EXISTENCE AND STABILITY FOR A CLASS OF VARIATIONAL-HEMIVARIATIONAL INEQUALITIES INVOLVING MULTIVALUED BREZIS PSEUDOMONOTONE OPERATORS

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ABSTRACT. In this paper, we provide existence and stability results for a new multivalued variational-hemivariational inequality (MVHVI, for short) involving Brezis pseudomonotone operators in a reflexive Banach space. First, we use the Moreau-Yosida approximation to introduce an approximated problem corresponding to (MVHVI), and apply an existence result for nonlinear equilibrium problems, convergence techniques as well as Sion's Minimax Theorem to prove the existence of solutions of (MVHVI). Then, a stability result for (MVHVI) is obtained via employing Tikhonov regularization and perturbed approach. The theoretical results established in this paper extend the ones in [J. Global Optim. 52 (2012), 743–756] and [J. Global Optim. 56 (2013), 605–622]. Finally, we study a stationary Navier-Stokes equation with nonmonotone and multivalued constitutive laws for illustrating the validity of the main theoretical results in this paper.

1. INTRODUCTION

It is well known that the models formulated by equations can be used to solve several problems, but in many cases, equations are not sufficient to describe complicated natural phenomena and motion behaviors such as Signorini contact conditions and multivalued constitutive laws. Therefore, the study of problems involving inequalities or inclusions attracts the attention of scientists because they can accurately describe various comprehensive problems arising in physical processes, engineering, economics and other fields, see, for examples the papers of Clason-Valkonen [10] Huang-Fang [17] as well as the famous monographs of Kinderlehrer-Stampacchia [19] and Naniewicz-Panagiotopoulos [30]. Usually, inequalities are mainly divided into two types of inequalities, namely variational inequalities and hemivariational inequalities. From a mathematical point of view, variational inequalities are formulated using convex energy functionals. After the pioneering work of Lions-Stampacchia [20], the study of variational inequalities became a highly interesting topic and a large number of excellent research papers have appeared in the past 60 years, covering both theoretical analysis and practical application in the fields of economics, transportation and operations research, see, for example, Giannessi-Maugeri [13], Nagurney [29] and Zeng-Migórski [40]. The notion of hemivariational inequalities was originally introduced by Panagiotopoulos [31] in the early 1980s for the study of contact problems with nonsmooth and nonconvex energy superpotentials. Essentially, hemivariational inequalities are based on the generalized subgradient and the generalized directional derivative in the sense of Clarke, see, for example, Liu [24], Naniewicz-Panagiotopoulos [30] and Zeng-Migórski-Khan [42].

An inequality problem is called to be a variational-hemivariational inequality, if it contains both convex functionals and locally Lipschitz functionals that are nonconvex in general, see Liu-Motreanu [23]. Such inequalities are powerful and useful models to explore the contact problems with multivalued nonmonotone and monotone constitutive laws formulated by Clarke's generalized subdifferentials and convex superpotentials. For instance, Han [14] applied a perturbed method to introduce a family of singularly perturbed problems for an elliptic variational-hemivariational inequality and proved that the solution of the singularly perturbed problem converges to the solution of the limiting problem when the singular perturbation parameter tends to zero. Furthermore, Xiao-Liu-Chen-Huang [38] performed

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a stability analysis for a variational-hemivariational inequality while Bartosz-Cheng-Kalita-Yu-Zheng [3] developed a version of Rothe's method to a parabolic variational-hemivariational inequality for establishing an existence theorem. A sub-supersolution approach to a class of parabolic variational-hemivariational inequalities has been done by Carl [8] in order to obtain existence of weak solutions and extremal solutions with respect to an appropriate pair of sub-supersolution. For more details on this topics, we refer to the works of Bonanno-Motreanu-Winkert [5], Bonanno-Winkert [6], Fang-Han-Migórski-Sofonea [12], Han-Migórski-Sofonea [15], Migórski-Ochal [26], Migórski-Ochal-Sofonea [27], Liu-Liu-Motreanu [21], Liu-Liu-Wen-Yao-Zeng [22], Papageorgiou-Rădulescu-Repovš [32, 33], Shillor-Sofonea-Telega [36], Zeng-Migórski [41] and the references therein.

Recently, Tang-Huang [37] and Costea-Rădulescu [11] considered the following hemivariational inequality with multivalued term: Find $x \in C$ and $x^* \in F(x)$ such that

$$\langle x^*, y - x\rangle + J^\circ(Tx, T(y - x)) \geq \langle f, y - x\rangle \quad \text{for all } y \in C, \tag{1.1}$$

and proved the existence of solutions of (1.1) under the assumptions that $F\colon X \to 2^{X^*}$ is lower semicontinuous and stably quasimonotone with respect to a set $W \subset X^*$, where X is a reflexive and separable Banach space, $T\colon X \to V$ is a linear and compact operator with V being a reflexive Banach space, $J\colon V \to \mathbb{R}$ is a locally Lipschitz function and $f \in X^*$. However, it is difficult to verify that a multivalued mapping F is lower semicontinuous, and in many applications F does not even satisfy monotonicity conditions, not even generalized monotonicity properties. This severely limits the scope of application of the hemivariational inequalities. In order to fill this gap, this paper is concerned with the study of a variational-hemivariational inequality involving a multivalued Brezis pseudomonotone operator that is not lower semicontinuous and monotone.

To be more precise, we consider the following variational-hemivariational inequality involving a pseudomonotone multivalued mapping in the sense of Brezis:

Problem P: Find $x \in C$ such that

$$\sup_{x^*\in F(x)} \langle x^*, y - x\rangle + \varphi(y) - \varphi(x) + j^\circ(\hat{x}, \hat{y} - \hat{x}) \geq 0 \quad \text{for all } y \in C,$$

where $C \subset X$ is a nonempty closed convex set of a real and uniformly convex Banach space X (without loss of generality, we can suppose that X is reflexive by the Milman-Pettis theorem) with dual space X^* being uniformly convex, $F\colon X \to 2^{X^*}$ is a multivalued mapping, $\varphi\colon X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous functional, and $j^\circ(Tu, T(v - u))$ represents Clarke's generalized directional derivative of the locally Lipschitz functional $j\colon V \to \mathbb{R}$ with V being a real reflexive Banach space, and $T\colon X \to V$ is a linear and compact operator with $\hat{x} := Tx$ for all $x \in X$. The first novelty of this paper is to show that **Problem P** is equivalent to the following inequality: Find $x \in C$ and $x^* \in F(x)$ such that

$$\langle x^*, y - x\rangle + \varphi(y) - \varphi(x) + j^\circ(\hat{x}, \hat{y} - \hat{x}) \geq 0 \quad \text{for all } y \in C. \tag{1.2}$$

Namely, $x \in C$ is a solution of **Problem P** if and only if it solves inequality (1.2). Then, we prove the existence of solutions to **Problem P** without the assumptions that F is lower semicontinuous and generalized monotone (for example, stably quasimonotone with respect to a set $W \subset X^*$). Here, we have to mention that the method and techniques used in this paper are completely different to the ones of Tang-Huang [37] and Costea-Rădulescu [11], in which they used the well-known KKM principle due to Ky Fan and the argument of generalized monotone operators. However, in this paper, we apply the theory of Brezis pseudomonotone operators, the Moreau-Yosida approximation technique, Sion's Minimax Theorem and a generalized existence theorem for nonlinear equilibrium problems for establishing our existence theorem. Moreover, it should be pointed out that when $j \equiv 0$, then **Problem P** reduces to the following mixed variational inequality with multivalued operator: Find $x \in C$ such that

$$\sup_{x^*\in F(x)} \langle x^*, y - x\rangle + \varphi(y) - \varphi(x) \geq 0 \quad \text{for all } y \in C, \tag{1.3}$$

where inequality (1.3) was recently studied by Bianchi-Kassay-Pini [4] with $\varphi \equiv 0$. Note that one cannot obtain similar results as the ones established in [4] for inequality (1.3), because the presence of

φ leads to the invalidity of the framework proposed in [4]. On the other hand, the second contribution of this paper is to explore the stability of **Problem P** via employing the Tikhonov regularization and a perturbed approach. The last goal of this paper is to apply the established theoretical results to study a Navier-Stokes equation with nonmonotone and multivalued constitutive law on boundary.

The rest of the paper is organized as follows. In Section 2, we recall some definitions and preliminary materials on multivalued B-pseudomonotone operators and nonsmooth analysis which are needed in the next sections. In Section 3, we focus our attention on the research of the existence of solutions to **Problem P** in which the method is based on the use of the Moreau-Yosida approximation approach and the properties of B-pseudomonotone operators. Section 4 is devoted to present a stability result for **Problem P**. Finally, in Section 5, a stationary Navier-Stokes equation with nonmonotone and multivalued constitutive laws on boundary is considered for illustrating the validity of the abstract results established in the present paper.

2. Mathematical background

In this section, we will briefly recall some important notations and necessary preliminary results which will be used in the next sections for obtaining the main results of the paper.

We start with the following definitions.

Definition 2.1. Let X be a Banach space. A single-valued operator $A\colon X \to X^*$ is said to be:

(i) monotone, if for all $u, v \in X$, we have $\langle Au - Av, u - v\rangle_{X^*\times X} \geq 0$;
(ii) bounded, if A maps bounded sets of X into bounded sets of X^*;
(iii) demicontinuous, if $u_n \to u$ in X implies $Au_n \rightharpoonup Au$ in X^*, where the symbol "$\rightharpoonup$" stands for the weak convergence.

Definition 2.2. We say that the operator $A\colon X \to 2^{X^*}$ is pseudomonotone in the sense of Brezis (B-pseudomonotone, for short) on a nonempty subset D of $\operatorname{dom}(A) := \{x\colon A(x) \neq \emptyset\}$, if for every $\{x_n\}_{n\in\mathbb{N}} \subseteq D$ such that $x_n \rightharpoonup x \in D$ and for every $x_n^* \in A(x_n)$ with $\limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle \leq 0$, one has that for every $y \in D$, there exists $x^*(y) \in A(x)$ such that $\langle x^*(y), x - y\rangle \leq \liminf_{n\to\infty}\langle x_n^*, x_n - y\rangle$.

The next proposition shows that the sum of two B-pseudomonotone operators is B-pseudomonotone as well, see Hu-Papageorgiou [16, Proposition 6.15].

Proposition 2.3. *If $A, B\colon X \to 2^{X^*}$ are B-pseudomonotone operators, then the sum $A+B\colon X \to 2^{X^*}$ is B-pseudomonotone.*

Next, let us recall some important definitions and properties concerning nonsmooth analysis. For more details, we refer to the monograph of Migórski-Ochal-Sofonea [28].

Definition 2.4. Let X be a Banach space and $j\colon X \to \mathbb{R}$ be a real-valued function. Then j is called locally Lipschitz, if for each $x \in X$ there exist a neighborhood $N(x)$ and $L_x > 0$ such that

$$|j(y) - j(z)| \leq L_x\|y - z\|_X \quad \text{for all } y, z \in N(x).$$

The generalized directional derivative (in the sense of Clarke) of the locally Lipschitz function j at the point $x \in X$ in the direction $y \in X$, denoted by $j^\circ(x, y)$, is defined by

$$j^\circ(x,y) = \limsup_{z\to x,\lambda\searrow 0}\frac{j(z+\lambda y) - j(z)}{\lambda}.$$

The generalized subdifferential of j (in the sense of Clarke) is defined by

$$\partial j(x) := \{x^* \in X^*\colon \langle x^*, y\rangle \leq j^\circ(x,y) \text{ for all } y \in X\}.$$

Lemma 2.5. *Let $j : X \to \mathbb{R}$ be a locally Lipschitz function. Then, we have*

(i) *$v \mapsto j^\circ(x, v)$ is finite, positively homogeneous and subadditive on X, and satisfies*

$$|j^\circ(x,v)| \leq L_x\|v\|_X \textit{for all } v \in X,$$

where L_x is the Lipschitz constant near x;

(ii) $j^\circ(x,v)$ *is upper semicontinuous as a function of* (x,v)*, and Lipschitz on* X *as a function of* v*.*
(iii) *For every* $v \in X, j^\circ(x,v) = \max\{\langle \xi, v\rangle_{X^*\times X} : \xi \in \partial j(x)\}$*.*

The following lemma gives sufficient conditions to determinate that a superposition of the Clarke subgradient with a bounded linear operator is B-pseudomonotone. The proof can be found in Bartosz [2, Proposition 5.6].

Lemma 2.6. *Let* X *and* V *be two reflexive Banach spaces,* $T\colon X \to V$ *be a linear and compact operator with* $T^*\colon V^* \to X^*$ *being its adjoint operator. Let* $j\colon V \to \mathbb{R}$ *be a locally Lipschitz function such that its subdifferential (in the sense of Clarke) satisfies*

$$\|\xi\|_{V^*} \le c(1+\|x\|_V) \quad \text{for all } \xi \in \partial j(x) \tag{2.1}$$

with $c>0$*. Then, the multivalued operator* $M\colon X \to 2^{X^*}$ *defined by*

$$M(x) = T^*\partial j(Tx) \quad \text{for all } x \in X$$

is B-pseudomonotone.

Additionally, we recall the definition of the Moreau envelope for convex functions, see, for example, Khanh-Nguyen [18, Definition 2.1].

Definition 2.7. Let X be a Banach space, $\varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function, and $\lambda>0$. The Moreau envelope of φ of parameter λ is defined by

$$\varphi_\lambda(x) = \inf_{y\in X}\left(\varphi(y) + \frac{1}{2\lambda}\|x-y\|^2\right) \quad \text{for all } x \in X.$$

The next result can be found in the book by Hu-Papageorgiou [16, p. 350].

Proposition 2.8. *Let* X *and* X^* *be both locally uniformly convex, and* $\varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ *be a proper, convex and lower semicontinuous function, and* φ_λ *be the Moreau envelope of* φ *with respect to the parameter* $\lambda>0$*. Then, the following statements hold:*

(i) φ_λ *is convex and continuous on* X*, and its effective domain* $\operatorname{dom}(\varphi_\lambda)$ *is the whole space* X*;*
(ii) *For each* $x \in X$*, there exists a unique point* $\tilde{x} \in X$ *such that*
$$\varphi_\lambda(x) = \varphi(\tilde{x}) + \frac{1}{2\lambda}\|x-\tilde{x}\|^2;$$
(iii) *For all* $x\in X$*, we have* $\varphi_\lambda(x) \le \varphi(x)$ *and*
$$\varphi_\lambda(x) \nearrow \varphi(x) \quad \text{as } \lambda \searrow 0;$$
(iv) *The differential operator* $\varphi_\lambda'\colon X \to X^*$ *of* φ_λ *is bounded, monotone and demicontinuous.*
(v) *If* $x_\lambda \rightharpoonup x$ *weakly in* X*, then we have*
$$\varphi(x) \le \liminf_{\lambda\to 0}\varphi_\lambda(x_\lambda).$$

Moreover, we need the following concept of Mosco convergence for the sequence of sets, see, for example Papageorgiou-Winkert [34].

Definition 2.9. Let X be a Banach space, and $S, S_k \subset X$ for all $k\in\mathbb{N}$. We say that $\{S_k\}_{k\in\mathbb{N}}$ converges to S in the Mosco sense, and write $S_k \xrightarrow{\text{M}} S$, if

(i) for every $y \in S$, there is a sequence $y_k \in S_k$ such that $y_k \to y$,
(ii) whenever $y_k \in S_k$ for all k and y is a weak limit point of $\{y_k\}_{k\in\mathbb{N}}$, then $y\in S$.

The following lemma gives several useful properties for the normalized duality mapping, see, for instance, Aliprantis-Border [1] and Cioranescu [9].

Lemma 2.10. *Let X be a normed space and $J\colon X \to 2^{X^*}$ be the normalized duality mapping defined by*

$$J(x) = \left\{x^* \in X^* \colon \langle x^*, x\rangle = \|x^*\|_{X^*}^2 = \|x\|^2\right\}.$$

Then, J has the following properties:

(i) *For every $x \in X$, $J(x)$ is nonempty, bounded, closed and convex.*
(ii) *J is norm-to-weak upper semicontinuous on X, when X is a reflexive Banach space.*
(iii) *If X is a locally uniformly convex space, then the dual mapping turns out to be single-valued, coercive, demicontinuous, and B-pseudomonotone.*

We end this section to recall the concept of Navier-Stokes type operators and point out several important properties of such operators.

Definition 2.11. Let V be a reflexive Banach space with its dual space V^*. An operator $N\colon V \to V^*$ is called a Navier-Stokes type operator if $Nv = Av + B[v]$ for all $v \in V$ such that

H(A): $A\colon V \to V^*$ is a linear, continuous, symmetric operator such that

$$\langle Av, v\rangle \geq \alpha\|v\|_V^2 \quad \text{for } v \in V$$

with some constant $\alpha > 0$ which is independent of $v \in V$;

H(B): $B[v] = B(v, v)$ with $B\colon V \times V \to V^*$ is a bilinear and continuous operator satisfying the following conditions:

(a) $\langle B(u, v), v\rangle = 0$ for $u, v \in V$,
(b) the map $B[\cdot]\colon V \to V^*$ is weakly continuous.

The following lemma indicates that a Navier-Stokes type operator is B-pseudomonotone, see Bianchi-Kassay-Pini [4].

Lemma 2.12. *Let $N\colon V \to V^*$ be a Navier-Stokes type operator. Then, N is coercive, bounded and B-pseudomonotone.*

3. Existence result

This section deals with the study of the existence of a solution to **Problem P** in which our main method is based on the Moreau-Yosida approximating technique and the following existence result for equilibrium problems in Hausdorff topological vector spaces, see Brezis-Nirenberg-Stampacchia [7, Theorem 1].

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a Hausdorff topological vector space E, and $f\colon C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

(i) *$f(x, x) \geq 0$ for all $x \in C$;*
(ii) *for every $x \in C$, the set $\{y \in C\colon f(x, y) < 0\}$ is convex;*
(iii) *for every $y \in C$, the function $f(\cdot, y)$ is upper semicontinuous on the intersection of C with any finite dimensional subspace Z of E;*
(iv) *whenever $x, y \in C, x_n \in C, x_n \to x$ and $f(x_n, (1-t)x + ty) \geq 0$ for all $t \in [0, 1]$ and for all n, then $f(x, y) \geq 0$;*
(v) *if C is unbounded, there exists a compact subset K of E, and $y_0 \in K \cap C$ such that $f(x, y_0) < 0$ for every $x \in C\backslash K$.*

Then, there exists $\overline{x} \in C \cap K$ such that

$$f(\overline{x}, y) \geq 0 \quad \textit{for all } y \in C.$$

Before we can state our main existence theorem, we need the following Berge-type lemma, see Proposition 3.3 in Hu-Papageorgiou [16].

Lemma 3.2. *Let E_1, E_2 be Hausdorff topological spaces, $u\colon E_1 \times E_2 \to \mathbb{R}$ be an upper semicontinuous function, and $F\colon E_2 \to 2^{E_1}$ be an upper semicontinuous map with nonempty and compact values. Then, the value function $v\colon E_2 \to \overline{\mathbb{R}}$ given by $v(y) = sup_{x\in F(y)} u(x, y)$ is upper semicontinuous.*

For any fixed $\lambda > 0$, we consider the following regularization problem of **Problem P**:
Problem P$_\lambda$: Find $x_\lambda \in C$ such that

$$\sup_{x_\lambda^* \in F(x_\lambda)} \langle x_\lambda^*, y - x_\lambda \rangle + \varphi_\lambda(y) - \varphi_\lambda(x_\lambda) + j^\circ(\hat{x}_\lambda, \hat{y} - \hat{x}_\lambda) \geq 0 \quad \text{for all } y \in C,$$

where φ_λ is the Moreau envelope of φ with respect to the parameter $\lambda > 0$. In the sequel, we denote by S_λ the solution set of **Problem P$_\lambda$**.

In order to obtain the existence of solution of **Problem P**, we shall establish the solvability of **Problem P$_\lambda$**. Then, passing to the limit as $\lambda \to 0$, we are going to prove the existence of solutions of **Problem P**.

From Lemma 2.5, it is not difficult to see that a solution of the inequality

$$\sup_{x_\lambda^* \in F(x_\lambda) + T^* \partial j(T x_\lambda)} \langle x_\lambda^*, y - x_\lambda \rangle + \varphi_\lambda(y) - \varphi_\lambda(x_\lambda) \geq 0 \quad \text{for all } y \in C \tag{3.1}$$

is also a solution of **Problem P$_\lambda$**

Following this important fact, it is enough to show the solvability of **Problem P$_\lambda$** by proving the existence of solutions of problem (3.1).

The existence result for **Problem P$_\lambda$** is stated in the following theorem.

Theorem 3.3. *Assume that $\varphi\colon X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, and C is a nonempty closed convex set in a reflexive Banach space X. Suppose that*

(i) *$F(x)$ is bounded, closed and convex for every $x \in C$;*
(ii) *$F\colon X \to 2^{X^*}$ satisfies the property that for every finite dimensional subspace Z of X, for every $\{x_k\}_{k\in\mathbb{N}} \subset C \cap Z, x_k \to x$, and $x_k^* \in F(x_k)$, there is a subsequence $\{x_{k_n}^*\}_{n\in\mathbb{N}}$ converging in the weak topology to some point in $F(x)$;*
(iii) *$F\colon X \to 2^{X^*}$ is B-pseudomonotone on C;*
(iv) *There exists $y_0 \in C$ such that*
$$\lim_{\|x\|\to\infty} \frac{\inf_{x^* \in F(x) + T^*\partial j(T(x))} \langle x^*, x - y_0 \rangle}{\|x\|} = +\infty;$$
(iv) *$j\colon V \to \mathbb{R}$ is a locally Lipschitz function such that (2.1) holds with some with $c > 0$;*
(v) *$T\colon X \to V$ is a linear and compact operator.*

Then, **Problem P$_\lambda$** *admits at least one solution. Moreover, for each solution $x_\lambda \in C$, there exists $\overline{x}_\lambda^* \in F(x_\lambda) + T^*\partial j(T x_\lambda)$ such that*

$$\langle \overline{x}_\lambda^*, y - x_\lambda \rangle + \langle \nabla \varphi_\lambda(x_\lambda), y - x_\lambda \rangle \geq 0 \quad \textit{for all } y \in C. \tag{3.2}$$

Proof. By the definition of the convex subgradient, we have

$$\langle \nabla \varphi_\lambda(x), y - x \rangle \leq \varphi_\lambda(y) - \varphi_\lambda(x) \quad \text{for all } y \in X.$$

We observe that a solution of the inequality

$$\sup_{x^* \in F(x) + T^*\partial j(Tx)} \langle x^*, y - x \rangle + \langle \nabla \varphi_\lambda(x), y - x \rangle \geq 0 \quad \text{for all } y \in X \tag{3.3}$$

is also a solution of problem (3.1). Therefore, we should focus our attention on proving the existence of solutions to the problem (3.3) and so we are going to show that all assumptions of Theorem 3.1 are satisfied for problem (3.3).

To this end, we set $G_F\colon X \times X \to \mathbb{R}$ defined by

$$G_F(x, y) = \sup_{x^* \in F(x) + T^*\partial j(Tx)} \langle x^*, y - x \rangle + \langle \nabla \varphi_\lambda(x), y - x \rangle.$$

From the definition of G_F, it is obvious that G_F satisfies conditions (i) and (ii) of Theorem 3.1. Let Z be a finite dimensional space of X. The compactness and continuity of T together with Lemma 2.6 and hypothesis (v) imply that $T^* \circ \partial j \circ T$ is u.s.c. with compact values. But conditions (i) and (ii) indicate that F has weakly* compact values and F is u.s.c. on $C \cap Z$. Then, we can conclude that $F + T^* \circ \partial j \circ T$ is upper semicontinuous on $C \cap Z$ with nonempty and weakly* compact values in X^*.

Additionally, the function $X^* \times X \ni (x^*, x) \mapsto \langle x^*, y - x\rangle$ is continuous for any $y \in X$. We can apply Lemma 3.2 to conclude that

$$x \mapsto \sup_{x^* \in F(x) + T^* \partial j(Tx)} \langle x^*, y - x\rangle$$

is upper semicontinuous on $C \cap Z$ for all $y \in X$. The latter and the continuity of $x \mapsto \nabla \varphi_\lambda(x)$ imply that $G_F(\cdot, y)$ is upper semicontinuous on $C \cap Z$ for all $y \in X$. This indicates that (iii) of Theorem 3.1 is fulfilled.

Next, we shall show that (iv) of Theorem 3.1 is satisfied as well. Let $x, y \in C$, $x_n \in C$, $x_n \to x$ and $G_F(x_n, (1-t)x + ty) \geq 0$ for all $t \in [0,1]$, that is

$$\inf_{t\in[0,1]} \Bigg(\sup_{x_n^* \in F(x_n) + T^* \partial j(T(x_n))} \langle x_n^*, (1-t)x + ty - x_n\rangle \\ + \langle \nabla \varphi_\lambda(x_n), (1-t)x + ty - x_n\rangle \Bigg) \geq 0.$$

The weak compactness of $F(x_n) + T^* \partial j(Tx_n)$ (see assumptions (i) and (v)) reveals that

$$\sup_{x_n^* \in F(x_n) + T^* \partial j(T(x_n))} \langle x_n^*, (1-t)x + ty - x_n\rangle = \max_{x_n^* \in F(x_n) + T^* \partial j(T(x_n))} \langle x_n^*, (1-t)x + ty - x_n\rangle.$$

Let

$$P(t, x_n^*) = \langle x_n^*, (1-t)x + ty - x_n\rangle + \langle \nabla \varphi_\lambda(x_n), (1-t)x + ty - x_n\rangle.$$

Clearly, $P(\cdot, x_n^*)$ and $P(t, \cdot)$ are both linear and continuous. Therefore, one could use Sion's Minimax Theorem to obtain

$$\begin{aligned} &\inf_{t\in[0,1]} \Bigg(\max_{x_n^* \in F(x_n) + T^* \partial j(T(x_n))} \langle x_n^*, (1-t)x + ty - x_n\rangle + \langle \nabla \varphi_\lambda(x_n), (1-t)x + ty - x_n\rangle \Bigg) \\ &= \max_{x_n^* \in F(x_n) + T^* \partial j(T(x_n))} \Bigg(\inf_{t\in[0,1]} \langle x_n^*, (1-t)x + ty - x_n\rangle + \langle \nabla \varphi_\lambda(x_n), (1-t)x + ty - x_n\rangle \Bigg) \geq 0. \end{aligned}$$

Then, there exists $\overline{x}_n^* \in F(x_n) + T^* \partial j(Tx_n)$ such that for all $t \in [0,1]$ it holds

$$\langle \overline{x}_n^*, (1-t)x + ty - x_n\rangle + \langle \nabla \varphi_\lambda(x_n), (1-t)x + ty - x_n\rangle \geq 0. \tag{3.4}$$

In particular, for $t = 0$, we get $\langle \overline{x}_n^*, x - x_n\rangle + \langle \nabla \varphi_\lambda(x_n), x - x_n\rangle \geq 0$ and thus

$$\limsup_{n\to\infty} \left(\langle \overline{x}_n^*, x_n - x\rangle + \langle \nabla \varphi_\lambda(x_n), x_n - x\rangle\right) \leq 0.$$

Whereas, by the boundedness of $\nabla \varphi_\lambda$ and $x_n \to x$, one gets

$$\limsup_{n\to\infty} \langle \overline{x}_n^*, x_n - x\rangle \leq 0.$$

From Proposition 2.3 and Lemma 2.6, we observe that $F + T^* \circ \partial j \circ T$ is B-pseudomonotone. Then, for every $y \in C$, there exists $x^*(y) \in F(x) + T^* \partial j(T(x))$ such that

$$\langle x^*(y), y - x\rangle \geq \limsup_{n\to\infty} \langle \overline{x}_n^*, y - x_n\rangle.$$

Hence,

$$\begin{aligned} \langle x^*(y), y - x\rangle + \langle \nabla \varphi_\lambda(x), y - x\rangle &\geq \limsup_{n\to\infty} \langle \overline{x}_n^*, y - x_n\rangle + \limsup_{n\to\infty} \langle \nabla \varphi_\lambda(x_n), y - x_n\rangle \\ &\geq \limsup_{n\to\infty} \left(\langle \overline{x}_n^*, y - x_n\rangle + \langle \nabla \varphi_\lambda(x_n), y - x_n\rangle\right) \geq 0, \end{aligned}$$

where the last inequality follows from (3.4) for $t = 1$. This means that $G_F(x, y) \geq 0$, thus, (iv) of Theorem 3.1 is also fulfilled.

It remains to check condition (v) of Theorem 3.1. It follows from assumption (iv) that there exists a function $h \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $h(r) \to +\infty$ as $r \to +\infty$ such that

$$\inf_{x^* \in F(x) + T^* \partial j(T(x))} \langle x^*, x - y_0\rangle \geq h(\|x - y_0\|)\|x - y_0\|.$$

Recall that φ_λ is convex, so, for each fixed $\lambda > 0$, there exist $\alpha_\lambda, \beta_\lambda \in \mathbb{R}$ satisfying $\varphi_\lambda(x) \geq \alpha_\lambda + \beta_\lambda \|x\|$. This leads to

$$\begin{aligned} G_F(x, y_0) &= \sup_{x^* \in F(x) + T^*\partial j(Tx)} \left(\langle x^*, y_0 - x\rangle + \langle \nabla \varphi_\lambda(x), y_0 - x\rangle \right) \\ &\leq \sup_{x^* \in F(x) + T^*\partial j(T(x))} \left(\langle x^*, y_0 - x\rangle + \varphi_\lambda(y_0) - \varphi_\lambda(x) \right) \\ &\leq -h(\|x - y_0\|)\|x - y_0\| + \varphi_\lambda(y_0) - \varphi_\lambda(x) \\ &\leq -h(\|x - y_0\|)\|x - y_0\| + \varphi_\lambda(y_0) - (\alpha_\lambda + \beta_\lambda\|x\|) \\ &\leq -h(\|x - y_0\|)\|x - y_0\| + |\beta_\lambda|\|x\| + \varphi_\lambda(y_0) - \alpha_\lambda. \end{aligned}$$

Then, we have

$$\limsup_{\|x\| \to +\infty} \frac{G_F(x, y_0)}{\|x - y_0\|} \leq \limsup_{\|x\| \to +\infty} \left(\frac{|\beta_\lambda|\|x\|}{\|x - y_0\|} - h(\|x - y_0\|) + \frac{\varphi_\lambda(y_0) - \alpha_\lambda}{\|x - y_0\|} \right) = -\infty.$$

We infer that there exists $M > \|y_0\|$ such that

$$G_F(x, y_0) < 0 \quad \text{for all } x \in C \text{ with } \|x\| > M.$$

Therefore, $G_F(x, y_0) < 0$ for every $x \in C \setminus K$, where $K := \overline{B}_X(0, M)$ is a weakly compact set with $y_0 \in K$. Applying Theorem 3.1, we find a solution $x \in C$ satisfying $G_F(x, y) \geq 0$ for all $y \in C$. Moreover, it is also a solution of 3.1, so a solution of **Problem** $\mathbf{P}_\lambda$.

Let x_λ be a solution of **Problem** $\mathbf{P}_\lambda$. By Lemma 2.5, we have

$$\inf_{y \in C} \left(\sup_{x_\lambda^* \in F(x_\lambda) + T^*\partial j(Tx_\lambda)} \langle x_\lambda^*, y - x_\lambda\rangle + \langle \nabla \varphi_\lambda(x_\lambda), y - x_\lambda\rangle \right) \geq 0.$$

Using the weak compactness of $F(x_\lambda) + T^*\partial j(Tx_\lambda)$ leads to

$$\inf_{y \in C} \left(\max_{x_\lambda^* \in F(x_\lambda) + T^*\partial j(Tx_\lambda)} \langle x_\lambda^*, y - x_\lambda\rangle + \langle \nabla \varphi_\lambda(x_\lambda), y - x_\lambda\rangle \right) \geq 0.$$

Then, applying again Sion's Minimax Theorem yields

$$\begin{aligned} &\inf_{y \in C} \max_{x_\lambda^* \in F(x_\lambda) + T^*\partial j(Tx_\lambda)} \left(\langle x_\lambda^*, y - x_\lambda\rangle + \langle \nabla \varphi_\lambda(x_\lambda), y - x_\lambda\rangle \right) \\ &= \max_{x_\lambda^* \in F(x_\lambda) + T^*\partial j(Tx_\lambda)} \inf_{y \in C} \left(\langle x_\lambda^*, y - x_\lambda\rangle + \langle \nabla \varphi_\lambda(x_\lambda), y - x_\lambda\rangle \right) \geq 0. \end{aligned}$$

Thus, we could find $\overline{x}_\lambda^* \in F(x_\lambda) + T^*\partial j(Tx_\lambda)$ such that

$$\langle \overline{x}_\lambda^*, y - x_\lambda\rangle + \langle \nabla \varphi_\lambda(x_\lambda), y - x_\lambda\rangle \geq 0 \quad \text{for all } y \in C.$$

This completes the proof of the theorem. □

Remark 3.4. Normally, an element x_λ solving inequality **Problem** $\mathbf{P}_\lambda$ is called to be a weak solution. But, if there exists $\overline{x}_\lambda^* \in F(x_\lambda) + T^*\partial j(Tx_\lambda)$ such that inequality (3.2) is satisfied, then x_λ is called to be a strong solution. From the proof of Theorem 3.3, we can see that if the inequality satisfies the framework of Sion's Minimax Theorem, then the weak solutions coincide with the strong solutions.

Proposition 3.5. *Suppose that all assumptions of Theorem 3.3 are satisfied and $C \subseteq \operatorname{dom}(\varphi)$. Then, the solution sequence $\{x_\lambda\}$ is bounded, where x_λ is a solution of* **Problem** $\mathbf{P}_\lambda$ *for $\lambda > 0$.*

Proof. Suppose by contradiction that there exists an unbounded subsequence of $\{x_\lambda\}$. Without any loss of generality, we may suppose that $\|x_\lambda\| \to +\infty$ as $\lambda \to 0$. From the proof of Theorem 3.3, it is easy to see that $x_\lambda \in C$ is a solution of (3.1), namely,

$$\sup_{x_\lambda^* \in F(x_\lambda) + T^*\partial j(Tx_\lambda)} \langle x_\lambda^*, y - x_\lambda\rangle + \varphi_\lambda(y) - \varphi_\lambda(x_\lambda) \geq 0 \quad \text{for all } y \in C.$$

Taking $y = y_0$ gives

$$\sup_{x_\lambda^* \in F(x_\lambda)+T^*\partial j(Tx_\lambda)} \langle x_\lambda^*, y_0 - x_\lambda\rangle + \varphi_\lambda(y_0) - \varphi_\lambda(x_\lambda) \geq 0.$$

Let us fixed $\lambda_0 > 0$. Since $y_0 \in \mathrm{dom}(\varphi)$, we can use Proposition 2.8 to infer that for any $\lambda_0 > \lambda$ it holds

$$\begin{aligned}\inf_{x_\lambda^* \in F(x_\lambda)+T^*\partial j(Tx_\lambda)} \langle x_\lambda^*, x_\lambda - y_0\rangle &\leq \varphi_\lambda(y_0) - \varphi_\lambda(x_\lambda)\\ &\leq \varphi(y_0) - \varphi_{\lambda_0}(x_\lambda)\\ &\leq \varphi(y_0) - (\alpha_{\lambda_0} + \beta_{\lambda_0}\|x_\lambda\|),\end{aligned}$$

where $\alpha_{\lambda_0}, \beta_{\lambda_0}$ are two constants which only depend on λ_0. Assumption (iv) reveals that

$$\begin{aligned}\varphi(y_0) - (\alpha_{\lambda_0} + \beta_{\lambda_0}\|x_\lambda\|) &\geq \inf_{x_\lambda^* \in F(x_\lambda)+T^*\partial j(Tx_\lambda)} \langle x_\lambda^*, x_\lambda - y_0\rangle\\ &\geq h(\|x_\lambda - y_0\|)\|x_\lambda - y_0\|.\end{aligned}$$

Dividing both sides of the above inequality by $\|x_\lambda - y_0\|$ simultaneously, and then passing to the limit as $\lambda \to 0$, we get a contradiction. Thus, the solution sequence $\{x_\lambda\}$ is uniformly bounded. □

We are now able to use Theorem 3.3 and Proposition 3.5 to obtain the following existence result for **Problem P**.

Theorem 3.6. *Suppose that all assumptions of Theorem 3.5 are fulfilled, then every weak cluster point of $\{x_\lambda\}$ is a weak solution of* **Problem P** *which is a strong solution of* **Problem P** *as well.*

Proof. From Proposition 3.5, we know that $\{x_\lambda\}$ is bounded. We may assume that $\overline{x} \in C$ is a weak cluster point of $\{x_\lambda\}$, thus, there exists a subsequence, not relabeled, such that $x_\lambda \rightharpoonup \overline{x} \in C$. Keeping in mind that x_λ is a strong solution of (3.1) (see Theorem 3.3 and Remark 3.4), there exists $x_\lambda^* \in F(x_\lambda) + T^*\partial j(Tx_\lambda)$ such that

$$\langle x_\lambda^*, y - x_\lambda\rangle + \varphi_\lambda(y) - \varphi_\lambda(x_\lambda) \geq 0 \quad \text{for all } y \in C. \tag{3.5}$$

Putting $y = \overline{x}$ in (3.5), one has

$$\langle x_\lambda^*, \overline{x} - x_\lambda\rangle + \varphi_\lambda(\overline{x}) - \varphi_\lambda(x_\lambda) \geq 0.$$

Hence

$$\begin{aligned}\limsup_{\lambda\to 0}\langle x_\lambda^*, x_\lambda - \overline{x}\rangle &\leq \limsup_{\lambda\to 0}\varphi_\lambda(\overline{x}) - \liminf_{\lambda\to 0}\varphi_\lambda(x_\lambda)\\ &\leq \varphi(\overline{x}) - \varphi(\overline{x}) = 0.\end{aligned}$$

Recall that $F + T^*\partial jT$ is B-pseudomonotone, so, there exists $x^*(y) \in F(\overline{x}) + T^*\partial j(T\overline{x})$ such that $\langle x^*(y), \overline{x} - y\rangle \leq \liminf_{\lambda\to 0}\langle x_\lambda^*, x_\lambda - y\rangle$. We take the lim sup as $\lambda \to 0$ for (3.5), then

$$0 \leq \limsup_{\lambda\to 0}\left(\langle x_\lambda^*, y - x_\lambda\rangle + \varphi_\lambda(y) - \varphi_\lambda(x_\lambda)\right) \leq \langle x^*(y), y - \overline{x}\rangle + \varphi(y) - \varphi(\overline{x}).$$

We get

$$\sup_{x^* \in F(\overline{x})+T^*\partial j(T\bar{x})} \langle x^*, y - \overline{x}\rangle + \varphi(y) - \varphi(\overline{x}) \geq 0 \quad \text{for all } y \in C. \tag{3.6}$$

Therefore,

$$\sup_{x^* \in F(\bar{x})} \langle x^*, y - \overline{x}\rangle + \varphi(y) - \varphi(\overline{x}) + j^\circ(\hat{x}, \hat{y} - \hat{x}) \geq 0 \quad \text{for all } y \in C.$$

This means that $\overline{x}$ is a weak solution of **Problem P**, namely, every weak cluster point of $\{x_\lambda\}$ is a weak solution of **Problem P**.

Finally, we shall prove that $\overline{x}$ is also a strong solution of **Problem P**. From (3.6), we have that

$$\inf_{y\in C}\left(\sup_{x^* \in F(\bar{x})+T^*\partial j(T\bar{x})} \langle x^*, y - \overline{x}\rangle + \varphi(y) - \varphi(\overline{x})\right) \geq 0.$$

Since $F(x)+T^*\partial j(Tx)$ is weak compact, it leads to

$$\inf_{y\in C}\left(\max_{x^*\in F(\bar{x})+T^*\partial j(T\bar{x})}\langle x^*, y-\overline{x}\rangle+\varphi(y)-\varphi(\overline{x})\right)\geq 0.$$

Using again Sion's Minimax Theorem yields

$$\begin{aligned}&\inf_{y\in C}\big(\max_{x^*\in F(\bar{x})+T^*\partial j(T\bar{x})}\langle x^*, y-\overline{x}\rangle+\varphi(y)-\varphi(\overline{x})\big)\\ &=\max_{x^*\in F(\bar{x})+T^*\partial j(T\bar{x})}\inf_{y\in C}\big(\langle x^*, y-\overline{x}\rangle+\varphi(y)-\varphi(\overline{x})\big)\geq 0.\end{aligned}$$

Thus, we can find $\overline{x}^*\in F(\overline{x})$ and $\xi\in T^*\partial j(T\overline{x})$ such that

$$\langle \overline{x}^*+T^*\xi, y-\overline{x}\rangle+\varphi(y)-\varphi(\overline{x})\geq 0\quad\text{for all } y\in C.$$

By the definition of Clarke's subgradient, we obtain

$$\langle \overline{x}^*, y-\overline{x}\rangle+j^\circ(T\overline{x}, Ty-T\overline{x})+\varphi(y)-\varphi(\overline{x})\geq 0\quad\text{for all } y\in C.$$

We finally conclude that $\overline{x}$ is a strong solution of **Problem P**. □

Remark 3.7. Note that **Problem P** is given in a very general form and it includes several interesting special cases which we want to state here:

(i) If $\varphi\equiv 0$, the **Problem P** reduces to the following problem:

$$\sup_{x^*\in F(x)}\langle x^*, y-x\rangle+j^\circ(\hat{x},\hat{y}-\hat{x})\geq 0\quad\text{for all } y\in C.$$

Applying Theorem 3.1, we can conclude that the problem above admits at least one solution and the weak solutions are equivalent to the strong solutions.

(ii) If $j\equiv 0$, **Problem P** and **Problem P**$_\lambda$ reduce to **Problem Q** and **Problem Q**$_\lambda$, respectively:

Problem Q $$\sup_{x^*\in F(x)}\langle x^*, y-x\rangle+\varphi(y)-\varphi(x)\geq 0\quad\text{for all } y\in C,$$

Problem Q$_\lambda$ $$\sup_{x^*\in F(x)}\langle x^*, y-x\rangle+\varphi_\lambda(y)-\varphi_\lambda(x)\geq 0\quad\text{for all } y\in C.$$

Applying Theorem 3.6, we see that cluster point of the solutions sequence of **Problem Q**$_\lambda$ is a strong solution of **Problem Q** which is also a weak solution of **Problem Q**.

(iii) If $j=\varphi\equiv 0$, the **Problem P** reduces to the following problem:

$$\sup_{x^*\in F(x)}\langle x^*, y-x\rangle\geq 0\quad\text{for all } y\in C.$$

Theorem 3.1 implies that the above problem admits at least one solution and the weak solutions are equivalent to the strong solutions. In fact, such problem has been studied by Bianchi-Kassay-Pini in [4].

4. Stability analysis

In this section, we are going to study the stability of **Problem P**. To this end, let us assume that F_k and C_k are the perturbed functions and perturbed constraint sets of F and C, respectively, and J is the normalized duality mapping. The main goal of this section is to prove the solvability of the following perturbed **Problem P**$_{\lambda,k}$, and further discuss the relationship between the solution sets of the perturbed **Problem P**$_{\lambda,k}$ and the original **Problem P**, where the perturbed **Problem P**$_{\lambda,k}$ is given as follows:

Problem P$_{\lambda,k}$: Find $x_{\lambda,k}\in C_k$ such that

$$\begin{aligned}\sup_{x^*_{\lambda,k}\in F_k(x_{\lambda,k})}&\langle x^*_{\lambda,k}, y-x_{\lambda,k}\rangle+\varphi_\lambda(y)-\varphi_\lambda(x_{\lambda,k})+\alpha_k\langle J(x_{\lambda,k}), y-x_{\lambda,k}\rangle\\ &+j^\circ(\hat{x}_{\lambda,k},\hat{y}-\hat{x}_{\lambda,k})\geq 0\quad\text{for all } y\in C_k.\end{aligned}$$

Here, $\alpha_k>0$ is a given regularization parameter which tends to 0 as $k\to\infty$. In the sequel, we denote by $S_{\lambda,k}$ the solution set of **Problem P**$_{\lambda,k}$.

From Lemma 2.5, it is not difficult to see that a solution of the inequality

$$\begin{aligned}\sup_{x^*_{\lambda,k}\in F_k(x_{\lambda,k})+T^*\partial j(Tx_{\lambda,k})} &\langle x^*_{\lambda,k}, y-x_{\lambda,k}\rangle+\varphi_\lambda(y)-\varphi_\lambda(x_{\lambda,k})\\ &+\alpha_k\langle J(x_{\lambda,k}), y-x_{\lambda,k}\rangle\geq 0\quad \text{for all } y\in C_k.\end{aligned}\tag{4.1}$$

is also a solution of **Problem** $\mathbf{P}_{\lambda,k}$. Following this observation, we could prove the solvability of **Problem** $\mathbf{P}_{\lambda,k}$ by proving the existence of solutions of problem (4.1).

The main result in this section is the following theorem.

Theorem 4.1. *Let X be a reflexive and separable Banach space, $C\subseteq X$ be a nonempty, closed and convex set, and $F\colon X\to 2^{X^*}$ be an operator such that $C\subseteq \operatorname{dom}(F)$. Suppose that the sets $\{C_k\}$ and the multivalued functions $\{F_k\}$ satisfy the following conditions:*

A *for every k, C_k is nonempty, closed and convex, and $\{C_k\}$ converges to C in the sense of Mosco;*

B (i) *for each $k\in\mathbb{N}$, $C_k\subseteq \operatorname{dom}(F_k)$, and $F_k(x)$ is bounded, closed and convex for every $x\in\operatorname{dom}(F_k)$;*

(ii) *for every finite dimensional subspace Z of X, for every $\{x_n\}_{n\in\mathbb{N}}\subset C_k\cap Z, x_n\to x$, and $x^*_n\in F_k(x_n)$, there is a subsequence $\{x^*_{n_j}\}$ converging in the weak topology to some point in $F_k(x)$;*

(iii) *F_k is B-pseudomonotone on C_k;*

(iv) *there exists $y_0\in\cap_{k\in\mathbb{N}}C_k$ such that for each $k\in\mathbb{N}$*

$$\lim_{\|x\|\to+\infty}\frac{\inf_{x^*\in F_k(x)+T^*\partial j(Tx)}\langle x^*, x-y_0\rangle}{\|x\|}=+\infty;$$

C (i) *$C\cup_k C_k\subset\operatorname{dom}(F)$, F is bounded on $\cup_k C_k$ and has nonempty, bounded, closed and convex values;*

(ii) *F is B-pseudomonotone on $C\cup_k C_k$;*

(iii) *$\operatorname{Haus}(F_k(x),F(x))\leq\frac{\beta_k}{\|x\|+1}$, for every $x\in\operatorname{dom}(F_k)\cap\operatorname{dom}(F)$, where $\beta_k>0$ is such that $\beta_k\to 0$ as $k\to\infty$ and* Haus *stands for the Hausdorff metric in X;*

(iv) *there exists $y_1\in\cap_k C_k\neq\emptyset$ (it is nonempty by virtue of* **B** *(iv)) such that*

$$\lim_{\|x\|\to+\infty}\frac{\inf_{x^*\in F(x)+T^*\partial j(Tx)}\langle x^*, x-y_1\rangle}{\|x\|}=+\infty;$$

(v) *$j\colon V\to\mathbb{R}$ is a locally Lipschitz function such that (2.1) holds with some with $c>0$;*

(vi) *$T\colon X\to V$ is a linear and compact operator.*

Then, **Problem** $\mathbf{P}_{\lambda,k}$ *admits at least one solution $x_{\lambda,k}$, and the weak solutions are the strong solutions of* **Problem** $\mathbf{P}_{\lambda,k}$*. Moreover, there exists a subsequence of $\{x_{\lambda,k}\}$ which converges weakly to a weak solution of* **Problem P** *which is also a strong solution of* **Problem P** *as $k\to\infty$ and $\lambda\to 0$.*

Proof. The proof of this theorem is divided into several steps.

Step 1. The solution set of **Problem** $\mathbf{P}_{\lambda,k}$ is nonempty, and all weak solutions are strong solutions.

The main idea is to apply Theorem 3.3. So, we will show that all conditions of Theorem 3.3 are satisfied. Set $G_{\mathcal{F}_k}\colon X\times X\to\mathbb{R}$ defined by

$$G_{\mathcal{F}_k}(x,y)=\sup_{x^*\in F_k(x)+T^*\partial j(Tx)}\langle x^*, y-x\rangle+\varphi_\lambda(y)-\varphi_\lambda(x)+\alpha_k\langle J(x), y-x\rangle.$$

Recall that J is continuous, bounded and B-pseudomonotone (see Lemma 2.10), so it is easy to see that for any $\alpha_k>0$ the multivalued mapping $\mathcal{F}_k=F_k+\alpha_k J$ satisfies the assumptions of Theorem 3.3 (see the proof of Theorem 3.6). By assumption **B**(iv), there exists a function $h\colon\mathbb{R}^+\to\mathbb{R}^+$ with $h_k(r)\to+\infty$ as $r\to+\infty$ such that

$$\inf_{x^*\in F_k(x)+T^*\partial j(T(x))}\langle x^*, x-y_0\rangle\geq h_k(\|x-y_0\|)\|x-y_0\|.$$

Hence,

$$\begin{aligned}G_{\mathcal{F}_k}(x,y_0) &= \sup_{x^*\in F_k(x)+T^*\partial j(Tx)} \langle x^*, y_0-x\rangle + \varphi_\lambda(y_0)-\varphi_\lambda(x)+\alpha_k\langle J(x),y_0-x\rangle\\ &\le -h_k(\|x-y_0\|)\|x-y_0\| + \varphi_\lambda(y_0)-\varphi_\lambda(x)+\alpha_k\langle J(x),y_0-x\rangle\\ &\le -h_k(\|x-y_0\|)\|x-y_0\| + \varphi_\lambda(y_0)-(\alpha_\lambda+\beta_\lambda\|x\|)+\alpha_k\langle J(x),y_0-x\rangle\\ &\le -h_k(\|x-y_0\|)\|x-y_0\| + |\beta_\lambda|\|x\| + (\varphi_\lambda(y_0)-\alpha_\lambda)+\alpha_k\langle J(x),y_0-x\rangle.\end{aligned}$$

We infer that

$$\begin{aligned}\limsup_{\|x\|\to+\infty}\frac{G_{\mathcal{F}_k}(x,y_0)}{\|x-y_0\|} &\le \limsup_{\|x\|\to+\infty}\left(\frac{|\beta_\lambda|\|x\|}{\|x-y_0\|}-h_k(\|x-y_0\|)\right)\\ &\quad+\limsup_{\|x\|\to+\infty}\left(\frac{\varphi_\lambda(y_0)-\alpha_\lambda}{\|x-y_0\|}+\frac{\alpha_k\langle J(x),y_0-x\rangle}{\|x-y_0\|}\right)\\ &= -\infty.\end{aligned}$$

This implies that for each $k\in\mathbb{N}$ there exists $M_k>\|y_0\|$, such that

$$G_{\mathcal{F}_k}(x,y_0)<0 \quad \text{for all}\, x\in C \text{ with } \|x\|>M.$$

Therefore, using Theorem 3.3, we see that the solution set of **Problem** $\mathbf{P}_{\lambda,k}$ is nonempty and all weak solutions are strong ones.

Step 2. $\{x_{\lambda,k}\}$ is uniformly bounded with respect to $\lambda>0$ and $k\in\mathbb{N}$.

Suppose by contradiction that there exists an unbounded subsequence of $\{x_{\lambda,k}\}$, not relabeled, such that $\|x_{\lambda,k}\|\to+\infty$ as $\lambda\to 0^+$ and $k\to\infty$. From assumption $\mathbf{C}$(iv), it follows that

$$\lim_{\|x_{\lambda,k}\|\to+\infty}\frac{\inf_{x^*_{\lambda,k}\in F(x_{\lambda,k})+T^*\partial j(Tx_{\lambda,k})}\langle x^*_{\lambda,k},x_{\lambda,k}-y_1\rangle}{\|x_{\lambda,k}\|}=+\infty.$$

Since $x_{\lambda,k}\in C_k$ is a solution of **Problem** $\mathbf{P}_{\lambda,k}$, which is also a strong solution, there exists $x^*_{\lambda,k}\in F_k(x_{\lambda,k})$ such that

$$\begin{aligned}&\langle x^*_{\lambda,k},y-x_{\lambda,k}\rangle+\varphi_\lambda(y)-\varphi_\lambda(x_{\lambda,k})+\alpha_k\langle J(x_{\lambda,k}),y-x_{\lambda,k}\rangle\\ &\qquad+j^\circ(\hat{x}_{\lambda,k},\hat{y}-\hat{x}_{\lambda,k})\ge 0\quad\text{for all } y\in C_k.\end{aligned}\tag{4.2}$$

Taking $y=y_1$ in inequality (4.2), we have

$$\langle x^*_{\lambda,k}+\alpha_kJ(x_{\lambda,k}),y_1-x_{\lambda,k}\rangle+\varphi_\lambda(y_1)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_1-\hat{x}_{\lambda,k})\ge 0.$$

By virtue of Lemma 3.76 of Aliprantis-Border [1], for any $x^*_{\lambda,k}\in F_k(x_{\lambda,k})$ and $\varepsilon>0$, there exists $\tilde{x}^*_{\lambda,k}\in F(x_{\lambda,k})$ such that

$$\|x^*_{\lambda,k}-\tilde{x}^*_{\lambda,k}\|<\mathrm{Haus}(F_k(x_{\lambda,k}),F(x_{\lambda,k}))+\varepsilon.$$

We take $\varepsilon=\frac{\beta_k}{\|x_{\lambda,k}-y_1\|}$ in the inequality above and use $\mathbf{C}$(iii), which leads to

$$\|x^*_{\lambda,k}-\tilde{x}^*_{\lambda,k}\|\le\frac{\beta_k}{\|x_{\lambda,k}\|+1}+\frac{\beta_k}{\|x_{\lambda,k}-y_1\|}.$$

Let $\lambda_0>0$ be fixed. For all $\lambda<\lambda_0$, we get the following estimates

$$\begin{aligned}\langle\tilde{x}^*_{\lambda,k},x_{\lambda,k}-y_1\rangle &\le \langle\tilde{x}^*_{\lambda,k},x_{\lambda,k}-y_1\rangle+\langle x^*_{\lambda,k}+\alpha_kJ(x_{\lambda,k}),y_1-x_{\lambda,k}\rangle\\ &\quad+\varphi_\lambda(y_1)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_1-\hat{x}_{\lambda,k})\\ &\le \langle\tilde{x}^*_{\lambda,k},x_{\lambda,k}-y_1\rangle+\langle x^*_{\lambda,k}+\alpha_kJ(x_{\lambda,k}),y_1-x_{\lambda,k}\rangle\\ &\quad+\varphi_\lambda(y_1)-\varphi_{\lambda_0}(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_1-\hat{x}_{\lambda,k})\\ &\le \langle x^*_{\lambda,k}-\tilde{x}^*_{\lambda,k},y_1-x_{\lambda,k}\rangle+\alpha_k\|x_{\lambda,k}\|(\|y_1\|-\|x_{\lambda,k}\|)\\ &\quad+\varphi_\lambda(y_1)-\alpha_{\lambda_0}+|\beta_{\lambda_0}|\|x_{\lambda,k}\|+\langle T^*\xi,y_1-x_{\lambda,k}\rangle\\ &\le\left(\frac{\beta_k}{\|x_{\lambda,k}\|+1}+\frac{\beta_k}{\|x_{\lambda,k}-y_1\|}\right)\|y_1-x_{\lambda,k}\|+\alpha_k\|x_{\lambda,k}\|(\|y_1\|-\|x_{\lambda,k}\|)\end{aligned}$$

$$
\begin{aligned}
&+\varphi_\lambda(y_1)-\alpha_{\lambda_0}+|\beta_{\lambda_0}|\|x_{\lambda,k}\|+\langle T^*\xi, y_1-x_{\lambda,k}\rangle\\
&\le \beta_k\left(\frac{\|y_1-x_{\lambda,k}\|}{\|x_{\lambda,k}\|+1}+1\right)+\|x_{\lambda,k}\|(\alpha_k\|y_1\|+|\beta_{\lambda_0}|-\alpha_k\|x_{\lambda,k}\|)\\
&+(\varphi_\lambda(y_1)-\alpha_{\lambda_0})+\langle T^*\xi, y_1-x_{\lambda,k}\rangle,
\end{aligned}
$$

where $\xi\in\partial j(Tx_{\lambda,k})$ is such that $\langle \xi, T(y_1-x_{\lambda,k})\rangle = j^\circ(\hat{x}_{\lambda,k},\hat{y}_1-\hat{x}_{\lambda,k})$ (see Lemma 2.5(ii)). Then, we obtain

$$
\begin{aligned}
\langle \tilde{x}^*_{\lambda,k}+T^*\xi, x_{\lambda,k}-y_1\rangle \le \beta_k\left(\frac{\|y_1-x_{\lambda,k}\|}{\|x_{\lambda,k}\|+1}+1\right)+\|x_{\lambda,k}\|(\alpha_k\|y_1\|\\
+|\beta_{\lambda_0}|-\alpha_k\|x_{\lambda,k}\|)+(\varphi_\lambda(y_1)-\alpha_{\lambda_0}).
\end{aligned}
$$

This implies

$$
+\infty=\lim_{\|x_{\lambda,k}\|\to+\infty}\frac{\langle \tilde{x}^*_{\lambda,k}+T^*\xi, x_{\lambda,k}-y_1\rangle}{\|x_{\lambda,k}\|}<\infty,
$$

so a contradiction.

Step 3. Each weak cluster point of $\{x_{\lambda,k}\}$ is a weak solution of **Problem P**.

Recall that $x_{\lambda,k}$ is a solution of **Problem** $\mathbf{P}_{\lambda,k}$ which is also a strong solution. So, we can find $x^*_{\lambda,k}\in S_k(x_{\lambda,k}):=F_k(x_{\lambda,k})+\alpha_k J(x_{\lambda,k})$ such that

$$
\langle x^*_{\lambda,k}, z-x_{\lambda,k}\rangle+\varphi_\lambda(z)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{z}-\hat{x}_{\lambda,k})\ge 0 \quad \text{for all } z\in C_k.
$$

Fix any $y\in C$. From assumption **A**, there exists $y_k\in C_k$ such that $y_k\to y$. Suppose that $x\in C$ is a weak cluster point of $\{x_{\lambda,k}\}$, that is, $x_{\lambda,k}\rightharpoonup x$ in X (due to boundedness of $\{x_{\lambda,k}\}$, so, there exists a subsequence, not relabeled, such that $x_{\lambda,k}\rightharpoonup x$ as $\lambda\to 0$ and $k\to\infty$). From the definition of S_k it follows that

$$
\mathrm{Haus}(S_k(x),F(x))\le \mathrm{Haus}(F_k(x),F(x))+\alpha_k\|x\| \quad \text{for all } x\in C_k \text{ and } k\in\mathbb{N}.
$$

Therefore, using condition **C**(iii), we obtain

$$
\mathrm{Haus}(S_k(x_{\lambda,k}),F(x_{\lambda,k}))\le(\alpha_k+\beta_k)\left(\frac{1}{\|x_{\lambda,k}\|+1}+\|x_{\lambda,k}\|\right).
$$

Taking $\tilde{x}^*_{\lambda,k}\in F(x_{\lambda,k})$ such that

$$
\|x^*_{\lambda,k}-\tilde{x}^*_{\lambda,k}\|\to 0, \tag{4.3}
$$

since $\tilde{x}^*_{\lambda,k}\in F(x_{\lambda,k})$ and $\{x_{\lambda,k}\}$ is bounded, it follows by $\mathbf{C}(i)$ that the sequence $\{\tilde{x}^*_{\lambda,k}\}$ is bounded in X^*. Keeping in mind,

$$
\begin{aligned}
0&\le\langle x^*_{\lambda,k}, y_k-x_{\lambda,k}\rangle+\varphi_\lambda(y_k)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_k-\hat{x}_{\lambda,k})\\
&=\langle x^*_{\lambda,k}-\tilde{x}^*_{\lambda,k}, y_k-x_{\lambda,k}\rangle+\langle \tilde{x}^*_{\lambda,k}, y_k-x_{\lambda,k}\rangle\\
&\quad+\varphi_\lambda(y_k)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_k-\hat{x}_{\lambda,k})\\
&=\langle \tilde{x}^*_{\lambda,k}, y-x_{\lambda,k}\rangle-\langle \tilde{x}^*_{\lambda,k}, y-y_k\rangle-\langle \tilde{x}^*_{\lambda,k}-x^*_{\lambda,k}, y_k-x_{\lambda,k}\rangle\\
&\quad+\varphi_\lambda(y_k)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_k-\hat{x}_{\lambda,k}),
\end{aligned}
$$

we get

$$
\begin{aligned}
\langle \tilde{x}^*_{\lambda,k}, y-x_{\lambda,k}\rangle+\varphi_\lambda(y_k)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_k-\hat{x}_{\lambda,k})\\
\ge\langle \tilde{x}^*_{\lambda,k}, y-y_k\rangle+\langle \tilde{x}^*_{\lambda,k}-x^*_{\lambda,n_k}, y_k-x_{\lambda,k}\rangle.
\end{aligned}
$$

Taking into account the boundedness of $\{\tilde{x}^*_{\lambda,k}\}$, the strong convergence of $y_k\to y$, (4.3) and the boundedness of $\{y_k-x_{\lambda,k}\}$, we deduce

$$
\liminf_{k\to\infty,\lambda\to 0}\left(\langle \tilde{x}^*_{\lambda,k}, y-x_{\lambda,k}\rangle+\varphi_\lambda(y_k)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_k-\hat{x}_{\lambda,k})\right)\ge 0.
$$

Applying the continuity of φ_λ (see Proposition 2.8), the compactness of T and the upper semicontinuity of $(u,v)\mapsto j^\circ(u,v)$, we get

$$\begin{aligned}\liminf_{k\to\infty,\lambda\to 0}\langle \tilde{x}^*_{\lambda,k}, y-x_{\lambda,k}\rangle &\geq -\limsup_{k\to\infty,\lambda\to 0}\left(\varphi_\lambda(y_k)-\varphi_\lambda(x_{\lambda,k})+j^\circ(\hat{x}_{\lambda,k},\hat{y}_k-\hat{x}_{\lambda,k})\right)\\ &\geq -\limsup_{k\to\infty,\lambda\to 0}\varphi_\lambda(y_k)+\liminf_{k\to\infty,\lambda\to 0}\varphi_\lambda(x_{\lambda,k})\\ &\quad -\limsup_{k\to\infty,\lambda\to 0} j^\circ(\hat{x}_{\lambda,k},\hat{y}_k-\hat{x}_{\lambda,k})\\ &\geq -\varphi(y)+\varphi(x)-j^\circ(\hat{x},\hat{y}-\hat{x}).\end{aligned} \tag{4.4}$$

Taking $y=x_\lambda$ in (4.4), one has

$$\liminf_{k\to\infty}\langle \tilde{x}^*_{\lambda,k}, x_\lambda-x_{\lambda,k}\rangle\geq 0.$$

Because F is B-pseudomonotone, for every $y\in C$, there exists $x^*(y)\in F(x)$ such that

$$\langle x^*(y), x-y\rangle \leq \liminf_{k\to\infty,\lambda\to 0}\langle \tilde{x}^*_{\lambda,k}, x_{\lambda,k}-y\rangle. \tag{4.5}$$

Note that $\{x_{\lambda,k}\}$ is a solution of $\textbf{Problem}\,\mathbf{P}_{\lambda,k}$. So this yields

$$\begin{aligned}\langle \tilde{x}^*_{\lambda,k}, x_{\lambda,k}-y\rangle &= \langle \tilde{x}^*_{\lambda,k}, x_{\lambda,k}-y_k\rangle+\langle \tilde{x}^*_{\lambda,k}, y_k-y\rangle\\ &=\langle \tilde{x}^*_{\lambda,k}-x^*_{\lambda,k}, x_{\lambda,k}-y_k\rangle+\langle x^*_{\lambda,k}, x_{\lambda,k}-y_k\rangle+\langle \tilde{x}^*_{\lambda,k}, y_k-y\rangle\\ &\leq \|\tilde{x}^*_{\lambda,k}-x^*_{\lambda,k}\|_{X^*}\|x_{\lambda,k}-y_k\|+\varphi_\lambda(y_k)-\varphi_\lambda(x_{\lambda,k})\\ &\quad + j^\circ(\hat{x}_{\lambda,k},\hat{y}_k-\hat{x}_{\lambda,k})+\langle \tilde{x}^*_{\lambda,k}, y_k-y\rangle.\end{aligned} \tag{4.6}$$

Taking the lim sup as $k\to\infty$ and $\lambda\to 0$ in (4.6), we obtain

$$\limsup_{k\to\infty,\lambda\to 0}\langle \tilde{x}^*_{\lambda,k}, x_{\lambda,k}-y\rangle \leq \varphi(y)-\varphi(x)+j^\circ(\hat{x},\hat{y}-\hat{x}),$$

where we have used Proposition 2.8(i), (4.3) and $\mathbf{C}$(vi). With help of (4.5), it results in

$$\langle x^*(y), x-y\rangle\leq \varphi(y)-\varphi(x)+j^\circ(\hat{x},\hat{y}-\hat{x})$$

for all $y\in C$. This means that x is a weak solution to **Problem P**. Employing the same arguments as in the proof of Theorem 3.6, we conclude that x is also a strong solution of **Problem P**. ☐

Remark 4.2. From the proof of Theorem 4.1, we can observe that for each $\lambda>0$ fixed it holds $w-\limsup_{k\to\infty} S_{\lambda,k}\subseteq S_\lambda$, where the set S_λ is the solution set of $\textbf{Problem}\,\mathbf{P}_\lambda$. In addition, it holds

$$w-\limsup_{\lambda\to 0} S_\lambda\subseteq S \quad\text{and}\quad w-\limsup_{\lambda\to 0}(w-\limsup_{k\to\infty} S_{\lambda,k})\subseteq w-\limsup_{\lambda\to 0} S_\lambda\subseteq S.$$

When the set $\bigcup_{\lambda>0,k\in\mathbb{N}} S_{\lambda,k}$ is bounded, where $S_{\lambda,k}$ is the solution set of $\textbf{Problem}\,\mathbf{P}_{\lambda,k}$, then hypothesis $\mathbf{C}$(iii) can be relaxed and we have the following corollary.

Corollary 4.3. *Under the assumptions of Theorem 4.1 without supposing condition* $\mathbf{C}$*(iii), if* $\bigcup_{\lambda>0,k\in\mathbb{N}} S_{\lambda,k}$ *is bounded and there exists a bounded function* $\tau\colon\mathbb{R}_+\to\mathbb{R}_+$ *such that for any* $x\in \operatorname{dom}(F_k)\cap\operatorname{dom}(F)$*, and for every* $x^*\in F(x)$ *there exists* $\tilde{x}^*\in F_k(x)$ *satisfying*

$$\|x^*-\tilde{x}^*\|\leq \beta_k\tau(\|x\|),$$

where $\beta_k>0$ *and* $\beta_k\to 0$*, as* $k\to\infty$*, then any sequence* $\{x_{\lambda,k}\}$ *with* $x_{\lambda,k}\in S_{\lambda,k}$ *has a subsequence that converges weakly to a weak solution of* **Problem P** *(which is also a strong solution* **Problem P** *).*

5. Application to a stationary incompressible fluid model

The goal of this section is to apply the obtained abstract results in Sections 3 and 4 to a stationary Navier-Stokes equation with nonmonotone and multivalued constitutive law on boundary. The main reason that we focus our attention to stationary and nonsmooth Navier-Stokes equations is the fact that the Navier-Stokes operator is pseudomonotone, but it is not monotone.

Before we give the classical formula of the considered fluid model, we describe its physical setting. Given a bounded domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ such that Ω has a Lipschitz continuous boundary $\Gamma = \partial\Omega$, we suppose that Γ is separated into four measurable disjoint parts Γ_0, Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_0) > 0$. In what follows, we use the symbol $\mathbb{S}^d$ for the space of second-order symmetric tensors, and the inner product and the norm of $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined, respectively, by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{u} = (u_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Moreover, the normal and tangential components of a vector $\boldsymbol{u}$ on Γ are given by $u_\nu = \boldsymbol{u}\cdot\boldsymbol{v}$ and $u_\tau = \boldsymbol{u} - u_\nu\nu$, where ν is the unit outward normal vector on the boundary Γ. Likewise, the normal and tangential components of a tensor $\boldsymbol{\sigma}$ on Γ are defined by $\sigma_\nu = \boldsymbol{\sigma}\nu\cdot\nu$ and $\sigma_\tau = \boldsymbol{\sigma}\nu - \sigma_\nu\nu$, respectively. The considered mathematical model of the stationary Navier-Stokes problem is stated as follows:

Problem Q. Find a velocity field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ and a pressure $p\colon \Omega \to \mathbb{R}$ such that

$$-\mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{5.1}$$

$$\text{div}\boldsymbol{u} = \boldsymbol{0} \qquad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_0, \tag{5.3}$$

$$\begin{cases} u_\nu = 0 \\ -\boldsymbol{\tau}_\tau(\boldsymbol{u}) \in \partial j(\boldsymbol{u}_\tau), \end{cases} \qquad \text{on } \Gamma_1, \tag{5.4}$$

$$\begin{cases} u_\nu + g \geq 0, \tau_\nu(\boldsymbol{u}, p) \geq 0 \\ (u_\nu + g)\tau_\nu(\boldsymbol{u}, p) = 0 \\ \boldsymbol{\tau}_\tau(\boldsymbol{u}) = 0 \end{cases} \qquad \text{on } \Gamma_2, \tag{5.5}$$

$$\begin{cases} -\tau_\nu(\boldsymbol{u}, p) \in \partial\psi(u_\nu) \\ \boldsymbol{u}_\tau = 0 \end{cases} \qquad \text{on } \Gamma_3, \tag{5.6}$$

where $\boldsymbol{\tau}(\boldsymbol{u}, p) := -p\mathbf{I} + 2\mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ is the full stress tensor.

We briefly discuss the equations and conditions in **Problem Q**, more details can be found in Migórski-Dudek [25]. The system describes a stationary flow of incompressible viscous liquid occupying the volume Ω acted to external volume force $\boldsymbol{f}$. Equation (5.1) is the conservation law, where $\mu > 0$ is the kinematic viscosity coefficient, and the nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is called the convective term with

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \left(\sum_{j=1}^d u_j \frac{\partial u_i}{\partial x_j}\right)_{i=1}^d.$$

Equation (5.2) reveals that the fluid is incompressible, whereas equation (5.3) states that the fluid adheres on the boundary Γ_0. Moreover, equation (5.4) points out that there is no fluid across the boundary Γ_1, and a multivalued and nonmonotone friction law holds. Further, equation (5.5) represents a generalized Signorini-type contact condition with frictional effect on Γ_2, where $g \in L^2(\Gamma_2;\mathbb{R})$ is such that $g \geq 0$, see, for example, Saito-Sugitani-Zhou [35]. Finally, equation (5.6) indicates that the tangential component of the velocity field vanishes, and the fluid fulfills a monotone and multivalued boundary condition on Γ_3.

In order to obtain the weak formulation of **Problem Q**, let us consider the following function spaces

$$\widetilde{X} = \{\boldsymbol{v} \in C^\infty(\overline{\Omega}; \mathbb{R}^d) \colon \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} = 0 \text{ on } \Gamma_0,\ v_\nu = 0 \text{ on } \Gamma_1,\ \boldsymbol{v}_\tau = 0 \text{ on } \Gamma_3\},$$

$$X = \text{ closure of } \widetilde{X} \text{ in } H^1(\Omega; \mathbb{R}^d).$$

Then, we can prove that X endowed with the norm $X \ni \boldsymbol{u} \mapsto \|(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)\|_{L^2(\Omega;\mathbb{S}^d)}$ is a reflexive and separable space. In addition, the trace operator from X to $V := L^2(\Gamma; \mathbb{R}^d)$ is denoted by $T\colon X \to V$ which turns out to be compact. Moreover, we define the admissible set to the velocity fields by

$$C = \{\boldsymbol{v} \in X \colon v_\nu + g \geq 0 \text{ on } \Gamma_2\}.$$

We suppose the following assumptions on the data of **Problem Q**.

$\mathbf{H}(\psi)$ $\psi\colon \mathbb{R} \to \mathbb{R}$ is a convex and lower semicontinuous function.

$\mathbf{H}(j)$ $j\colon \Gamma_1 \times \mathbb{R}^d \to \mathbb{R}$ is a function such that

(i) $j(\cdot, \boldsymbol{r})$ is measurable on Γ_1 for all $\boldsymbol{r} \in \mathbb{R}^d$;

(ii) $j(\boldsymbol{x}, \cdot)$ is locally Lipschitz for a.e. $\boldsymbol{x} \in \Gamma_1$;

(iii) there exists $\overline{c} \geq 0$ such that

$$\|\partial j(\boldsymbol{x}, \boldsymbol{r})\|_{\mathbb{R}^d} \leq \overline{c}(1 + \|\boldsymbol{r}\|_{\mathbb{R}^d})$$

for a.e. $\boldsymbol{x} \in \Gamma_1$ and for all $\boldsymbol{r} \in \mathbb{R}^d$;

(iv) either $j(\boldsymbol{x}, \cdot)$ or $-j(\boldsymbol{x}, \cdot)$ is regular on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_1$;

$\mathbf{H}_0$ $\mu > \overline{c}\|T\|^2$.

Using a standard way (see for instance Migórski-Dudek [25]), we obtain the weak variational formulation of **Problem Q** as follows: Find a velocity field $\boldsymbol{u} \in C$ such that

$$\mu \int_\Omega \nabla(\boldsymbol{u}) : \nabla(\boldsymbol{v} - \boldsymbol{u}) \,\mathrm{d}x + \int_\Omega ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot (\boldsymbol{v} - \boldsymbol{u}) \,\mathrm{d}x$$
$$+ \int_{\Gamma_1} j^0(\boldsymbol{u}_\tau, \boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \,\mathrm{d}\Gamma + \int_{\Gamma_3} \big(\psi(v_\nu) - \psi(u_\nu)\big) \,\mathrm{d}\Gamma \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}) \,\mathrm{d}x$$

for all $\boldsymbol{v} \in C$.

The main existence theorem for **Problem Q** is stated as follows.

Theorem 5.1. *Assume that* $\mathbf{H}(\psi), \mathbf{H}(j)$ *and* $\mathbf{H}_0$ *hold, then* **Problem Q** *has a weak solution.*

Proof. First, we introduce the following operators:

$$F\colon X \to X^*, \quad \langle F\boldsymbol{u}, \boldsymbol{v}\rangle_{X^* \times X} = \langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{X^* \times X} + \langle B[\boldsymbol{u}], \boldsymbol{v}\rangle_{X^* \times X} \quad \text{for } u, v \in X,$$

$$A\colon X \to X^*, \quad \langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{X^* \times X} = \mu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \,\mathrm{d}x,$$

$$B[\cdot]\colon X \to X^*, \quad B[\boldsymbol{u}] := B(\boldsymbol{u}, \boldsymbol{u}), \quad \langle B(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{v}\rangle_{X^* \times X} = \int_\Omega ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}x,$$

$$\varphi\colon X \to \mathbb{R}, \quad \varphi(\boldsymbol{v}) = \int_{\Gamma_3} \psi(v_\nu) \,\mathrm{d}\Gamma,$$

$$\boldsymbol{f} \in X^*, \quad \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{X^* \times X} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x.$$

Therefore, $\boldsymbol{u} \in C$ is a weak solution of **Problem Q** if and only if, $\boldsymbol{u} \in C$ solves the inequality

$$\langle F\boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{X^* \times X} + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}) + j^\circ(T\boldsymbol{u}, T\boldsymbol{v} - T\boldsymbol{u}) \geq 0 \quad \text{for all } \boldsymbol{v} \in C. \tag{5.7}$$

We can easily prove that F is a Navier-Stoke type operator in the sense of Definition 2.11. So, from Lemma 2.12, we know that F is coercive, continuous and B-pseudomonotone. Moreover, we could apply the same arguments as in Migórski-Dudek [25] in order to see that all conditions of Theorem 3.3 are fulfilled. Using this theorem, we conclude that the inequality (5.7) is solvable, namely, **Problem Q** has a weak solution. $\square$

Furthermore, let us study the stability of **Problem Q**. For this purpose, we assume that $\{\mu_k\}$ and $\{g_k\} \subset L^2(\Gamma_2;\mathbb{R})$ are such that $\mu_k, \mu > 0$ and $\mu_k \to \mu$ as $k \to \infty$, and $g_k \geq 0$ and $g_k \to g$ in $L^2(\Gamma_2;\mathbb{R})$ as $k \to \infty$. The perturbed problem of **Problem Q** is given by

Problem $\mathbf{Q_k}$. Find a velocity field $\boldsymbol{u}_k \colon \Omega \to \mathbb{R}^d$ and a pressure $p_k \colon \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&-\mu\Delta \boldsymbol{u}_k + (\boldsymbol{u}_k \cdot \nabla)\boldsymbol{u}_k + \nabla p_k = \boldsymbol{f} && \text{in} \quad \Omega,\\
&\mathrm{div}\boldsymbol{u}_k = \boldsymbol{0} && \text{in} \quad \Omega,\\
&\boldsymbol{u}_k = \boldsymbol{0} && \text{on} \quad \Gamma_0,\\
&\begin{cases} u_{k,\nu} = 0 \\ -\boldsymbol{\tau}_\tau(\boldsymbol{u}_k) \in \partial j(\boldsymbol{u}_{k,\tau}), \end{cases} && \text{on} \quad \Gamma_1,\\
&\begin{cases} u_{k,\nu} + g_k \geq 0, \tau_\nu(\boldsymbol{u}_k, p_k) \geq 0 \\ (u_{k,\nu} + g_k)\tau_\nu(\boldsymbol{u}_k, p_k) = 0 \\ \boldsymbol{\tau}_\tau(\boldsymbol{u}_k) = 0 \end{cases} && \text{on} \quad \Gamma_2,\\
&\begin{cases} -\tau_\nu(\boldsymbol{u}_k, p_k) \in \partial\psi(u_{k,\nu}) \\ \boldsymbol{u}_{k,\tau} = 0 \end{cases} && \text{on} \quad \Gamma_3.
\end{aligned}$$

Then, the weak formulation of **Problem $\mathbf{Q_k}$** is given in the following form:

Problem $\mathbf{Q}_k^V$. Find a displacement $\boldsymbol{u} \in C_k$ such that

$$\langle F_k\boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}) + j^\circ(T\boldsymbol{u}, T\boldsymbol{v} - T\boldsymbol{u}) \geq 0 \quad \text{for all } \boldsymbol{v} \in C_k,$$

where $F_k \colon X \to X^*$ is defined by

$$\langle F_k\boldsymbol{u}, \boldsymbol{v}\rangle_{X^*\times X} = \langle A_k\boldsymbol{u}, \boldsymbol{v}\rangle_{X^*\times X} + \langle B[\boldsymbol{u}], \boldsymbol{v}\rangle_{X^*\times X} \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in X,$$

with

$$A_k \colon X \to X^*, \quad \langle A_k\boldsymbol{u}, \boldsymbol{v}\rangle_{X^*\times X} = \mu_k \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\,\mathrm{d}x,$$

and the set C_k is given by

$$C_k = \{\boldsymbol{v} \in X \colon v_\nu + g_k \geq 0 \text{ on } \Gamma_2\}.$$

We are now in a position to give the following stability result for **Problem Q** which shows that the weak solution set of **Problem $\mathbf{Q}_k$** converges to the weak solution set of **Problem Q** in the sense of Kuratowski as $k \to \infty$.

Theorem 5.2. *Suppose that all assumptions of Theorem 5.1 hold and further assume the following hypotheses:*

- $\mu_k, \mu > 0$ *and* $\mu_k \to \mu$ *as* $k \to \infty$;
- $g_k, g \in L^2(\Gamma_2; [0, +\infty))$ *are such that* $g_k \to g$ *in* $L^2(\Gamma_2; [0, +\infty))$ *as* $k \to \infty$.

Then we have the following assertions:

(i) *for each* $k \in \mathbb{N}$ *the solution set of* **Problem $\mathbf{Q}_k^V$**, *denoted by* $\mathcal{S}_k$, *is nonempty,*
(ii) $w - \limsup_{k\to\infty} \mathcal{S}_k \subset \mathcal{S}$,

where $\mathcal{S}$ *is the solution set of* **Problem Q**.

Proof. (i) The solvability of **Problem $\mathbf{Q}_k^V$** is a direct consequence of Theorem 5.1.

(ii) Arguing as in the proof of Theorem 5.1, one can show that the set $\bigcup_{k\in\mathbb{N}} \mathcal{S}_k$ is bounded in X. Also, we can apply the same arguments as in the proof of Theorem 16 by Xiao-Sofonea [39] in order to prove that C_k converges to C in the sense of Mosco and F_k, F satisfy the framework of Corollary 4.3. So, the desired conclusion can be obtained by applying Corollary 4.3. □

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