DISCONTINUOUS VARIATIONAL-HEMIVARIATIONAL INEQUALITIES INVOLVING THE P-LAPLACIAN

PATRICK WINKERT

ABSTRACT. We deal with discontinuous quasilinear elliptic variational-hemivariational inequalities. By using the method of sub- and supersolution and based on the results of S. Carl, we extend the theory for discontinuous problems. The proof of the existence of extremal solutions within a given order interval of sub- and supersolutions is the main goal of this paper. In the last part, we give an example of the construction of sub- and supersolutions.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. As $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega), 1 < p < \infty$, we denote the usual Sobolev spaces with their dual spaces $V^* = (W^{1,p}(\Omega))^*$ and $V_0^* = W^{-1,q}(\Omega)$, respectively (q is the Hölder conjugate of p). In this paper, we consider the following elliptic variational-hemivariational inequality

$$u \in K: \quad \langle -\Delta_p u + F(u), v - u \rangle + \int_\Omega j^\circ(u; v - u) dx \geq 0, \quad \forall v \in K, \qquad (1.1)$$

where $j^\circ(s; r)$ denotes the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ at s in the direction r given by

$$j^\circ(s; r) = \limsup_{y \to s, t \downarrow 0} \frac{j(y + tr) - j(y)}{t},$$

(cf. [7, Chapter 2]), and $K \subset V_0$ is some closed and convex subset. The operator $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < \infty$, and F denotes the Nemytskij operator related to the function $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$F(u)(x) = f(x, u(x), u(x)).$$

In [3] the method of sub- and supersolution was developed for variational-hemivariational inequalities of the form (1.1) with $F(u) \equiv f \in V_0^*$. The aim of this paper is the generalization for discontinuous Nemytskij operators $F : L^p(\Omega) \to L^q(\Omega)$. Let us consider some special cases of problem (1.1) as follows.

(i) For $f \in V_0^*$, (1.1) is also a variational-hemivariational inequality which is discussed in [3].

(ii) If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying some growth condition and $j = 0$, then (1.1) is a classical variational inequality of the form

$$u \in K: \quad \langle -\Delta_p u + F(u), v - u \rangle \geq 0, \quad \forall v \in K,$$

for which the method of sub- and supersolution has been developed in [5, Chapter 5].

(iii) For $K = V_0$, $f \in V_0^*$ and $j : \mathbb{R} \to \mathbb{R}$ smooth, (1.1) becomes a variational equality of the form

$$u \in V_0 : \quad \langle -\Delta_p u + f + j'(u), \varphi \rangle = 0, \quad \forall \varphi \in V_0,$$

for which the sub-supersolution method is well known.

2. Notations and hypotheses

For functions $u, v : \Omega \to \mathbb{R}$, we use the notation $u \wedge v = \min(u, v), u \vee v = \max(u, v), K \wedge K = \{u \wedge v : u, v \in K\}, K \vee K = \{u \vee v : u, v \in K\}$, and $u \wedge K = \{u\} \wedge K, u \vee K = \{u\} \vee K$ and introduce the following definitions.

Definition 2.1. *A function $\underline{u} \in V$ is called a subsolution of (1.1) if the following holds:*

(1) $\underline{u} \leq 0$ *on* $\partial\Omega$ *and* $F(\underline{u}) \in L^q(\Omega)$;
(2) $\langle -\Delta_p \underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_\Omega j^\circ(\underline{u}; w - \underline{u})dx \geq 0, \quad \forall w \in \underline{u} \wedge K$.

Definition 2.2. *A function $\overline{u} \in V$ is called a supersolution of (1.1) if the following holds:*

(1) $\overline{u} \geq 0$ *on* $\partial\Omega$ *and* $F(\overline{u}) \in L^q(\Omega)$;
(2) $\langle -\Delta_p \overline{u} + F(\overline{u}), w - \overline{u} \rangle + \int_\Omega j^\circ(\overline{u}; w - \overline{u})dx \geq 0, \quad \forall w \in \overline{u} \vee K$.

Definition 2.3. *The multivalued operator $\partial j : \mathbb{R} \to 2^\mathbb{R} \setminus \{\emptyset\}$ is called Clarke's generalized gradient of j defined by*

$$\partial j(s) := \{\xi \in \mathbb{R} : j^\circ(s; r) \geq \xi r, \forall r \in \mathbb{R}\}.$$

We impose the following hypotheses for j and the nonlinearity f in problem (1.1).

(A) There exists a constant $c_1 \geq 0$ such that

$$\xi_1 \leq \xi_2 + c_1(s_2 - s_1)^{p-1}$$

for all $\xi_i \in \partial j(s_i), i = 1, 2$, and for all s_1, s_2 with $s_1 < s_2$.

(B) There is a constant $c_2 \geq 0$ such that

$$\xi \in \partial j(s) : |\xi| \leq c_2(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R}.$$

(C) (i) $x \mapsto f(x, r, u(x))$ is measurable for all $r \in \mathbb{R}$ and for all measurable functions $u : \Omega \to \mathbb{R}$.
(ii) $r \mapsto f(x, r, s)$ is continuous for all $s \in \mathbb{R}$ and for almost all $x \in \Omega$.
(iii) $s \mapsto f(x, r, s)$ is decreasing for all $r \in \mathbb{R}$ and for almost all $x \in \Omega$.
(iv) For a given ordered pair of sub- and supersolutions $\underline{u}, \overline{u}$ of problem (1.1), there exists a function $k_1 \in L_+^q(\Omega)$ such that $|f(x, r, s)| \leq k_1(x)$ for all $r, s \in [\underline{u}(x), \overline{u}(x)]$ and for almost all $x \in \Omega$.

By [2] the mapping $x \mapsto f(x, u(x), u(x))$ is measurable for $x \mapsto u(x)$ measurable, but the associated Nemytskij operator $F : L^p(\Omega) \to L^q(\Omega)$ needs not necessarily be continuous. In this paper we assume K has lattice structure, that is, K fulfills

$$K \vee K \subset K, \qquad K \wedge K \subset K. \tag{2.1}$$

We recall that the normed space $L^p(\Omega)$ is equipped with the natural partial ordering of functions defined by $u \leq v$ if and only if $v - u \in L_+^p(\Omega)$, where $L_+^p(\Omega)$ is the set of all nonnegative functions of $L^p(\Omega)$.

3. Preliminaries

Here we consider (1.1) for a Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (i.e., $x \mapsto h(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$ and $s \mapsto h(x, s)$ is continuous on $\mathbb{R}$ for almost all $x \in \Omega$), which fulfills the following growth condition:

$$|h(x, s)| \le k_2(x) \quad \forall s \in [\underline{u}(x), \overline{u}(x)] \text{ and for a.e. } x \in \Omega, \tag{3.1}$$

where $k_2 \in L_+^q(\Omega)$ and $[\underline{u}, \overline{u}]$ is some ordered pair in $L^p(\Omega)$, specified later. Note that the associated Nemytskij operator H defined by $H(u)(x) = h(x, u(x))$ is continuous and bounded from $[\underline{u}, \overline{u}] \subset L^p(\Omega)$ to $L^q(\Omega)$ (cf. [9]). Next we introduce the indicator function $I_K : V_0 \to \mathbb{R} \cup \{+\infty\}$ related to the closed convex set $K \ne \emptyset$ given by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K, \end{cases}$$

which is known to be proper, convex and lower semicontinuous. The variational-hemivariational inequality (1.1) can be rewritten as follows: Find $u \in V_0$ such that

$$\langle -\Delta_p u + H(u), v - u \rangle + I_K(v) - I_K(u) + \int_\Omega j^o(u; v - u) dx \ge 0, \quad \forall v \in V_0. \tag{3.2}$$

If $H(u) \equiv h \in V_0^*$, problem (3.2) is a special case of the elliptic variational-hemivariational inequality in [5, Corollary 7.15] for which the method of sub- and supersolutions was developed. In the next result, we show the existence of extremal solutions of (3.2) for a Carathéodory function $h = h(x, s)$.

Lemma 3.1. *Let hypotheses (A),(B) and (2.1) be satisfied and assume the existence of sub- and supersolutions $\underline{u}$ and $\overline{u}$ satisfying $\underline{u} \le \overline{u}$, $\underline{u} \vee K \subset K$ and $\overline{u} \wedge K \subset K$. Furthermore we suppose that the Carathédory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (3.1). Then, (3.2) has a greatest solution u^* and a smallest solution u_* such that*

$$\underline{u} \le u_* \le u^* \le \overline{u}, \tag{3.3}$$

that is, u_* and u^* are solutions of (3.2) that satisfy (3.3), and if u is any solution of (3.2) such that $\underline{u} \le u \le \overline{u}$, then $u_* \le u \le u^*$.

Proof. The proof follows the same ideas as in the proof for $H(u) \equiv h \in V_0^*$ with an additional modification. We only introduce a truncation operator related to the functions $\underline{u}$ and $\overline{u}$ defined by

$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x). \end{cases}$$

The mapping T is continuous and bounded from V into V which follows from the fact that the functions $\min(\cdot, \cdot)$ and $\max(\cdot, \cdot)$ are continuous from V to itself and that T can be represented as $Tu = \max(u, \underline{u}) + \min(u, \overline{u}) - u$ (cf. [8]). In the auxiliary problems of the proof of [5, Corollary 7.15], we replace $h \in V_0^*$ by $(H \circ T)(u)$ and argue in an analogous way. $\square$

An important tool in extending the previous result to discontinuous Nemytskij operators is the next fixed point result. The proof of this Lemma can be found in [4, Theorem 1.1.1].

Lemma 3.2. *Let P be a subset of an ordered normed space, $G : P \to P$ an increasing mapping and $G[P] = \{Gx \mid x \in P\}$.*

(1) *If $G[P]$ has a lower bound in P and the increasing sequences of $G[P]$ converge weakly in P, then G has the least fixed point x_*, and $x_* = \min\{x \mid Gx \leq x\}$.*

(2) *If $G[P]$ has an upper bound in P and the decreasing sequences of $G[P]$ converge weakly in P, then G has the greatest fixed point x^*, and $x^* = \max\{x \mid x \leq Gx\}$.*

4. Main results

One of our main results is the following theorem.

Theorem 4.1. *Let hypotheses (A)–(C), (2.1) be satisfied and assume the existence of sub- and supersolutions $\underline{u}$ and $\overline{u}$ satisfying $\underline{u} \leq \overline{u}$, $\underline{u} \vee K \subset K$ and $\overline{u} \wedge K \subset K$. If f is right-continuous (resp., left-continuous) in the third argument, then there exists a greatest solution u^* (resp., a smallest solution u_*) of (1.1) in the order interval $[\underline{u}, \overline{u}]$.*

Proof. We choose a fixed element $z \in [\underline{u}, \overline{u}]$ which is a supersolution of (1.1) satisfying $z \wedge K \subset K$ and consider the following auxiliary problem:

$$u \in K : \quad \langle -\Delta_p u + F_z(u), v - u \rangle + \int_\Omega j^\circ(u; v - u)dx \geq 0, \quad \forall v \in K, \qquad (4.1)$$

where $F_z(u)(x) = f(x, u(x), z(x))$. It is readily seen that the mapping $(x, u) \mapsto f(x, u, z(x))$ is a Carathéodory function satisfying some growth condition as in (3.1). Since $F_z(z) = F(z)$, z is also a supersolution of (4.1). By Definition 2.1, we have for a given subsolution $\underline{u}$ of (1.1)

$$\langle -\Delta_p \underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_\Omega j^\circ(\underline{u}; w - \underline{u})dx \geq 0, \quad \forall w \in \underline{u} \wedge K.$$

Setting $w = \underline{u} - (\underline{u} - v)^+$ for all $v \in K$ and using the monotonicity of f with respect to s, we get

$$0 \geq \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_\Omega j^\circ(\underline{u}; -(\underline{u} - v)^+)dx$$

$$\geq \langle -\Delta_p \underline{u} + F_z(\underline{u}), (\underline{u} - v)^+ \rangle - \int_\Omega j^\circ(\underline{u}; -(\underline{u} - v)^+)dx, \quad \forall v \in K,$$

which shows that $\underline{u}$ is also a subsolution of (4.1). Lemma 3.1 implies the existence of a greatest solution $u^* \in [\underline{u}, z]$ of (4.1). Now we introduce the set A given by $A := \{z \in V : z \in [\underline{u}, \overline{u}] \text{ and } z \text{ is a supersolution of (1.1) satisfying } z \wedge K \subset K\}$ and define the operator $L : A \to K$ by $z \mapsto u^* =: Lz$. This means that the operator L assigns to each $z \in A$ the greatest solution u^* of (4.1) in $[\underline{u}, z]$. In the next step we construct a decreasing sequence as follows:

$$\begin{aligned}
u_0 &:= \overline{u} \\
u_1 &:= Lu_0 \quad \text{with} \quad u_1 \in [\underline{u}, u_0] \\
u_2 &:= Lu_1 \quad \text{with} \quad u_2 \in [\underline{u}, u_1] \\
&\vdots \\
u_n &:= Lu_{n-1} \quad \text{with} \quad u_n \in [\underline{u}, u_{n-1}].
\end{aligned} \qquad (4.2)$$

As $u_n \in [\underline{u}, u_{n-1}]$, we get $u_n(x) \searrow u(x)$ for a.e. $x \in \Omega$. Furthermore, the sequence u_n is bounded in V_0, that is, $\|u_n\|_{V_0} \le C$ for all n and due to the monotony of u_n and the compact embedding $V_0 \hookrightarrow L^p(\Omega)$, we obtain

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } V_0, \\ u_n &\to u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega. \end{aligned} \tag{4.3}$$

The fact that u_n is a solution of (4.1) with $z = u_{n-1}$ and $v = u \in K$ results in

$$\langle -\Delta_p u_n, u_n - u \rangle \le \langle F_{u_{n-1}}(u_n), u - u_n \rangle + \int_\Omega j^\circ(\underline{u}; u - u_n) dx.$$

Applying Fatou's Lemma, (4.3), and the upper semicontinuity of $j^\circ(\cdot, \cdot)$ yields

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u \rangle$$

$$\le \underbrace{\limsup_{n\to\infty} \|k\|_{L^q(\Omega)} \|u - u_n\|_{L^p(\Omega)}}_{\to 0} + \int_\Omega \underbrace{\limsup_{n\to\infty} j^\circ(\underline{u}; u - u_n)}_{\le j^\circ(u;0)=0} dx \le 0,$$

which by the S_+-property of $-\Delta_p$ on V_0 along with (4.3) implies

$$u_n \to u \quad \text{in } V_0.$$

The right-continuity of f and the strong convergence of the decreasing sequence (u_n) along with the upper semicontinuity of $j^\circ(\cdot; \cdot)$ allow us to pass to the $\limsup$ in (4.1), where u (resp., z) is replaced by u_n (resp., u_{n-1}). We have

$$0 \le \limsup_{n\to\infty} \langle -\Delta_p u_n + F_{u_{n-1}}(u_n), v - u_n \rangle + \limsup_{n\to\infty} \int_\Omega j^\circ(u_n; v - u_n) dx$$

$$\le \lim_{n\to\infty} \langle -\Delta_p u_n + F_{u_{n-1}}(u_n), v - u_n \rangle + \int_\Omega \limsup_{n\to\infty} j^\circ(u_n; v - u_n) dx$$

$$\le \langle -\Delta_p u + F_u(u), v - u \rangle + \int_\Omega j^\circ(u; v - u) dx, \quad \forall v \in K.$$

This shows that u is a solution of (1.1) in the order interval $[\underline{u}, \overline{u}]$. Now, we still have to prove that u is the greatest solution of (1.1) in $[\underline{u}, \overline{u}]$. Let $\widetilde{u}$ be any solution of (1.1) in $[\underline{u}, \overline{u}]$. Because of the fact that K has lattice structure, $\widetilde{u}$ is also a subsolution of (1.1), respectively, a subsolution of (4.1). By the same construction as in (4.2) we obtain.

$$\begin{aligned} \widetilde{u_0} &:= \overline{u} \\ \widetilde{u_1} &:= L u_0 \quad \text{with } \widetilde{u_1} \in [\widetilde{u}, u_0] \\ \widetilde{u_2} &:= L u_1 \quad \text{with } \widetilde{u_2} \in [\widetilde{u}, u_1] \\ &\vdots \\ \widetilde{u_n} &:= L u_{n-1} \quad \text{with } \widetilde{u_n} \in [\widetilde{u}, u_{n-1}]. \end{aligned} \tag{4.4}$$

Obviously, the sequences in (4.2) and (4.4) create the same extremal solutions u_n and $\widetilde{u_n}$, which implies that $\widetilde{u} \le \widetilde{u_n} = u_n$ for all n. Passing to the limit delivers the assertion. The existence of a smallest solution can be shown in a similar way. $\square$

In the next theorem we will prove that only the monotony of f in the third argument is sufficient for the existence of extremal solutions. The function f needs neither be right-continuous nor left-continuous.

Theorem 4.2. *Assume that hypotheses (A)–(C), (2.1) are valid and let $\underline{u}$ and $\overline{u}$ be sub- and supersolutions of (1.1) satisfying $\underline{u} \leq \overline{u}$, $\underline{u} \vee K \subset K$ and $\overline{u} \wedge K \subset K$. Then there exist extremal solutions u^* and u_* of (1.1) with $\underline{u} \leq u_* \leq u^* \leq \overline{u}$.*

Proof. As in the proof of Theorem 4.1, we consider the following auxiliary problem:

$$u \in K : \quad \langle -\Delta_p u + F_z(u), v - u \rangle + \int_\Omega j^\circ(u; v - u) dx \geq 0, \quad \forall v \in K, \qquad (4.5)$$

where $F_z(u)(x) = f(x, u(x), z(x))$. We define again the set $A := \{ z \in V : z \in [\underline{u}, \overline{u}]$ and z is a supersolution of (1.1) satisfying $z \wedge K \subset K \}$ and introduce the fixed point operator $L : A \to K$ by $z \mapsto u^* =: Lz$. For a given supersolution $z \in A$, the element Lz is the greatest solution of (4.5) in $[\underline{u}, z]$, and thus it holds $\underline{u} \leq Lz \leq z$ for all $z \in A$ which implies $L : A \to [\underline{u}, \overline{u}]$. Because of (2.1), Lz is also a supersolution of (4.5) satisfying

$$\langle -\Delta_p Lz + F_z(Lz), w - Lz \rangle + \int_\Omega j^\circ(Lz; w - Lz) dx \geq 0, \quad \forall w \in Lz \vee K.$$

By the monotonicity of f with respect to $Lz \leq z$ and using the representation $w = Lz + (v - Lz)^+$ for any $v \in K$, we obtain

$$0 \leq \langle -\Delta_p Lz + F_z(Lz), (v - Lz)^+ \rangle + \int_\Omega j^\circ(Lz; (v - Lz)^+) dx$$

$$\leq \langle -\Delta_p Lz + F_{Lz}(Lz), (v - Lz)^+ \rangle + \int_\Omega j^\circ(Lz; (v - Lz)^+) dx, \quad \forall v \in K.$$

Consequently, Lz is a supersolution of (1.1). This shows $L : A \to A$.
Let $v_1, v_2 \in A$ and assume that $v_1 \leq v_2$. Then we have

$\quad Lv_1 \in [\underline{u}, v_1]$ is the greatest solution of

$$\langle -\Delta_p u + F_{v_1}(u), v - u \rangle + \int_\Omega j^\circ(u; v - u) dx \geq 0, \quad \forall v \in K, \qquad (4.6)$$

$\quad Lv_2 \in [\underline{u}, v_2]$ is the greatest solution of

$$\langle -\Delta_p u + F_{v_2}(u), v - u \rangle + \int_\Omega j^\circ(u; v - u) dx \geq 0, \; \forall v \in K. \qquad (4.7)$$

Since $v_1 \leq v_2$, it follows that $Lv_1 \leq v_2$ and due to (2.1), Lv_1 is also a subsolution of (4.6), that is, (4.6) holds, in particular, for $v \in Lv_1 \wedge K$, that is,

$$\langle -\Delta_p Lv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_\Omega j^\circ(Lv_1; -(Lv_1 - v)^+) dx \leq 0, \quad \forall v \in K.$$

Using the monotonicity of f with respect to s yields

$$0 \geq \langle -\Delta_p Lv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_\Omega j^\circ(Lv_1; -(Lv_1 - v)^+) dx$$

$$\geq \langle -\Delta_p Lv_1 + F_{v_2}(Lv_1), (Lv_1 - v)^+ \rangle - \int_\Omega j^\circ(Lv_1; -(Lv_1 - v)^+) dx, \quad \forall v \in K.$$

and hence Lv_1 is a subsolution of (4.7). By Lemma 3.1, we know there exists a greatest solution of (4.7) in $[Lv_1, v_2]$. But Lv_2 is the greatest solution of (4.7) in $[\underline{u}, v_2] \supseteq [Lv_1, v_2]$ and therefore, $Lv_1 \leq Lv_2$. This shows that L is increasing.
In the last step we have to prove that any decreasing sequence of $L(A)$ converges weakly in A. Let $(u_n) = (Lz_n) \subset L(A) \subset A$ be a decreasing sequence. The same argument as in the proof of Theorem 4.1 delivers $u_n(x) \searrow u(x)$ a.e. in Ω. The

boundedness of u_n in V_0, and the compact imbedding $V_0 \hookrightarrow L^p(\Omega)$ along with the monotony of u_n implies

$$u_n \rightharpoonup u \quad \text{in } V_0,$$

$$u_n \to u \quad \text{in } L^p(\Omega) \text{ and a.e. } x \text{ in } \Omega.$$

Since $u_n \in K$ solves (4.5), it follows $u \in K$. From (4.5) with u replaced by u_n and v by u and with the fact that $(s, r) \mapsto j^\circ(s; r)$ is upper semicontinuous, we obtain by applying Fatou's Lemma

$$\limsup_{n \to \infty} \langle -\Delta_p u_n, u_n - u \rangle \le \limsup_{n \to \infty} \langle F_{z_n}(u_n), u - u_n \rangle + \limsup_{n \to \infty} \int_\Omega j^\circ(u_n; u - u_n) dx$$

$$\le \underbrace{\limsup_{n \to \infty} \langle F_{z_n}(u_n), u - u_n \rangle}_{\to 0} + \int_\Omega \underbrace{\limsup_{n \to \infty} j^\circ(u_n; u - u_n)}_{\le j^\circ(u;0)=0} dx$$

$$\le 0.$$

The S_+-property of $-\Delta_p$ provides the strong convergence of (u_n) in V_0. As $Lz_n = u_n$ is also a supersolution of (4.5), Definition 2.2 yields

$$\langle -\Delta_p u_n + F_{z_n}(u_n), (v - u_n)^+ \rangle + \int_\Omega j^\circ(u_n; (v - u_n)^+) dx \ge 0, \quad \forall v \in K.$$

Due to $z_n \ge u_n \ge u$ and the monotonicity of f, we get

$$0 \le \langle -\Delta_p u_n + F_{z_n}(u_n), (v - u_n)^+ \rangle + \int_\Omega j^\circ(u_n; (v - u_n)^+) dx$$

$$\le \langle -\Delta_p u_n + F_u(u_n), (v - u_n)^+ \rangle + \int_\Omega j^\circ(u_n; (v - u_n)^+) dx, \quad \forall v \in K,$$

and, since the mapping $u \mapsto u^+ = \max(u, 0)$ is continuous from V_0 to itself (cf. [8]), we can pass to the upper limit on the right hand side for $n \to \infty$. This yields

$$\langle -\Delta_p u + F_u(u), (v - u)^+ \rangle + \int_\Omega j^\circ(u; (v - u)^+) dx \ge 0, \quad \forall v \in K,$$

which shows that u is a supersolution of (1.1), that is, $u \in A$. As $\overline{u}$ is an upper bound of $L(A)$, we can apply Lemma 3.2, which yields the existence of a greatest fixed point u^* of L in A. This implies that u^* must be the greatest solution of (1.1) in $[\underline{u}, \overline{u}]$. By analogous reasoning, one shows the existence of a smallest solution u_* of (1.1). This completes the proof of the theorem. $\qquad \square$

APPLICATION

In the last part, we give an example of the construction of sub- and supersolutions of problem (1.1). We denote by $\lambda_1 > 0$ the first eigenvalue of $(-\Delta_p, V_0)$ and by φ_1 the eigenfunction of $(-\Delta_p, V_0)$ corresponding to λ_1 satisfying $\varphi_1 \in \text{int}(C_0^1(\overline{\Omega})_+)$ and $\|\varphi\|_p = 1$ (cf. [1]). Here, $\text{int}(C_0^1(\overline{\Omega})_+)$ describes the interior of the positive cone $C_0^1(\overline{\Omega})_+$ given by

$$\text{int}(C_0^1(\overline{\Omega})_+) = \left\{ u \in C_0^1(\overline{\Omega}) : u(x) > 0, \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial n}(x) < 0, \forall x \in \partial\Omega \right\}.$$

We suppose the following conditions for f and Clarke's generalized gradient of j, where $\lambda > \lambda_1$ is any fixed constant:

(D)   (i) $\displaystyle\lim_{|s|\to\infty}\frac{f(x,s,s)}{|s|^{p-2}s}=+\infty,$ uniformly with respect to a.a. $x\in\Omega$.

 (ii) $\displaystyle\lim_{s\to 0}\frac{f(x,s,s)}{|s|^{p-2}s}=-\lambda,$ uniformly with respect to a.a. $x\in\Omega$.

 (iii) $\displaystyle\lim_{s\to 0}\frac{\xi}{|s|^{p-2}s}=0,$ uniformly with respect to a.a. $x\in\Omega$, for all $\xi\in\partial j(s)$.

 (iv) f is bounded on bounded sets.

Proposition 4.3. *Assume hypotheses (A), (B), (C)(i)–(iv), and (D). Then there exists a constant a_λ such that $a_\lambda e$ and $-a_\lambda e$ are supersolution and subsolution of problem (1.1), where $e\in\mathrm{int}(C_0^1(\overline{\Omega})_+)$ is the unique solution of $-\Delta_p u=1$ in V_0. Moreover, $-\varepsilon\varphi_1$ is a supersolution and $\varepsilon\varphi_1$ is a subsolution of (1.1) provided that $\varepsilon>0$ is sufficiently small.*

Proof. A sufficient condition for a subsolution $\underline{u}\in V$ of problem (1.1) is $\underline{u}\le 0$ on $\partial\Omega$, $F(\underline{u})\in L^q(\Omega)$ and

$$-\Delta_p\underline{u}+F(\underline{u})+\xi\le 0\quad\text{in }V_0^*,\quad\forall\xi\in\partial j(\underline{u}).\tag{4.8}$$

Multiplying (4.8) with $(\underline{u}-v)^+\in V_0\cap L_+^p(\Omega)$ and using the fact $j^{\circ}(\underline{u};-1)\ge-\xi$, for all $\xi\in\partial j(\underline{u})$, yield

$$0\ge\langle-\Delta_p\underline{u}+F(\underline{u})+\xi,(\underline{u}-v)^+\rangle$$

$$=\langle-\Delta_p\underline{u}+F(\underline{u}),(\underline{u}-v)^+\rangle+\int_\Omega\xi(\underline{u}-v)^+dx$$

$$\ge\langle-\Delta_p\underline{u}+F(\underline{u}),(\underline{u}-v)^+\rangle-\int_\Omega j^{\circ}(\underline{u};-1)(\underline{u}-v)^+dx$$

$$=\langle-\Delta_p\underline{u}+F(\underline{u}),(\underline{u}-v)^+\rangle-\int_\Omega j^{\circ}(\underline{u};-(\underline{u}-v)^+)dx,\quad\forall v\in K,$$

and thus, $\underline{u}$ is a subsolution of (1.1). Analogously, $\overline{u}\in V$ is a supersolution of problem (1.1) if $\overline{u}\ge 0$ on $\partial\Omega$, $F(\overline{u})\in L^q(\Omega)$, and if the following inequality is satisfied,

$$-\Delta_p\overline{u}+F(\overline{u})+\xi\ge 0\quad\text{in }V_0^*,\quad\forall\xi\in\partial j(\overline{u}).$$

The main idea of this proof is to show the applicability of [6, Lemma 2.1–2.3]. We put $g(x,s)=f(x,s,s)+\xi+\lambda|s|^{p-2}s$ for $\xi\in\partial j(s)$ and notice that in our considerations the nonlinearity g needs not be a continuous function. In view of assumption (B), we see at once that

$$\frac{|\xi|}{|s|^{p-1}}\le c,\quad\text{for }|s|\ge k>0,\forall\xi\in\partial j(s),$$

where c is a positive constant. This fact and the condition (D) yield the following limit values:

$$\lim_{|s|\to\infty}\frac{g(x,s)}{|s|^{p-2}s}=+\infty,\qquad\lim_{s\to 0}\frac{g(x,s)}{|s|^{p-2}s}=0.$$

By [6, Lemma 2.1–2.3], we obtain a pair of positive sub- and supersolutions given by $\underline{u}=\varepsilon\varphi_1$ and $\overline{u}=a_\lambda e$, respectively, a pair of negative sub- and supersolutions given by $\underline{u}=-a_\lambda e$ and $\overline{u}=-\varepsilon\varphi_1$.   □

In order to apply Theorem 4.2, we need to satisfy the assumptions

$$\underline{u} \vee K \subset K, \quad \overline{u} \wedge K \subset K, \quad K \vee K \subset K, \quad K \wedge K \subset K, \qquad (4.9)$$

which depend on the specific K. For example, we consider an obstacle problem given by

$$K = \{v \in V_0 : v(x) \leq \psi(x) \text{ for a.e. } x \in \Omega\}, \quad \psi \in L^\infty(\Omega), \quad \psi \geq C > 0, \quad (4.10)$$

where C is a positive constant. One can show that for the positive pair of sub- and supersolutions in Proposition 4.3, all these conditions in (4.9) with respect to the closed convex set K defined in (4.10) can be satisfied.

Example 4.4. *The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by*

$$f(r,s) = \begin{cases} -(\lambda + 1)|s|^{p-2}s + |r|^{p-1}r & \text{for } s < -1, \\ -\lambda|s|^{p-2}s + |r|^{p-1}r & \text{for } -1 \leq s \leq 1, \\ -(\lambda + 1)|s|^{p-2}s + |r|^{p-1}r & \text{for } s > 1 \end{cases}$$

fulfills the assumption (C)(i)–(iv) with respect to $\underline{u}, \overline{u}$ defined in Proposition 4.3. Moreover f satisfies the conditions (D)(i)–(ii),(D)(iv), where $\lambda > \lambda_1$ is fixed.

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Martin-Luther-Universität Halle-Wittenberg, Institut für Mathematik, Theodor-Lieser-Strasse 5, 06120 Halle, Germany

E-mail address: `patrick.winkert@mathematik.uni-halle.de`