*$ and problem (P_λ) admits a greatest negative solution $v_\lambda^* \in \tilde{\mathcal{S}}(\lambda) \subseteq -D_+$ for every $\lambda \in \tilde{\mathcal{L}}$.

Let $\lambda_0^* = \max\{\lambda^*, \tilde{\lambda}^*\}$. Then problem ($P_\lambda$) has a smallest positive solution $u_\lambda^* \in D_+$ and a greatest negative solution $v_\lambda^* \in -D_+$ for all $\lambda \geq \lambda_0^*$. Using them we can generate a nodal solution when $\lambda \geq \lambda_0^*$ is large enough.

Theorem 4.1. *If hypotheses $H(a)$, $H(\beta)$ and $H(f)_1$ hold, then there exists $\lambda_1^* \geq \lambda_0^*$ such that problem (P_λ) admits a nodal solution $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$ for all $\lambda \geq \lambda_1^*$.*

Proof. We consider the following truncation perturbation of the right-hand side nonlinearity of problem (P_λ)

$$\hat{f}_\lambda(x,s) = \begin{cases} f\left(x, v_\lambda^*(x), \lambda\right) + |v_\lambda^*(x)|^{p-2}(x)|v_\lambda^*(x) & \text{if } s < v_\lambda^*(x), \\ f(x,s,\lambda) + |s|^{p-2}s & \text{if } v_\lambda^*(x) \leq s \leq u_\lambda^*(x), \\ f\left(x, u_\lambda^*(x), \lambda\right) + (u_\lambda^*(x))^{p-1} & \text{if } u_\lambda^*(x) < s. \end{cases} \tag{4.1}$$

Of course, $\hat{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function for every $\lambda \geq \lambda_0^*$. We set $\hat{F}_\lambda(x,s) = \int_0^s \hat{f}_\lambda(x,t)dt$ and consider the C^1-functional $\hat{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \int_\Omega G(x, \nabla u)dx + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u|^p d\sigma - \int_\Omega \hat{F}_\lambda(x,u)dx.$$

Furthermore, let $\hat{f}_\lambda^\pm$ be the positive and negative truncations of $\hat{f}_\lambda(x,\cdot)$, that is, $\hat{f}_\lambda^\pm(x,s) = \hat{f}_\lambda(x, \pm s^\pm)$. Both are Carathéodory functions. We set $\hat{F}_\lambda^\pm(x,s) = \int_0^s \hat{f}_\lambda^\pm(x,t)dt$ and consider the C^1-functionals $\hat{\psi}_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda^\pm(u) = \int_\Omega G(x, \nabla u)dx + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u|^p d\sigma - \int_\Omega \hat{F}_\lambda^\pm(x,u)dx.$$

Claim 1: $K_{\hat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$, $K_{\hat{\psi}^+} = \{0, u_\lambda^*\}$, $K_{\hat{\psi}_\lambda^-} = \{0, v_\lambda^*\}$ for all $\lambda \geq \lambda_0^*$

Let $u \in K_{\hat{\psi}_\lambda}$. Then

$$\langle A(u), h\rangle + \int_\Omega |u|^{p-2}uhdx + \int_{\partial\Omega} \beta(x)|u|^{p-2}uhd\sigma = \int_\Omega \hat{f}_\lambda(x,u)hdx$$

for all $h \in W^{1,p}(\Omega)$. First, let $h = (u - u_\lambda^*)^+ \in W^{1,p}(\Omega)$ in the equality above. Applying (4.1) and the fact that $u_\lambda^* \in \mathcal{S}(\lambda)$ gives

$$\begin{aligned} &\langle A(u), (u-u_\lambda^*)^+\rangle + \int_\Omega u^{p-1}(u-u_\lambda^*)^+dx + \int_{\partial\Omega} \beta(x)u^{p-1}(u-u_\lambda^*)^+d\sigma \\ &= \int_\Omega \left[f\left(x, u_\lambda^*, \lambda\right) + (u_\lambda^*)^{p-1}\right](u-u_\lambda^*)^+dx \\ &= \left\langle A\left(u_\lambda^*\right), (u-u_\lambda^*)^+\right\rangle + \int_\Omega \left(u_\lambda^*\right)^{p-1}(u-u_\lambda^*)^+dx + \int_{\partial\Omega} \beta(x)\left(u_\lambda^*\right)^{p-1}(u-u_\lambda^*)^+d\sigma. \end{aligned}$$

As before, see the proof of Proposition 3.5 or Proposition 3.7, this shows that $u \leq u_\lambda^*$. Similarly, using $h = (v^* - u)^+ \in W^{1,p}(\Omega)$, we obtain $v_\lambda^* \leq u$. Hence, $u \in [v_\lambda^*, u_\lambda^*]$. Taking (4.1) into account, we see that u is a solution of (P_λ). Then, as before, the nonlinear regularity theory implies that $u \in C^1(\overline{\Omega})$. Hence, $K_{\hat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$.

In a similar way we prove that

$$K_{\hat{\psi}_\lambda^+} \subseteq \left[0, u_\lambda^*\right] \cap C^1(\overline{\Omega}) \quad \text{and} \quad K_{\hat{\psi}_\lambda^-} \subseteq \left[v_\lambda^*, 0\right] \cap C^1(\overline{\Omega}).$$

The extremality of u_λ^* and v_λ^*, see Proposition 3.11 and recall $\lambda \geq \lambda_0^*$, implies that

$$K_{\hat{\psi}_\lambda^+} = \left\{0, u_\lambda^*\right\} \quad \text{and} \quad K_{\hat{\psi}_\lambda^-} = \left\{v_\lambda^*, 0\right\}.$$

This proves Claim 1.

Claim 2: There exists $\lambda_1^* \geq \lambda_0^*$ such that $u_\lambda^* \in D_+$ and $v_\lambda^* \in -D_+$ are local minimizers for the functional $\hat{\psi}_\lambda$ for all $\lambda \geq \lambda_1^*$.

From (4.1) we see that $\hat{\psi}_\lambda$ is coercive. Moreover, it is sequentially weakly lower semicontinuous. So, by the Weierstraß-Tonelli theorem there exists $\tilde{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_\lambda^+\left(\tilde{u}_\lambda^*\right) = \inf \left[\hat{\psi}_\lambda(u) : u \in W^{1,p}(\Omega)\right]. \tag{4.2}$$

Recall that $F(x, \eta, \lambda) > 0$ for a.a. $x \in \Omega$, for all $\lambda > 0$ and for all $\eta \in (0, +\infty)$. Hence

$$\int_\Omega F(x, \eta, \lambda) dx \to +\infty \quad \text{as } \lambda \to +\infty. \tag{4.3}$$

Fix $\tilde{\lambda} > 0$, recall that $u_{\tilde{\lambda}}^* \in D_+$ and choose $\eta \in (0, \min_{\overline{\Omega}} u_{\tilde{\lambda}}^*)$. On account of Proposition 3.12 we have

$$\eta < \min_{\overline{\Omega}} u_{\tilde{\lambda}}^* \leq \min_{\overline{\Omega}} u_\lambda^* \quad \text{for all } \lambda \geq \tilde{\lambda}.$$

This fact along with (4.1) yields

$$\hat{\psi}_\lambda^+(\eta) = \frac{\eta^p}{p} \|\beta\|_{L^1(\partial\Omega)} - \int_\Omega F(x, \eta, \lambda) dx \quad \text{for all } \lambda \geq \tilde{\lambda}.$$

From (4.3) we see that by choosing $\tilde{\lambda} > 0$ large enough we obtain $\hat{\psi}_\lambda^+(\eta) < 0$ for all $\lambda \geq \tilde{\lambda}$. Then

$$\hat{\psi}_\lambda^+\left(\tilde{u}_\lambda^+\right) < 0 = \hat{\psi}_\lambda^+(0) \quad \text{for all } \lambda \geq \lambda_1^* = \max\left\{\tilde{\lambda}, \lambda_0^*\right\},$$

see (4.2). Therefore, $\tilde{u}_\lambda^* \neq 0$ for all $\lambda \geq \lambda_1^*$.

Since $\tilde{u}_\lambda^* \in K_{\hat{\psi}_\lambda^+}$, invoking Claim 1, we have

$$\tilde{u}_\lambda^* = u_\lambda^* \in D_+ \quad \text{for all } \lambda \geq \lambda_1^*. \tag{4.4}$$

Note that

$$\hat{\psi}_\lambda\big|_{C^1(\overline{\Omega})_+} = \hat{\psi}_\lambda^+\big|_{C^1(\overline{\Omega})_+}.$$

Thus, u_λ^* is a local $C^1(\overline{\Omega})$-minimizer of $\hat{\psi}_\lambda$ for all $\lambda \geq \lambda_1^*$, see (4.4). Then, Proposition 2.6 implies that u_λ^* is a local $W^{1,p}(\Omega)$-minimizer of $\hat{\psi}_\lambda$ for all $\lambda \geq \lambda_1^*$. This proves Claim 2.

We may assume that

$$\psi_\lambda\left(v_\lambda^*\right) \leq \psi_\lambda\left(u_\lambda^*\right).$$

The reasoning is similar if the opposite inequality holds. Moreover, we assume that $K_{\hat{\psi}_\lambda}$ is finite, otherwise we already have an infinite number of nodal solutions on account of Claim 1 and the extremality of u_λ^* and v_λ^*. This fact and Claim 2 imply that there exists $\rho \in (0,1)$ such that

$$\hat{\psi}_\lambda(v_\lambda^*) \leq \hat{\psi}_\lambda(u_\lambda^*) < \inf\left[\hat{\psi}_\lambda(u) : \|u-u_\lambda^*\| = \rho\right] = m_\lambda, \quad \|v_\lambda^* - u_\lambda^*\| > \rho \tag{4.5}$$

for $\lambda \geq \lambda_1^*$.

Because of (4.1) we know that $\hat{\psi}_\lambda$ is coercive for all $\lambda \geq \lambda_0^*$ and this implies that

$$\hat{\psi}_\lambda \text{ satisfies the PS-condition for all } \lambda \geq \lambda_0^*. \tag{4.6}$$

Then (4.5) and (4.6) permit the use of the mountain pass theorem stated as Theorem 2.1. Therefore, we find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\hat{\psi}_\lambda} \quad \text{and} \quad m_\lambda \leq \hat{\psi}_\lambda(y_\lambda). \tag{4.7}$$

From (4.5) and (4.7) it follows that

$$y_\lambda \notin \{u_\lambda^*, v_\lambda^*\} \quad \text{for all } \lambda \geq \lambda_1^*. \tag{4.8}$$

In addition, from (4.7) and Claim 1, we have

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}) \quad \text{for all } \lambda \geq \lambda_1^*. \tag{4.9}$$

From (4.8) and (4.9) we see that if we can show the nontriviality of y_λ, then this will be a nodal solution of (P_λ). To this end, note that y_λ is a critical point of mountain pass type for $\hat{\psi}_\lambda$. Therefore, we obtain

$$C_1\left(\hat{\psi}_\lambda, y_\lambda\right) \neq 0 \quad \text{for all } \lambda \geq \lambda_1^*, \tag{4.10}$$

see Motreanu-Motreanu-Papageorgiou [17, Corollary 6.81, p. 168]. We consider the homotopy $h_\lambda : [0,1] \times W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$h_\lambda(t,u) = (1-t)\hat{\psi}_\lambda(u) + t\varphi_\lambda(u) \quad \text{for all } \lambda \geq \lambda_1^*.$$

Suppose we could find sequences $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$t_n \to t, \quad u_n \to 0 \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad (h_\lambda)'_u(t_n,u_n) = 0 \quad \text{for all } n \in \mathbb{N}, \tag{4.11}$$

which results in

$$\begin{aligned}
&\langle A(u_n), h\rangle + (1-t_n)\int_\Omega |u_n|^{p-2}u_n h dx + \int_{\partial\Omega} \beta(x)|u_n|^{p-2}u_n h d\sigma \\
&= \int_\Omega \left[(1-t_n)\hat{f}_{\lambda_n}(x,u_n) + t_n f(x,u_n,\lambda_n)\right] h dx \quad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned}$$

This means

$$\begin{aligned}
&-\operatorname{div} a(x,\nabla u_n(x)) + (1-t_n)|u_n(x)|^{p-2}u_n(x) \\
&\quad = (1-t_n)\hat{f}_{\lambda_n}(x,u_n(x)) + t_n f(x,u_n(x),\lambda_n) \quad \text{for a.a. } x \in \Omega, \\
&\frac{\partial u}{\partial n_a} + \beta(x)|u_n|^{p-2}u_n = 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

From Winkert [23] we know that there exists $c_{20} > 0$ such that $\|u_n\|_\infty \leq c_{20}$ for all $n \in \mathbb{N}$. Then the nonlinear regularity theory of Lieberman [16] implies that we can find $\eta \in (0,1)$ and $c_{21} > 0$ such that

$$u_n \in C^{1,\eta}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\eta}(\overline{\Omega})} \leq c_{21} \quad \text{for all } n \in \mathbb{N}.$$

From (4.11) and the compact embedding of $C^{1,\eta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ it follows that $u_n \to 0$ in $C^1(\overline{\Omega})$ as $n \to \infty$. Hence, $u_n \in [v_\lambda^*, u_\lambda^*]$ for all $n \geq n_0$ and so, due to Claim 1, $\{u_n\}_{n \geq n_0} \subseteq K_{\hat{\psi}_\lambda}$. This contradicts the finiteness of $K_{\hat{\psi}_\lambda}$. Therefore, (4.11) cannot occur and so from the homotopy invariance of critical groups, see Gasiński-Papageorgiou [12, Theorem 5.125, p. 836], we have

$$C_k\left(\hat{\psi}_\lambda, 0\right) = C_k\left(\varphi_\lambda, 0\right) \quad \text{for all } k \in \mathbb{N}_0 \text{ and for all } \lambda \geq \lambda_1^*.$$

Since $u = 0$ is a local minimizer of φ_λ, see the proof of Proposition 3.7, we obtain

$$C_k\left(\hat{\psi}_\lambda, 0\right) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.12}$$

Comparing (4.10) and (4.12) we see that $y_\lambda \neq 0$. Hence

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$$

is a nodal solution of problem (P_λ) for all $\lambda \geq \lambda_1^*$. □

We can improve the conclusion of the previous theorem by strengthening the conditions on the map $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$. The new conditions read as follows.

H(a)$_1$: $a(x,\xi) = a_0\left(x, \|\xi\|_{\mathbb{R}^N}\right)\xi$ with $a_0 \in C(\overline{\Omega} \times \mathbb{R}_+)$ for all $\xi \in \mathbb{R}^N$ where $\mathbb{R}_+ = [0,+\infty)$ and with $a_0(x,t) > 0$ for all $x \in \overline{\Omega}$, for all $t > 0$, hypotheses H(a)$_1$(i), (ii), (iv) are the same as the corresponding hypotheses H(a)(i), (ii), (iv) and

(iii) $(\nabla_\xi a(x,\xi)y, y)_{\mathbb{R}^N} \geq c_{22}\|y\|^2$ for all $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N \setminus \{0\}$, for all $y \in \mathbb{R}^N$ and for some $c_{22} > 0$.

Remark 4.2. *So, we have strengthened the coercivity condition on $\nabla_\xi a(x,\cdot)$.*

Example 4.3. *Let $\hat{a} \in C^1(\overline{\Omega})$ be such that*

$$0 < \eta_0 \leq \hat{a}(x) \leq \eta_1 \quad \text{and} \quad 0 < \eta_0 \leq \|\nabla \hat{a}(x)\|_{\mathbb{R}^N} \leq \eta_1 \quad \text{for all } x \in \overline{\Omega}.$$

Then the following maps satisfy hypotheses H(a)$_1$:

$$\begin{aligned}
&a_1(x,\xi) = \hat{a}\|\xi\|_{\mathbb{R}^N}^{p-2}\xi + \ln\left(c + \|\xi\|_{\mathbb{R}^N}\right)\xi \quad \text{with} \quad 2 \leq p < \infty,\ c > 1,\\
&a_2(x,\xi) = \hat{a}(x)\|\xi\|_{\mathbb{R}^N}^{p-2}\xi + \xi \quad \text{with} \quad 2 < p < \infty,\\
&a_3(x,\xi) = A_0(x)\xi \quad \text{with} \quad A_0 \in C^1\left(\overline{\Omega}, \mathbb{R}^{N\times N}\right) \text{ positive definite.}
\end{aligned}$$

We have the following result.

Proposition 4.4. *If hypotheses H(a)$_1$, H(β) and H(f)$_1$ hold, then there exists $\lambda_1^* \geq \lambda_0^*$ such that problem (P_λ) has a nodal solution*

$$y_\lambda \in \operatorname*{int}_{C^1(\overline{\Omega})} [v_\lambda^*, u_\lambda^*]$$

for all $\lambda \geq \lambda_1^*$.

Proof. From Theorem 4.1 we know that there exists $\lambda_1^* \geq \lambda_0^*$ such that problem (P_λ) admits a nodal solution

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$$

for all $\lambda \geq \lambda_1^*$. Since $y_\lambda \leq u_\lambda^*$ we obtain

$$\begin{aligned} -\operatorname{div} a(x, \nabla y_\lambda(x)) + |y_\lambda(x)|^{p-2} y_\lambda(x) &= f\left(x, y_\lambda(x), \lambda\right) + |y_\lambda(x)|^{p-2} y_\lambda(x) \\ &\leq f\left(x, u_\lambda^*(x), \lambda\right) + \left(u_\lambda^*(x)\right)^{p-1} \\ &= -\operatorname{div} a\left(x, \nabla u_\lambda^*(x)\right) + \left(u_\lambda^*(x)\right)^{p-1}. \end{aligned} \tag{4.13}$$

Hypotheses $H(a)_1$(iii) and the tangency principle of Pucci-Serrin [21, Theorem 2.5.2, p. 35] imply that

$$y_\lambda(x) < u_\lambda^*(x) \quad \text{for all } x \in \Omega. \tag{4.14}$$

We set

$$\begin{aligned} h_1(x) &= f\left(x, y_\lambda(x), \lambda\right) + |y_\lambda(x)|^{p-2} y_\lambda(x), \\ h_2(x) &= f\left(x, u_\lambda^*(x), \lambda\right) + \left(u_\lambda^*(x)\right)^{p-1}. \end{aligned}$$

Evidently, $h_1, h_2 \in L^\infty(\Omega)$ and since $y_\lambda, u_\lambda^* \in C^1(\overline{\Omega})$ we infer from (4.14) that $h_1 \prec h_2$. Then because of (4.13) and Proposition 2.9 we obtain

$$u_\lambda^* - y_\lambda \in \hat{D}_+.$$

In a similar way we prove that

$$y_\lambda - v_\lambda^* \in \hat{D}_+.$$

Therefore, we conclude that $y_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]$. □

References

[1] S. Aizicovici, N. S. Papageorgiou, V. Staicu, "Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints", Mem. Amer. Math. Soc., Vol. 196, no. 915, 2008.

[2] A. Ambrosetti, H. Brezis, G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), no. 2, 519–543.

[3] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.

[4] D. Arcoya, D. Ruiz, *The Ambrosetti-Prodi problem for the p-Laplacian operator*, Comm. Partial Differential Equations **31** (2006), no. 4-6, 849–865.

[5] F. Brock, L. Iturriaga, P. Ubilla, *A multiplicity result for the p-Laplacian involving a parameter*, Ann. Henri Poincaré **9** (2008), no. 9, 1371–1386

[6] T. Cardinali, N. S. Papageorgiou, P. Rubbioni, *Bifurcation phenomena for nonlinear superdiffusive Neumann equations of logistic type*, Ann. Mat. Pura Appl. (4) **193** (2014), no. 1, 1–21.

[7] M. Cuesta, P. Takáč, *A strong comparison principle for positive solutions of degenerate elliptic equations*, Differential Integral Equations **13** (2000), no. 4-6, 721–746.

[8] G. Fragnelli, D. Mugnai, N. S. Papageorgiou, *Positive and nodal solutions for parametric nonlinear Robin problems with indefinite potential*, Discrete Contin. Dyn. Syst. **36** (2016), no. 11, 6133–6166.

[9] J. P. García Azorero, I. Peral Alonso, J. J. Manfredi, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp. Math. **2** (2000), no. 3, 385–404.

[10] L. Gasiński, N. S. Papageorgiou, "Nonlinear Analysis", Chapman & Hall/CRC, Boca Raton, 2006.

[11] L. Gasiński, N. S. Papageorgiou, *Bifurcation-type results for nonlinear parametric elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 3, 595–623.

[12] L. Gasiński, N. S. Papageorgiou, "Exercises in Analysis. Part 2: Nonlinear Analysis", Springer, Heidelberg, 2016.
[13] Z. M. Guo, *Some existence and multiplicity results for a class of quasilinear elliptic eigenvalue problems*, Nonlinear Anal. **18** (1992), no. 10, 957–971.
[14] Z. Guo, Z. Zhang, *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **286** (2003), no. 1, 32–50.
[15] S. Hu, N. S. Papageorgiou, "Handbook of Multivalued Analysis. Vol. I: Theory", Kluwer Academic Publishers, Dordrecht, 1997.
[16] G. M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural′tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), no. 2-3, 311–361.
[17] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou, "Topological and variational methods with applications to nonlinear boundary value problems", Springer, New York, 2014.
[18] N. S. Papageorgiou, V. D. Rădulescu, *Multiple solutions with precise sign for nonlinear parametric Robin problems*, J. Differential Equations **256** (2014), no. 7, 2449–2479.
[19] N. S. Papageorgiou, V. D. Rădulescu, *Nonlinear nonhomogeneous Robin problems with superlinear reaction term*, Adv. Nonlinear Stud. **16** (2016), no. 4, 737–764.
[20] N. S. Papageorgiou, V. D. Rădulescu, D. D. Repovš, *Positive solutions for perturbations of the Robin eigenvalue problem plus an indefinite potential*, Discrete Contin. Dyn. Syst. **37** (2017), no. 5, 2589–2618.
[21] P. Pucci, J. Serrin, "The Maximum Principle", Birkhäuser Verlag, Basel, 2007.
[22] S. Takeuchi, *Positive solutions of a degenerate elliptic equation with logistic reaction*, Proc. Amer. Math. Soc. **129** (2001), no. 2, 433–441.
[23] P. Winkert, *L^∞-estimates for nonlinear elliptic Neumann boundary value problems*, NoDEA Nonlinear Differential Equations Appl. **17** (2010), no. 3, 289–302.
[24] E. Zeidler, "Nonlinear Functional Analysis and its Applications. II/B", Springer-Verlag, New York, 1990.
[25] Q. Zhang, *A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions*, J. Math. Anal. Appl. **312** (2005), no. 1, 24–32.

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