VARIATIONAL-HEMIVARIATIONAL INEQUALITIES WITH NONHOMOGENEOUS NEUMANN BOUNDARY CONDITION

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Abstract. The aim of this paper is the study of variational-hemivariational inequalities with nonhomogeneous Neumann boundary condition. We prove sufficient conditions for the existence of a whole sequence of solutions which is either unbounded or converges to zero. For homogeneous Neumann boundary condition, results of this type have been obtained in Marano and Motreanu [3]. Our approach is based on abstract nonsmooth critical point results given in [3]. The applicability of our results is demonstrated by providing two verifiable criteria which address problems with nonsmooth potential and nonzero Neumann boundary condition.

1. Introduction

The present paper is devoted to the study of variational-hemivariational inequalities involving boundary integral terms. Specifically, given a bounded domain Ω in $\mathbb{R}^N$ with a C^1-boundary $\partial\Omega$ and $p \in]N, +\infty[$, we consider the problem: Find $u \in K$ such that, for all $v \in K$,

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(v-u)dx + \int_\Omega a|u|^{p-2}u(v-u)dx \\ &\quad + \int_\Omega \alpha F^\circ(u; v-u)dx + \int_{\partial\Omega} \theta H^\circ(\gamma u; \gamma v - \gamma u)d\sigma \geq 0, \end{aligned} \tag{1.1}$$

where K is a closed convex subset of $W^{1,p}(\Omega)$ containing the constant functions. The data in (1.1) are supposed to satisfy: $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} a(x) > 0$, $\alpha \in L^1(\Omega)$ and $\theta \in L^1(\partial\Omega)$ fulfilling

$$\alpha(x) \geq 0, \quad \text{for a.a. } x \in \Omega, \qquad \theta(x) \geq 0, \quad \text{for a.a. } x \in \partial\Omega, \tag{1.2}$$

F° and H° stand for Clarke's generalized directional derivatives of locally Lipschitz functions $F, H : \mathbb{R} \to \mathbb{R}$ given by

$$F(\xi) := \int_0^\xi f(t)dt, \qquad H(\xi) := \int_0^\xi h(t)dt,$$

where $f, h : \mathbb{R} \to \mathbb{R}$ are locally essentially bounded functions, and $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator. We endow the space $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_\Omega (|\nabla u|^p + a|u|^p)\, dx \right)^{1/p},$$

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which is equivalent to the usual one. Since $p > N$, there is the compact embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$. For a later use, let

$$c := \sup\{\|u\|^{-1}_{W^{1,p}(\Omega)}\|u\|_{C^0(\overline{\Omega})} : u \in W^{1,p}(\Omega), u \not\equiv 0\} < \infty, \tag{1.3}$$

be the best embedding constant, where $\|u\|_{C^0(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |u(x)|$. The expression of c implies that

$$c^p\|a\|_{L^1(\Omega)} \geq 1.$$

Problem (1.1) differs from the corresponding problem studied in Marano and Motreanu [3] by the fact that the integral term

$$\int_\Omega \beta G^o(u; v-u)dx$$

in [3], with a locally Lipschitz function $G : \mathbb{R} \to \mathbb{R}$ and some $\beta \in L^1(\Omega)$, is replaced in our formulation with the boundary term

$$\int_{\partial\Omega} \theta H^o(\gamma u; \gamma v - \gamma u)d\sigma.$$

Actually, this expresses the passage in the Neumann boundary condition from the homogeneous situation (i.e., $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$) to the possibly nonhomogeneous case in (1.1). A prototype of (1.1), taking for simplicity $K = W^{1,p}(\Omega)$, is the following boundary value problem with nonsmooth potential and nonhomogeneous, nonsmooth Neumann boundary condition:

$$\begin{aligned} \Delta_p u - a(x)|u|^{p-2}u &\in \alpha(x)\partial F(u) && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} &\in -\theta(x)\partial H(\gamma u) && \text{on } \partial\Omega, \end{aligned}$$

where $n(x)$ is the outward unit normal at $x \in \partial\Omega$, $\frac{\partial u}{\partial n}$ denotes the corresponding normal derivative of u on $\partial\Omega$, while ∂F and ∂H represent the generalized gradients of F and H, respectively.

Our main results are Theorems 3.1 and 3.2 providing sufficient conditions that problem (1.1) admit a whole sequence of solutions which is either unbounded or converges to zero. Theorems 3.1 and 3.2 correspond to Theorems 2.1 and 2.2 in [3] which hold for the homogeneous Neumann boundary condition. However, Theorems 3.1 and 3.2 are not extensions of Theorems 2.1 and 2.2 in [3] due to the presence therein of a term involving a locally Lipschitz function $G : \mathbb{R} \to \mathbb{R}$ that cannot be incorporated in the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ in view of the imposed assumptions on F and G. We illustrate the applicability of Theorems 3.1 and 3.2 by two results stated as Theorems 4.1 and 4.2, which give verifiable criteria to fulfill the hypotheses of Theorems 3.1 and 3.2 in the case where the function H describing the generalized Neumann boundary condition is not trivial. Our approach is variational relying on a nonsmooth critical point theorem that guarantees the existence of infinitely many critical points in our nonsmooth setting with suitable convergence properties (see [3]).

The rest of the paper is organized as follows. Section 2 presents some notions and results in the nonsmooth critical point theory which are needed in the sequel. Section 3 contains our main results, while Section 4 sets forth our applications.

2. Preliminaries

In this section, we give a brief overview on some prerequisites of nonsmooth analysis which are needed in the sequel. Let X be a real normed space with the norm $||\cdot||$. Given a locally Lipschitz function $j : X \to \mathbb{R}$ on a Banach space X, we denote by $j^{\mathrm{o}}(u; v)$ the generalized directional derivative of j at the point $u \in X$ in direction $v \in X$, which is defined by

$$j^{\mathrm{o}}(u; v) = \limsup_{x \to u, t \downarrow 0} \frac{j(x + tv) - j(x)}{t},$$

(see [2, Chapter 2]). If $j_1, j_2 : X \to \mathbb{R}$ are locally Lipschitz functions, there holds

$$(j_1 + j_1)^{\mathrm{o}}(u; v) \leq j_1^{\mathrm{o}}(u; v) + j_2^{\mathrm{o}}(u; v), \qquad \forall u, v \in X.$$

Now we consider a function $\Phi : X \to \mathbb{R}$ which satisfies the structure hypothesis:

(S) $\Phi = j + I$, with $j : X \to \mathbb{R}$ locally Lipschitz and $I : X \to \mathbb{R} \cup \{+\infty\}$ convex, proper (i.e., $I \not\equiv +\infty$), and lower semicontinuous.

A point $u \in X$ is called a critical point of Φ if the following inequality is valid

$$j^{\mathrm{o}}(u; v - u) + I(v) - I(u) \geq 0, \quad \forall v \in X$$

(see [5, Chapter 3]). In the case where j is continuously differentiable, this definition reduces to the one of Szulkin in [7] and in the case where $I \equiv 0$ it coincides with the notion of critical point introduced by Chang (cf. [1]). By [4, Proposition 2.1], we know that each local minimum of Φ is a critical point.

Let $(X, \|\cdot\|)$ and $\widetilde{X}$ be real Banach spaces such that X is compactly embedded in $\widetilde{X}$. Further, let $j_1 : \widetilde{X} \to \mathbb{R}$ and $j_2 : X \to \mathbb{R}$ be locally Lipschitz, and let $I : X \to \mathbb{R} \cup \{+\infty\}$ be convex, proper, and lower semicontinuous. By $D(I)$ we denote the effective domain of I, that is $D(I) = \{u \in X : I(u) < +\infty\}$. Set

$$\Phi(u) = j_1(u) + I(u), \qquad \Psi(u) = j_2(u) \quad \text{ for all } u \in X.$$

We note that Φ and Ψ satisfy the structure condition (S). We assume that

$$\Psi^{-1}(]-\infty, \varrho[) \cap D(I) \neq \emptyset, \quad \forall \varrho > \inf_X \Psi, \tag{2.1}$$

and define for every $\varrho > \inf_X \Psi$ the nonnegative number

$$\varphi(\varrho) := \inf_{u \in \Psi^{-1}(]-\infty,\varrho[)} \frac{\Phi(u) - \inf\limits_{v \in \overline{(\Psi^{-1}(]-\infty,\varrho[))}_w} \Phi(v)}{\varrho - \Psi(u)}, \tag{2.2}$$

where $\overline{(\Psi^{-1}(]-\infty, \varrho[))}_w$ denotes the weak closure of $\Psi^{-1}(]-\infty, \varrho[)$. Finally, we introduce

$$\delta_1 := \liminf_{\varrho \to +\infty} \varphi(\varrho), \qquad \delta_2 := \liminf_{\varrho \to (\inf_X \Psi)^+} \varphi(\varrho).$$

Keeping the notation above, we state the following nonsmooth version of a critical point result due to Ricceri [6, Theorem 2.5], which was established in [3, Theorem 1.1].

Theorem 2.1. *Assume that X is a reflexive Banach space and that the function Ψ is weakly sequentially lower semicontinuous, coercive and satisfies (2.1). Then the following properties hold:*

(a) *For every $\varrho > \inf_X \Psi$ and every $\lambda > \varphi(\varrho)$, the function $\Phi + \lambda\Psi$ has a critical point (local minimum) lying in $\Psi^{-1}(]-\infty, \varrho[)$.*

(b) *If $\delta_1 < +\infty$, then for every $\lambda > \delta_1$, either*
  (b1) *$\Phi + \lambda\Psi$ possesses a global minimum, or*
  (b2) *there is a sequence (u_n) of critical points (local minima) of $\Phi + \lambda\Psi$ such that $\lim_{n\to+\infty} \Psi(u_n) = +\infty$.*

(c) *If $\delta_2 < +\infty$, then for every $\lambda > \delta_2$, either*
  (c1) *$\Phi + \lambda\Psi$ has a local minimum, which is also a global minimum of Ψ, or*
  (c2) *there exists a sequence (u_n) of pairwise distinct critical points (local minima) of $\Phi + \lambda\Psi$, with $\lim_{n\to+\infty} \Psi(u_n) = \inf_X \Psi$, weakly converging to a global minimum of Ψ.*

3. Main Results

Our first main result reads as follows.

Theorem 3.1. *Assume that*

$$\inf_{\xi\in\mathbb{R}} H(\xi) \geq 0 \tag{3.1}$$

and there exist two sequences $(r_n) \subset \mathbb{R}_+$ and $(\xi_n) \subset \mathbb{R}$ such that

$$\lim_{n\to+\infty} r_n = +\infty, \tag{3.2}$$

$$F(\xi_n) = \inf_{|\xi|\leq c(pr_n)^{1/p}} F(\xi), \qquad \forall n \in \mathbb{N}, \tag{3.3}$$

$$\frac{1}{p}\|a\|_{L^1(\Omega)}|\xi_n|^p + H(\xi_n)\|\theta\|_{L^1(\partial\Omega)} < r_n, \qquad \forall n \in \mathbb{N}, \tag{3.4}$$

$$\liminf_{|\xi|\to+\infty} \frac{F(\xi)\|\alpha\|_{L^1(\Omega)} + H(\xi)\|\theta\|_{L^1(\partial\Omega)}}{|\xi|^p} < -\frac{1}{p}\|a\|_{L^1(\Omega)}. \tag{3.5}$$

Then problem (1.1) possesses an unbounded sequence of solutions.

Proof. In order to apply Theorem 2.1, we set $X := W^{1,p}(\Omega)$ and $\widetilde{X} = C^0(\overline{\Omega})$, which guarantees that X is compactly embedded in $\widetilde{X}$ due to $p > N$. Let the functions $j_1 : \widetilde{X} \to \mathbb{R}$ and $j_2 : X \to \mathbb{R}$ be defined by

$$j_1(u) := \int_\Omega \alpha F(u)dx, \qquad j_2(u) := \frac{1}{p}\|u\|_X^p + \int_{\partial\Omega} \theta H(\gamma u)d\sigma.$$

They are locally Lipschitz since $f, h \in L^\infty_{\text{loc}}(\mathbb{R})$. Let $I : X \to \mathbb{R} \cup \{+\infty\}$ be the indicator function of the set K, that is

$$I(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

which is convex, proper, and lower semicontinuous. Next, we introduce

$$\Phi(u) = j_1(u) + I(u), \qquad \Psi(u) = j_2(u), \text{ for all } u \in X.$$

Hypotheses (1.2) and (3.1) imply the estimate

$$\Psi(u) = \frac{1}{p}\|u\|_X^p + \int_{\partial\Omega} \theta H(\gamma u)d\sigma \geq \frac{1}{p}\|u\|_X^p, \qquad \forall u \in X, \tag{3.6}$$

which proves the coercivity of Ψ and that $\inf_X \Psi = \Psi(0) = 0$. Recalling that K contains the constant functions, it follows that $0 \in K = D(I)$, thereby

$$0 \in \Psi^{-1}(]-\infty, \varrho[) \cap D(I), \qquad \forall \varrho > \inf_X \Psi.$$

Using the compactness of the trace operator γ, we can show that Ψ is weakly sequentially lower semicontinuous on X.
Taking into account (3.6), we get for each $v \in \overline{(\Psi^{-1}(]-\infty,\varrho[))}_w$ the estimate $\|v\|_X \leq (p\varrho)^{1/p}$. Then, in view of (2.2), we infer that

$$0 \leq \varphi(\varrho) \leq \inf_{u\in\Psi^{-1}(]-\infty,\varrho[)} \frac{\Phi(u) - \inf\limits_{\|v\|_X\leq(p\varrho)^{1/p}} \Phi(v)}{\varrho - \Psi(u)}, \quad \forall \varrho > \inf_X \Psi = 0. \tag{3.7}$$

Let $n \in \mathbb{N}$ be fixed. The definition of the embedding constant c in (1.3) ensures that for each $v \in X$ satisfying $\|v\|_X \leq (pr_n)^{1/p}$ we have $|v(x)| \leq c(pr_n)^{1/p}$ for all $x \in \Omega$. From (1.2) and (3.3), we obtain

$$\Phi(\xi_n) \leq \inf_{\|v\|_X\leq(pr_n)^{1/p}} \Phi(v). \tag{3.8}$$

Applying (3.4) yields

$$\Psi(\xi_n) = \frac{1}{p}\|\xi_n\|_X^p + \int_{\partial\Omega} \theta H(\xi_n)d\sigma = \frac{1}{p}\|a\|_{L^1(\Omega)}|\xi_n|^p + H(\xi_n)\|\theta\|_{L^1(\partial\Omega)} < r_n,$$

which proves that $\xi_n \in \Psi^{-1}(]-\infty,r_n[)$. By virtue of (3.2), we may insert $\varrho = r_n$ in (3.7) provided n is sufficiently large. Combining with (3.8) it results in

$$0 \leq \varphi(r_n) \leq \inf_{u\in\Psi^{-1}(]-\infty,r_n[)} \frac{\Phi(u) - \Phi(\xi_n)}{r_n - \Psi(u)} \leq \frac{\Phi(\xi_n) - \Phi(\xi_n)}{r_n - \Psi(\xi_n)} = 0. \tag{3.9}$$

It turns out from (3.2) and (3.9) that $\liminf_{\varrho\to+\infty} \varphi(\varrho) = 0$ meaning $\delta_1 = 0$. We are thus allowed to apply part (b) of Theorem 2.1 with $\lambda = 1$.

We claim that the function $\Phi + \Psi$ is unbounded from below. According to (3.5), we can choose $\eta > 0$ such that

$$\eta \in \left]\frac{1}{p}\|a\|_{L^1(\Omega)}, -\liminf_{|\xi|\to+\infty} \frac{F(\xi)\|\alpha\|_{L^1(\Omega)} + H(\xi)\|\theta\|_{L^1(\partial\Omega)}}{|\xi|^p}\right[.$$

This allows us to select a sequence $(\sigma_n) \subset \mathbb{R}$ satisfying

$$\lim_{n\to+\infty} |\sigma_n| = +\infty, \quad F(\sigma_n)\|\alpha\|_{L^1(\Omega)} + H(\sigma_n)\|\theta\|_{L^1(\partial\Omega)} < -\eta|\sigma_n|^p, \quad \forall n \in \mathbb{N}.$$

Then we derive

$$\begin{aligned}\Phi(\sigma_n) + \Psi(\sigma_n) &= F(\sigma_n)\|\alpha\|_{L^1(\Omega)} + \frac{1}{p}\|a\|_{L^1(\Omega)}|\sigma_n|^p + H(\sigma_n)\|\theta\|_{L^1(\partial\Omega)} \\ &< (\frac{1}{p}\|a\|_{L^1(\Omega)} - \eta)|\sigma_n|^p, \quad n \in \mathbb{N}.\end{aligned}$$

Hence, by the choice of η, $\lim_{n\to+\infty}(\Phi(\sigma_n) + \Psi(\sigma_n)) \to -\infty$, which justifies our claim. Therefore, assertion (b2) in Theorem 2.1 yields a sequence $(u_n) \subset X$ of critical points of $\Phi + \Psi$ satisfying $\lim_{n\to+\infty} \Psi(u_n) = +\infty$. As Ψ is bounded on bounded sets, we deduce that the sequence (u_n) is unbounded in X. The fact that u_n is a critical point of $\Phi + \Psi$ means that

$$(j_1 + j_2)^{\text{o}}(u_n; v - u_n) + I(v) - I(u_n) \geq 0, \qquad \forall v \in X. \tag{3.10}$$

Then (3.10) entails that $u_n \in K$ and

$$j_1^{\text{o}}(u_n; v - u_n) + j_2^{\text{o}}(u_n; v - u_n) \geq 0, \qquad \forall v \in K. \tag{3.11}$$

A basic result on the generalized directional derivative of an integral functional (see [2, p. 77]) shows that

$$j_1^{\circ}(u_n; v - u_n) \le \int_\Omega \alpha(x) F^{\circ}(u_n; v - u_n) dx, \qquad \forall v \in K, \tag{3.12}$$

and, for all $v \in K$,

$$\begin{aligned} j_2^{\circ}(u_n; v - u_n) \le & \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (v - u_n) dx + \int_\Omega a |u_n|^{p-2} u_n (v - u_n) dx \\ & + \int_{\partial\Omega} \theta H^{\circ}(\gamma u_n; \gamma v - \gamma u_n) d\sigma. \end{aligned} \tag{3.13}$$

Combining (3.11), (3.12), and (3.13) leads to

$$\begin{aligned} & \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (v - u_n) dx + \int_\Omega a |u_n|^{p-2} u_n (v - u_n) dx \\ & + \int_\Omega \alpha F^{\circ}(u_n; v - u_n) dx + \int_{\partial\Omega} \theta H^{\circ}(\gamma u_n; \gamma v - \gamma u_n) d\sigma \ge 0, \end{aligned}$$

for all $v \in K$, which completes the proof. □

Our second main result is the following theorem. Since its proof can be carried out along the same lines as for Theorem 3.1, we omit it.

Theorem 3.2. *Assume that*

$$\inf_{\xi \in \mathbb{R}} H(\xi) \ge 0$$

and there exist two sequences $(r_n) \subset \mathbb{R}_+$ and $(\xi_n) \subset \mathbb{R}$ such that

$$\begin{aligned} & \lim_{n \to +\infty} r_n = 0, \\ & F(\xi_n) = \inf_{|\xi| \le c(pr_n)^{1/p}} F(\xi), \qquad \forall n \in \mathbb{N}, \\ & \frac{1}{p} \|a\|_{L^1(\Omega)} |\xi_n|^p + H(\xi_n) \|\theta\|_{L^1(\partial\Omega)} < r_n, \qquad \forall n \in \mathbb{N}, \\ & \liminf_{\xi \to 0} \frac{F(\xi)\|\alpha\|_{L^1(\Omega)} + H(\xi)\|\theta\|_{L^1(\partial\Omega)}}{|\xi|^p} < -\frac{1}{p} \|a\|_{L^1(\Omega)}. \end{aligned}$$

Then problem (1.1) admits a sequence of nontrivial solutions converging to zero.

4. Applications

First, we present an application of Theorem 3.1 with a function H which is not zero.

Theorem 4.1. *Assume that $\alpha \not\equiv 0$ in (1.2). Let $(\xi_n) \subset \mathbb{R}$ be a sequence with $\lim_{n \to +\infty} \xi_n = +\infty$ and let $F : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}_+$ be locally Lipschitz functions such that for n sufficiently large the following conditions hold:*

$$\xi_{n-1}^2 < \xi_n; \tag{4.1}$$

$$F|_{[-\xi_n^2, -\xi_{n-1}^2] \cup [\xi_{n-1}^2,\ \xi_n^2]} \ge F(\xi_n) = -\frac{\xi_n^{2p}}{p\|\alpha\|_{L^1(\Omega)} c^p}; \tag{4.2}$$

$$\|\theta\|_{L^1(\partial\Omega)} H(\xi_n) < \frac{1}{p} \left(\frac{1}{c^p} \xi_n^{2p} - \|a\|_{L^1(\Omega)} (1 + \varepsilon) |\xi_n|^p \right) \tag{4.3}$$

with a constant $\varepsilon > 0$. Then problem (1.1) possesses an unbounded sequence of solutions.

Proof. Let us check that Theorem 3.1 applies. Set

$$r_n = \frac{1}{pc^p}\xi_n^{2p}. \tag{4.4}$$

By (4.4) we see that hypothesis (3.2) is fulfilled. Notice that $c(pr_n)^{\frac{1}{p}} = \xi_n^2$. Then, by (4.1) and (4.2), we have

$$\inf_{|\xi|\le c(pr_n)^{\frac{1}{p}}} F(\xi) = \inf_{|\xi|\le \xi_n^2} F(\xi) = \inf_{\xi\in[-\xi_n^2,-\xi_{n-1}^2]\cup[\xi_{n-1}^2,\ \xi_n^2]} F(\xi) = F(\xi_n).$$

This shows that hypothesis (3.3) is satisfied. Using (4.3) and (4.4), it follows readily that the inequality required in hypothesis (3.4) is true. Finally, we note that hypothesis (3.5) is also verified because through (4.3) and (4.2) we arrive at

$$\begin{aligned}
&\liminf_{|\xi|\to+\infty} \frac{F(\xi)\|\alpha\|_{L^1(\Omega)} + H(\xi)\|\theta\|_{L^1(\partial\Omega)}}{|\xi|^p} \\
&\le \liminf_{n\to+\infty} \frac{F(\xi_n)\|\alpha\|_{L^1(\Omega)} + H(\xi_n)\|\theta\|_{L^1(\partial\Omega)}}{|\xi_n|^p} \\
&\le \liminf_{n\to+\infty} \frac{F(\xi_n)\|\alpha\|_{L^1(\Omega)} + \frac{1}{p}\left(\frac{1}{c^p}\xi_n^{2p} - \|a\|_{L^1(\Omega)}(1+\varepsilon)|\xi_n|^p\right)}{|\xi_n|^p} \\
&= -\frac{\|a\|_{L^1(\Omega)}(1+\varepsilon)}{p}.
\end{aligned}$$

Applying Theorem 3.1 we achieve the desired conclusion. □

Now we present an application of Theorem 3.2 involving a function H which is not zero.

Theorem 4.2. *Assume that $\alpha \not\equiv 0$ in (1.2). Let (ξ_n) be a sequence of positive real numbers with $\xi_n \downarrow 0$ and let $F : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}_+$ be locally Lipschitz functions such that for n sufficiently large the following conditions hold:*

$$\sqrt{\xi_n} < \xi_{n-1}; \tag{4.5}$$

$$F|_{[-\sqrt{\xi_n},-\sqrt{\xi_{n+1}}]\cup[\sqrt{\xi_{n+1}},\sqrt{\xi_n}]} \ge F(\xi_n) = -\frac{\xi_n^{\frac{p}{2}}}{p\|\alpha\|_{L^1(\Omega)}c^p}; \tag{4.6}$$

$$\|\theta\|_{L^1(\partial\Omega)}H(\xi_n) < \frac{1}{p}\left(\frac{1}{c^p}\xi_n^{\frac{p}{2}} - \|a\|_{L^1(\Omega)}(1+\varepsilon)\xi_n^p\right) \tag{4.7}$$

with a constant $\varepsilon > 0$. Then problem (1.1) possesses a sequence of solutions converging to zero.

Proof. The proof proceeds in the same way as for Theorem 4.1, this time applying Theorem 3.2. To this end, we set

$$r_n = \frac{1}{pc^p}\xi_n^{\frac{p}{2}}.$$

By (4.5) and (4.6), we have

$$\inf_{|\xi|\le c(pr_n)^{\frac{1}{p}}} F(\xi) = \inf_{[-\sqrt{\xi_n},-\sqrt{\xi_{n+1}}]\cup[\sqrt{\xi_{n+1}},\sqrt{\xi_n}]} F(\xi) = F(\xi_n).$$

Using (4.7) and (4.6) we see that

$$
\begin{aligned}
&\liminf_{\xi\to 0} \frac{F(\xi)\|\alpha\|_{L^1(\Omega)} + H(\xi)\|\theta\|_{L^1(\partial\Omega)}}{|\xi|^p} \\
&\leq \liminf_{n\to+\infty} \frac{F(\xi_n)\|\alpha\|_{L^1(\Omega)} + H(\xi_n)\|\theta\|_{L^1(\partial\Omega)}}{|\xi_n|^p} \\
&\leq \liminf_{n\to+\infty} \frac{F(\xi_n)\|\alpha\|_{L^1(\Omega)} + \frac{1}{p}\left(\frac{1}{c^p}\xi_n^{\frac{p}{2}} - \|a\|_{L^1(\Omega)}(1+\varepsilon)|\xi_n|^p\right)}{|\xi_n|^p} \\
&= -\frac{\|a\|_{L^1(\Omega)}(1+\varepsilon)}{p}.
\end{aligned}
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We may apply Theorem 3.2, which completes the proof. $\square$

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