SINGULAR p-LAPLACIAN EQUATIONS WITH SUPERLINEAR PERTURBATION

NIKOLAOS S. PAPAGEORGIOU AND PATRICK WINKERT

Abstract. We consider a nonlinear Dirichlet problem driven by the p-Laplace operator and with a right-hand side which has a singular term and a parametric superlinear perturbation. We are interested in positive solutions and prove a bifurcation-type theorem describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. In addition, we show that for every admissible parameter $\lambda > 0$ the problem has a smallest positive solution $\overline{u}_\lambda$ and we establish the monotonicity and continuity properties of the map $\lambda \to \overline{u}_\lambda$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we deal with the following nonlinear parametric singular problem

$$\begin{aligned} -\Delta_p u &= u^{-\gamma} + \lambda f(x,u) && \text{in } \Omega,\\ u &> 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{P$_\lambda$}$$

where $1 < p < \infty$, $0 < \gamma < 1$ and Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

In the right-hand side of (P$_\lambda$), $u^{-\gamma}$ is the singular term while λf is the parametric term with $\lambda > 0$ and a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, that is, $x \to f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \to f(x,s)$ is continuous for a.a. $x \in \Omega$. We assume that $f(x,\cdot)$ exhibits $(p-1)$-superlinear growth near $+\infty$ but without satisfying the usual Ambrosetti-Rabinowitz condition, AR-condition for short. We are interested in finding positive solutions and our goal is to determine how the set of positive solutions of (P$_\lambda$) changes as the parameter $\lambda > 0$ varies. We are going to prove a bifurcation-type result which produces a critical parameter value $\lambda^* > 0$ such that

- problem (P$_\lambda$) has at least two positive solutions for all $\lambda \in (0, \lambda^*)$;
- problem (P$_\lambda$) has at least one positive solution for $\lambda = \lambda^*$;
- problem (P$_\lambda$) has no positive solutions for all $\lambda > \lambda^*$.

This result was motivated by the work of Papageorgiou-Smyrlis [15] who proved such a theorem for problem (P$_\lambda$) under the hypotheses that the perturbation term $f(x,\cdot)$ is $(p-1)$-linear near 0^+. This condition removes from consideration nonlinearities with a concave term near 0^+. Our framework removes this restriction and incorporates perturbations which exhibit the competing effects of concave and

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convex terms. This changes the geometry of the problem. Moreover, our growth condition on $f(x,\cdot)$ is more general than that in Papageorgiou-Smyrlis [15].

Nonlinear singular Dirichlet problems were also investigated in the papers of Giacomoni-Schindler-Takáč [5], Papageorgiou-Rădulescu-Repovš [14] and Perera-Zhang [16] for different settings and conditions.

2. Preliminaries and Hypotheses

Let X be a Banach space and let X^* be its topological dual. We denote by $\langle\cdot,\cdot\rangle$ the duality brackets to the pair (X^*,X). Given $\varphi\in C^1(X,\mathbb{R})$ we say that φ satisfies the Cerami condition, C-condition for short, if every sequence $\{u_n\}_{n\geq 1}\subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1}\subseteq\mathbb{R}$ is bounded and such that $(1+\|u_n\|_X)\,\varphi'(u_n)\to 0$ in X^* as $n\to\infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ and leads to following minimax theorem known as the mountain pass theorem.

Theorem 2.1. *Let $\varphi\in C^1(X,\mathbb{R})$ be a functional satisfying the C-condition and let $u_1,u_2\in X$, $\|u_2-u_1\|_X>\rho>0$,*

$$\max\{\varphi(u_1),\varphi(u_2)\}<\inf\{\varphi(u):\|u-u_1\|_X=\rho\}=:\eta_\rho$$

and $c=\inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1}\varphi(\gamma(t))$ with $\Gamma=\{\gamma\in C([0,1],X):\gamma(0)=u_1,\gamma(1)=u_2\}$. Then $c\geq\eta_\rho$ with c being a critical value of φ, that is, there exists $\hat{u}\in X$ such that $\varphi'(\hat{u})=0$ and $\varphi(\hat{u})=c$.

By $W^{1,p}_0(\Omega)$ we denote the usual Sobolev space with norm $\|\cdot\|$. Thanks to the Poincaré inequality we have

$$\|u\|=\|\nabla u\|_p\quad\text{for all } u\in W^{1,p}_0(\Omega),$$

where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$ and $L^p\left(\Omega;\mathbb{R}^N\right)$, respectively. Furthermore, we need the ordered Banach space $C^1_0(\overline{\Omega})=\{u\in C^1(\overline{\Omega}):u\big|_{\partial\Omega}=0\}$ and its positive cone

$$C^1_0(\overline{\Omega})_+=\left\{u\in C^1_0(\overline{\Omega}):u(x)\geq 0\text{ for all } x\in\overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C^1_0(\overline{\Omega})_+\right)=\left\{u\in C^1_0(\overline{\Omega})_+:u(x)>0\text{ for all } x\in\Omega\text{ and }\frac{\partial u}{\partial n}\bigg|_{\partial\Omega}<0\right\},$$

where n is the outward unit normal on $\partial\Omega$.

The norm of $\mathbb{R}^N$ is denoted by $|\cdot|$ and "$\cdot$" stands for the inner product in $\mathbb{R}^N$. For $s\in\mathbb{R}$, we set $s^\pm=\max\{\pm s,0\}$ and for $u\in W^{1,p}_0(\Omega)$ we define $u^\pm(\cdot)=u(\cdot)^\pm$. It is well known that

$$u^\pm\in W^{1,p}_0(\Omega),\quad |u|=u^++u^-,\quad u=u^+-u^-.$$

For $u,v\in W^{1,p}_0(\Omega)$ with $u(x)\leq v(x)$ for a.a. $x\in\Omega$ we define

$$[u,v]=\left\{y\in W^{1,p}_0(\Omega):u(x)\leq y(x)\leq v(x)\text{ for a.a. } x\in\Omega\right\},$$

$$\operatorname*{int}_{C^1_0(\overline{\Omega})}[u,v]=\text{the interior in } C^1_0(\overline{\Omega})\text{ of } [u,v]\cap C^1_0(\overline{\Omega}),$$

$$[u)=\left\{y\in W^{1,p}_0(\Omega):u(x)\leq y(x)\text{ for a.a. } x\in\Omega\right\}.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. By $p^* > 1$ we denote the Sobolev critical exponent for p defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$$

Finally, if $h_1, h_2 \in L^\infty(\Omega)$, then we write $h_1 \prec h_2$ if and only if for every compact $K \subseteq \Omega$ we have $0 < m_K \leq h_2(x) - h_1(x)$ for a.a. $x \in K$.

Let $A : W_0^{1,p}(\Omega) \to W^{-1.p'}(\Omega) = W_0^{1,p}(\Omega)^*$ with $\frac{1}{p} + \frac{1}{p'} = 1$ be defined by

$$\langle A(u), \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx \quad \text{for all } u, \varphi \in W_0^{1,p}(\Omega). \tag{2.1}$$

The next proposition states the main properties of this map and it can be found in Gasiński-Papageorgiou [4, Problem 2.192, p. 279].

Proposition 2.2. *The map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined in (2.1) is bounded, that is, it maps bounded sets to bounded sets, continuous, strictly monotone, hence maximal monotone and it is of type* $(\mathrm{S})_+$*, that is,*

$$u_n \overset{\mathrm{w}}{\to} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

imply $u_n \to u$ in $W_0^{1,p}(\Omega)$.

Moreover, we denote by $\hat{\lambda}_1$ the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and by $\hat{u}_1 \in W_0^{1,p}(\Omega)$ the corresponding positive, L^p-normalized, that is, $\|\hat{u}_1\|_p = 1$, eigenfunction. We know that $\hat{\lambda}_1 > 0$ and $\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Gasiński-Papageorgiou [3].

Also, for a given $\varphi \in C^1(X, \mathbb{R})$ we denote by K_φ the critical set of φ, that is, $K_\varphi = \{u \in X : \varphi'(u) = 0\}$.

Now we introduce the hypotheses on the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.a. $x \in \Omega$ and

(i) if $a \in L^s(\Omega)$ with $s > N$, then

$$0 < f(x, s) \leq a(x) \left(1 + s^{r-1}\right)$$

for a.a. $x \in \Omega$, for all $s > 0$ and for $p < r < p^*$;

(ii) if $F(x, s) = \int_0^s f(x, t) dt$, then

$$\lim_{s\to+\infty} \frac{F(x, s)}{s^p} = +\infty \quad \text{uniformly for a.a. } x \in \Omega;$$

(iii) if

$$\hat{\eta}_\lambda(x, s) = \left[1 - \frac{p}{1-\gamma}\right] s^{1-\gamma} + \lambda \left[f(x, s)s - pF(x, s)\right]$$

with $\lambda > 0$, then

$$\hat{\eta}_\lambda(x, s_1) \leq \hat{\eta}_\lambda(x, s_2) + \tau_\lambda(x)$$

for a.a. $x \in \Omega$, for all $0 \leq s_1 \leq s_2$ with $\tau_\lambda \in L^1(\Omega)$ and $\lambda \to \tau_\lambda$ is nondecreasing from $(0, +\infty)$ into $L^1(\Omega)$;

(iv) there exist $c_1 > 0$ and $q \leq p$ such that
$$f(x,s) \leq c_1 \left[s^{r-1} + s^{q-1}\right]$$
for a.a. $x \in \Omega$ and for all $s \geq 0$;

(v) for every $\eta > 0$ there exists $m_\eta > 0$ such that
$$f(x,s) \geq m_\eta$$
for a.a. $x \in \Omega$ and for all $s \geq \eta$;

(vi) for every $\rho > 0$ there exists $\hat{\xi}_\rho > 0$ such that the function
$$s \to f(x,s) + \hat{\xi}_\rho s^{p-1}$$
is nondecreasing on $[0,\rho]$ for a.a. $x \in \Omega$.

Remark 2.3. *Since we are interested on positive solutions and the hypotheses above concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality, we may assume that*
$$f(x,s) = 0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \leq 0. \tag{2.2}$$
Hypotheses H(ii), H(iii) imply that
$$\lim_{s \to +\infty} \frac{f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$
Hence, the perturbation term in (P_λ) is $(p-1)$-superlinear in the second variable. However, we do not employ the usual AR-condition for superlinear problems. Recall that this condition says that there exist $\tau > p$ and $M > 0$ such that
$$0 < \tau F(x,s) \leq f(x,s)s \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M, \tag{2.3}$$
$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M). \tag{2.4}$$
In fact this is a unilateral version of the AR-condition on account of (2.2). Integrating (2.3) and using (2.4) we obtain the weaker condition
$$c_2 s^\tau \leq F(x,s) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \geq M \text{ and for some } c_2 > 0.$$
Hence, the AR-condition implies that $f(x,\cdot)$ exhibits at least $(\tau-1)$-polynomial growth. This excludes superlinear nonlinearities with slower growth near $+\infty$ from consideration. Instead we employ the quasimonotonicity condition on $\eta_\lambda(x,\cdot)$ in hypothesis H(iii). This condition is a slight generalization of a hypothesis introduced by Li-Yang [11]. This superlinearity hypothesis is different from the one used by Papageorgiou-Smyrlis [15]. There are easy ways to verify H(iii). For example, condition H(iii) holds if there exists $M > 0$ such that
$$s \to \frac{s^{-\gamma} + \lambda f(x,s)}{s^{p-1}}$$
is nondecreasing on $[M,+\infty)$ for a.a. $x \in \Omega$ or
$$s \to \hat{\eta}_\lambda(x,s)$$
is nondecreasing on $[M,+\infty)$, see Li-Yang [11].

Hypothesis H(iv) allows perturbations which have concave terms. This is excluded from the hypotheses of Papageorgiou-Smyrlis [15]. Hypothesis H(iv) is satisfied if, *for example, $f(x,\cdot)$ is differentiable for a.a. $x \in \Omega$ and for every $\rho > 0$ there exists $c_\rho > 0$ such that*

$$f'_s(x,s) \geq -c_\rho s^{p-1}$$

for a.a. $x \in \Omega$ and for all $0 \leq s \leq \rho$.

Example 2.4. *For the sake of simplicity we drop the x-dependence. The following functions satisfy hypotheses H:*

$$f_1(s) = s^{\tau-1} \text{ with } p < \tau < p^*,$$

$$f_2(s) = \begin{cases} (s^+)^{\vartheta-1} & \text{if } s \leq 1, \\ s^{p-1}[\ln s + 1] & \text{if } 1 < s \end{cases} \quad \text{with } 1 < \vartheta < p < \infty.$$

Note that f_2 fails to satisfy the AR-condition and it is outside the framework of Papageorgiou-Smyrlis [15].

3. Positive Solutions

We introduce the following two sets

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (\mathrm{P}_\lambda) \text{ has a positive solution}\},$$

$$\mathcal{S}_\lambda = \{u : u \text{ is a positive solution of problem } (\mathrm{P}_\lambda)\}.$$

Proposition 3.1. *If hypotheses H hold, then $\mathcal{L} \neq \emptyset$.*

Proof. We consider the following purely singular Dirichlet problem

$$-\Delta_p u = u^{-\gamma} \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0, \quad u > 0. \tag{3.1}$$

From Papageorgiou-Smyrlis [15, Proposition 5] we know that problem (3.1) has a unique positive solution $\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Moreover, we consider the following auxiliary Dirichlet problem

$$-\Delta_p u = 1 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{3.2}$$

Problem (3.2) has a unique solution $e \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ which can be shown easily. For $1 < \tau < +\infty$, we have $e^\tau \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and using Proposition 2.1 of Marano-Papageorgiou [12], see also Gasiński-Papageorgiou [4, Problem 4.180, p. 680], there exists $c_3 > 0$ such that $\hat{u}_1 \leq c_3 e^\tau$ and so

$$\hat{u}_1^{\frac{1}{\tau}} \leq c_3^{\frac{1}{\tau}} e,$$

which implies

$$e^{-\gamma} \leq c_4 \hat{u}_1^{-\frac{\gamma}{\tau}} \tag{3.3}$$

for some $c_4 > 0$. From the Lemma in Lazer-McKenna [9] we know that

$$\hat{u}_1^{-\frac{\gamma}{\tau}} \in L^\tau(\Omega).$$

This fact along with (3.3) gives

$$e^{-\gamma} \in L^\tau(\Omega) \quad \text{and} \quad \left\|e^{-\gamma}\right\|_\tau \leq c_4 \left\|\hat{u}_1^{-\gamma}\right\|_1^{\frac{1}{\tau}}.$$

Hence

$$\limsup_{\tau \to +\infty} \left\|e^{-\gamma}\right\|_\tau \leq c_4. \tag{3.4}$$

On the other hand, from the Chebyshev inequality, we have

$$\eta^\tau \left|\left\{e^{-\gamma} \geq \eta\right\}\right|_N \leq \left\|e^{-\gamma}\right\|_\tau^\tau$$

with $\eta > 0$, or equivalently,

$$\eta \left|\left\{e^{-\gamma} \geq \eta\right\}\right|_N^{\frac{1}{\tau}} \leq \left\|e^{-\gamma}\right\|_\tau .$$

This facts yields

$$\eta \leq \liminf_{\tau \to +\infty} \left\|e^{-\gamma}\right\|_\tau \quad \text{provided} \quad \left|\left\{e^{-\gamma} \geq \eta\right\}\right|_N > 0. \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$e^{-\gamma} \in L^\infty(\Omega) \quad \text{and} \quad \left\|e^{-\gamma}\right\|_\tau \to \left\|e^{-\gamma}\right\|_\infty \quad \text{as } \tau \to +\infty.$$

Now let $c_5 > \|e^{-\gamma}\|_\infty$ and $m_0 = \|e\|_\infty$. For $t > 0$ we consider the function

$$\begin{aligned}
\vartheta(t) &= \frac{t^{p-1} - c_5 t^{-\gamma}}{c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1}\right]}\\
&= \frac{t^{p+\gamma-1} - c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1}\right]}\\
&= \frac{1}{c_1 \left[m_0^{r-1} t^{r-p} + m_0^{q-1} t^{q-p}\right]} - \frac{c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1}\right]}\\
&= \frac{t^{p-q}}{c_1 \left[m_0^{r-1} t^{r-q} + m_0^{q-1}\right]} - \frac{c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1}\right]}.
\end{aligned}$$

Since $q \leq p < r$ we see that

$$\vartheta(t) \to -\infty \text{ as } t \to 0^+ \quad \text{and} \quad \vartheta(t) \to 0^+ \text{ as } t \to +\infty.$$

Therefore, there exists $t_0 > 0$ such that

$$\lambda_0 = \vartheta(t_0) = \max\left[\vartheta(t) : t > 0\right] > 0.$$

Let $\lambda \in (0, \lambda_0)$. We can find $t > 0$ such that $\vartheta(t) \geq \lambda$. Hence

$$t^{p-1} \geq c_5 t^{-\gamma} + \lambda c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1}\right]. \tag{3.6}$$

We set $\overline{u} = te \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Then, because of (3.6), hypothesis H(iv) and the choice of c_5, m_0, we obtain

$$\begin{aligned}
-\Delta_p \overline{u} &= t^{p-1}\left[-\Delta_p e\right]\\
&= t^{p-1}\\
&\geq c_5 t^{-\gamma} + \lambda c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1}\right]\\
&\geq \overline{u}^{-\gamma} + \lambda c_1 \left[\overline{u}^{r-1} + \overline{u}^{q-1}\right]\\
&\geq \overline{u}^{-\gamma} + \lambda f(x, \overline{u}) \quad \text{for a.a. } x \in \Omega.
\end{aligned} \tag{3.7}$$

Since $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, as before, there exists $\vartheta \in (0,1)$ small enough such that $\vartheta \tilde{u} \leq \overline{u}$. If $\tilde{u}_0 = \vartheta \tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, then

$$-\Delta_p \tilde{u}_0 = -\Delta_p\left(\vartheta \tilde{u}\right) = \vartheta^{p-1}\left(-\Delta_p \tilde{u}\right) = \vartheta^{p-1} \tilde{u}^{-\gamma} \leq \left(\vartheta \tilde{u}\right)^{-\gamma} = \tilde{u}_0^{-\gamma} \tag{3.8}$$

since $\vartheta \in (0,1)$. Using the functions $\tilde{u}_0, \overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, we introduce the following truncation of the reaction of problem (P_λ)

$$g_\lambda(x,s)=\begin{cases}\tilde{u}_0(x)^{-\gamma}+\lambda f\left(x,\tilde{u}_0(x)\right) & \text{if } s<\tilde{u}_0(x),\\ s^{-\gamma}+\lambda f(x,s) & \text{if } \tilde{u}_0(x)\leq s\leq \overline{u}(x),\\ \overline{u}(x)^{-\gamma}+\lambda f\left(x,\overline{u}(x)\right) & \text{if } \overline{u}(x)<s,\end{cases} \tag{3.9}$$

with $\lambda \in (0,\lambda_0)$. Evidently, $g_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We set $G_\lambda(x,s)=\int_0^s g_\lambda(x,t)dt$ and consider the functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u)=\frac{1}{p}\|\nabla u\|_p^p-\int_\Omega G_\lambda(x,u)dx.$$

On account of Proposition 3 of Papageorgiou-Smyrlis [15] we have that $\psi_\lambda \in C^1(W_0^{1,p}(\Omega))$. Moreover, from (3.9) it is clear that ψ_λ is coercive. The Sobolev embedding theorem implies that ψ_λ is sequentially weakly lower semicontinuous. So, by the Weierstraß-Tonelli theorem, there exists $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_\lambda)=\inf\left[\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)\right].$$

Since u_λ is a global minimizer, it fulfills $\psi_\lambda'(u_\lambda)=0$, which is equivalent to

$$\langle A(u_\lambda),h\rangle=\int_\Omega g_\lambda(x,u_\lambda)hdx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.10}$$

Taking $h=(\tilde{u}_0-u_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.10) gives, thanks to (3.9), (3.8) and the fact that $f \geq 0$,

$$\begin{aligned}\left\langle A(u_\lambda),(\tilde{u}_0-u_\lambda)^+\right\rangle&=\int_\Omega\left[\tilde{u}_0^{-\gamma}+\lambda f\left(x,\tilde{u}_0\right)\right](\tilde{u}_0-u_\lambda)^+dx\\&\geq\int_\Omega \tilde{u}_0^{-\gamma}\left(\tilde{u}_0-u_\lambda\right)^+dx\\&\geq\left\langle A\left(\tilde{u}_0\right),\left(\tilde{u}_0-u_\lambda\right)^+\right\rangle.\end{aligned}$$

Because of the monotonicity of A, see Proposition 2.2, we obtain that $\tilde{u}_0 \leq u_\lambda$. Next, we choose $h=(u_\lambda-\overline{u})^+ \in W_0^{1,p}(\Omega)$ in (3.10). This gives, by applying (3.9) and (3.7), that

$$\left\langle A\left(u_\lambda\right),\left(u_\lambda-\overline{u}\right)^+\right\rangle=\int_\Omega\left[\overline{u}^{-\gamma}+\lambda f\left(x,\overline{u}\right)\right]\left(u_\lambda-\overline{u}\right)^+dx\leq\left\langle A\left(\overline{u}\right),\left(u_\lambda-\overline{u}\right)^+\right\rangle.$$

As before, by applying Proposition 2.2, it follows that $u_\lambda \leq \overline{u}$. So, we have proved that

$$u_\lambda \in \left[\tilde{u}_0,\overline{u}\right]. \tag{3.11}$$

From (3.9), (3.10), (3.11), it follows that

$$\langle A(u_\lambda),h\rangle=\int_\Omega\left[u_\lambda^{-\gamma}+\lambda f(x,u_\lambda)\right]hdx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.12}$$

Since $\tilde{u}_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, as before, we have that $\tilde{u}_0^{-\gamma} \in L^s(\Omega)$ for $s>N$ and since $0 \leq u_\lambda^{-\gamma} \leq \tilde{u}_0^{-\gamma}$, see (3.11), one has that $u_\lambda^{-\gamma} \in L^s(\Omega)$. From (3.12) it follows that

$$-\Delta_p u_\lambda(x)=u_\lambda(x)^{-\gamma}+\lambda f(x,u_\lambda(x)) \quad \text{for a.a. } x \in \Omega, \quad u_\lambda\big|_{\partial\Omega}=0. \tag{3.13}$$

From (3.13) and Proposition 1.3 of Guedda-Véron [7] we have that $u_\lambda \in L^\infty(\Omega)$. Let $\xi_\lambda(x) = u_\lambda(x)^{-\gamma} + \lambda f(x, u_\lambda(x))$. Then $\xi_\lambda \in L^s(\Omega)$, see hypothesis H(i). We consider now the following linear Dirichlet problem

$$-\Delta v = \xi_\lambda \quad \text{in } \Omega, \quad v\big|_{\partial\Omega}.$$

This problem has a unique solution v_λ which by the linear regularity theory belongs to $W^{2,s}(\Omega)$, see Gilbarg-Trudinger [6, Theorem 9.9, p. 230]. Then, since $s > N$, the Sobolev embedding theorem implies that

$$v_\lambda \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{with} \quad \alpha = 1 - \frac{N}{s}. \tag{3.14}$$

We set $k_\lambda(x) = \nabla v_\lambda(x)$. Then $k_\lambda \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$, see (3.14). From (3.13) we obtain

$$-\operatorname{div}\left(|\nabla u_\lambda(x)|^{p-2}\nabla u_\lambda(x) - k_\lambda(x)\right) = 0 \quad \text{for a.a. } x \in \Omega, \quad u_\lambda\big|_{\partial\Omega} = 0.$$

Invoking Theorem 1 of Lieberman [10], we infer that $u_\lambda \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Finally from (3.13) and the nonlinear maximum principle, see for example, Gasinski-Papageorgiou [3, Theorem 6.2.8, p. 738] and Pucci-Serrin [17, p. 120], we conclude that $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. It follows that $(0, \lambda_0) \subseteq \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$. □

From the proof above we infer the following corollary.

Corollary 3.2. *If hypotheses H hold and $\lambda \in \mathcal{L}$, then $\mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

In the next proposition we show that $\mathcal{L}$ is in fact an interval.

Proposition 3.3. *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$ there exists $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Corollary 3.2. Since $\mu < \lambda$ and $f \geq 0$, we have

$$-\Delta_p u_\lambda(x) = u_\lambda(x)^{-\gamma} + \lambda f(x, u_\lambda(x)) \geq u_\lambda(x)^{-\gamma} + \mu f(x, u_\lambda(x))$$

for a.a. $x \in \Omega$. Recall that $\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique solution of (3.1). Since $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ there exists $t \in (0,1)$ small enough such that $t\tilde{u} \leq u_\lambda$. We set $\tilde{u}_* = t\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and introduce the following truncation nonlinearity

$$\hat{g}_\mu(x,s) = \begin{cases} \tilde{u}_*(x)^{-\gamma} + \mu f\left(x, \tilde{u}_*(x)\right) & \text{if } s < \tilde{u}_*(x), \\ s^{-\gamma} + \mu f(x,s) & \text{if } \tilde{u}_*(x) \leq s \leq u_\lambda(x), \\ u_\lambda(x)^{-\gamma} + \mu f(x, u_\lambda(x)) & \text{if } u_\lambda(x) < s, \end{cases} \tag{3.15}$$

which is a Carathéodory function. We set $\hat{G}_\mu(x,s) = \int_0^s \hat{g}_\mu(x,t)dt$ and consider the functional $\hat{\psi}_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\mu(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \hat{G}_\mu(x,u)dx.$$

As before, we have $\hat{\psi}_\mu \in C^1(W_0^{1,p}(\Omega))$, see Papageorgiou-Smyrlis [15, Proposition 3]. From (3.15) it is clear that $\hat{\psi}_\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstraß-Tonelli theorem there exists $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_\mu\left(u_\mu\right) = \inf\left[\hat{\psi}_\mu(u) : u \in W_0^{1,p}(\Omega)\right].$$

Hence, $\hat{\psi}'_\mu(u_\mu) = 0$ which is equivalent to

$$\langle A(u_\mu), h\rangle = \int_\Omega \hat{g}_\mu(x, u_\mu) h dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.16}$$

We choose $h = (\tilde{u}_* - u_\mu)^+ \in W_0^{1,p}(\Omega)$ in (3.16). Then, using (3.15), $f \geq 0$, (3.1) and $\tilde{u}_* = t\tilde{u}$ for $0 < t < 1$, we obtain

$$\begin{aligned}
\langle A(u_\mu), (\tilde{u}_* - u_\mu)^+\rangle &= \int_\Omega \left[\tilde{u}_*^{-\gamma} + \mu f(x, \tilde{u}_*)\right](\tilde{u}_* - u_\mu)^+ dx \\
&\geq \int_\Omega \tilde{u}_*^{-\gamma}(\tilde{u}_* - u_\mu)^+ dx \\
&\geq \left\langle A(\tilde{u}_*), (\tilde{u}_* - u_\mu)^+\right\rangle.
\end{aligned}$$

Hence, by Proposition 2.2, $\tilde{u}_* \leq u_\mu$. Next, we choose $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.16). Then, as before, by applying (3.15) and since $f \geq 0$, $\mu < \lambda$ and $u_\lambda \in \mathcal{S}_\lambda$ we obtain

$$\begin{aligned}
\left\langle A(u_\mu), (u_\mu - u_\lambda)^+\right\rangle &= \int_\Omega \left[u_\lambda^{-\gamma} + \mu f(x, u_\mu)\right](u_\mu - u_\lambda)^+ dx \\
&\leq \left[u_\lambda^{-\gamma} + \lambda f(x, u_\lambda)\right](u_\mu - u_\lambda)^+ dx \\
&= \left\langle A(u_\lambda), (u_\mu - u_\lambda)^+\right\rangle.
\end{aligned}$$

Using Proposition 2.2 we see that $u_\mu \leq u_\lambda$.

So, we have proved that

$$u_\mu \in [\tilde{u}_*, u_\lambda]. \tag{3.17}$$

From (3.15), (3.16) and (3.17) we infer that $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so $\mu \in \mathcal{L}$. ☐

A useful byproduct of the proof above is the following corollary.

Corollary 3.4. *If hypotheses H hold,* $0 < \mu < \lambda \in \mathcal{L}$ *and* $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, *then* $\mu \in \mathcal{L}$ *and there exists* $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *such that* $u_\lambda - u_\mu \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$.

In fact using hypotheses H(v), (vi) we can improve the conclusion of the corollary above.

Proposition 3.5. *If hypotheses H hold,* $0 < \mu < \lambda \in \mathcal{L}$ *and if* $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, *then* $\mu \in \mathcal{L}$ *and there exists* $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *such that* $u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Proof. From Corollary 3.4 we already know that $\mu \in \mathcal{L}$ and we can find $u_\lambda \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_\lambda - u_\mu \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(vi). Since $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), u_\lambda \in \mathcal{S}_\lambda \subseteq$

int $\left(C_0^1(\overline{\Omega})_+\right)$, $u_\mu \leq u_\lambda$ and because of hypotheses H(v), (vi) we derive

$$\begin{aligned}
&-\Delta_p u_\mu(x) + \lambda\hat{\xi}_\rho u_\mu(x)^{p-1} - u_\mu(x)^{-\gamma}\\
&= \mu f(x, u_\mu(x)) + \lambda\hat{\xi}_\rho u_\mu(x)^{p-1}\\
&= \lambda f(x, u_\mu(x)) + \lambda\hat{\xi}_\rho u_\mu(x)^{p-1} - (\lambda-\mu) f(x, u_\mu(x))\\
&< \lambda f(x, u_\lambda(x)) + \lambda\hat{\xi}_\rho u_\lambda(x)^{p-1}\\
&= -\Delta_p u_\lambda(x) + \lambda\hat{\xi}_\rho u_\lambda^{p-1} - u_\lambda(x)^{-\gamma}
\end{aligned} \tag{3.18}$$

for a.a. $x \in \Omega$. Let $\hat{h}_0(x) = (\lambda-\mu) f(x, u_\mu(x))$. Since $u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and using hypothesis H(v), we see that $0 \prec \hat{h}_0$. Therefore, from (3.18) and the singular strong comparison principle, see Papageorgiou-Smyrlis [15, Proposition 4], we conclude that $u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

We set $\lambda^* = \sup \mathcal{L}$.

Proposition 3.6. *If hypotheses H hold, then $\lambda^* < \infty$.*

Proof. Recall that

$$\lim_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,$$

see hypotheses H(ii), (iii). Therefore, for a given $k > \hat{\lambda}_1$, there exists $M > 0$ such that

$$f(x,s) \geq k s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M. \tag{3.19}$$

On the other hand, we have

$$s^{-\gamma} + \lambda f(x,s) \geq M^{-\gamma} + \lambda f(x,s) \tag{3.20}$$

for a.a. $x \in \Omega$, for all $0 \leq s \leq M$ and for all $\lambda > 0$. Note that, since $f \geq 0$,

$$\lim_{s\to 0^+} \frac{M^{-\gamma} + \lambda f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,$$

which implies that there exists $\delta_\lambda > 0$ such that

$$M^{-\gamma} + \lambda f(x,s) \geq \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \delta_\lambda.$$

Combining this with (3.20) we see that

$$s^{-\gamma} + \lambda f(x,s) \geq \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \delta_\lambda. \tag{3.21}$$

Finally, note that on account of hypothesis H(v), there exists $\tilde{\lambda} \geq 1$ large enough such that

$$s^{-\gamma} + \tilde{\lambda} f(x,s) \geq M^{-\gamma} + \tilde{\lambda} m_{\delta_{\tilde{\lambda}}} \geq \hat{\lambda}_1 M^{p-1} \geq \hat{\lambda}_1 s^{p-1} \tag{3.22}$$

for a.a. $x \in \Omega$ and for all $\delta_{\tilde{\lambda}} \leq s \leq M$. Combining (3.19), (3.21), and (3.22) we conclude that

$$s^{-\gamma} + \tilde{\lambda} f(x,s) \geq \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq 0. \tag{3.23}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. There exists $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Let $t > 0$ be such that

$$t\hat{u}_1 \leq u_\lambda. \tag{3.24}$$

Assume that $t > 0$ is the largest positive real number for which (3.24) holds. Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(vi). Applying (3.24), hypothesis H(vi) and (3.23) gives

$$\begin{aligned}
&-\Delta_p u_\lambda(x) + \lambda\hat{\xi}_\rho u_\lambda(x)^{p-1} - u_\lambda(x)^{-\gamma}\\
&\quad = \lambda f\left(x, u_\lambda(x)\right) + \lambda\hat{\xi}_\rho u_\lambda(x)^{p-1}\\
&\quad \geq \lambda f\left(x, t\hat{u}_1(x)\right) + \lambda\hat{\xi}_\rho \left(t\hat{u}_1(x)\right)^{p-1}\\
&\quad = \tilde{\lambda} f\left(x, t\hat{u}_1(x)\right) + \lambda\hat{\xi}_\rho \left(t\hat{u}_1(x)\right)^{p-1} + \left(\lambda - \tilde{\lambda}\right) f\left(x, t\hat{u}_1(x)\right)\\
&\quad \geq \hat{\lambda}_1 \left(t\hat{u}_1(x)\right)^{p-1} + \lambda\hat{\xi}_\rho \left(t\hat{u}_1(x)\right)^{p-1}\\
&\quad \geq -\Delta_p \left(t\hat{u}_1(x)\right) + \lambda\hat{\xi}_\rho \left(t\hat{u}_1(x)\right)^{p-1} - \left(t\hat{u}_1(x)\right)^{-\gamma} \quad \text{for a.a. } x \in \Omega.
\end{aligned} \tag{3.25}$$

We set $\tilde{h}_0(x) = \left(\lambda - \tilde{\lambda}\right) f\left(x, t\hat{u}_1(x)\right)$. We see that since $\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and because of hypothesis H(v), we have $0 \prec \tilde{h}_0$. Therefore, from (3.25) and Papageorgiou-Smyrlis [15, Proposition 4] we infer that $u_\lambda - t\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ which contradicts the maximality of $t > 0$, see (3.24). This shows that $\lambda \notin \mathcal{L}$ and so $\lambda^* \leq \tilde{\lambda} < +\infty$. □

Next we show that the critical parameter $\lambda^* > 0$ is admissible.

Proposition 3.7. *If hypotheses H hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Consider a sequence $\{\lambda_n\}_{n\geq 1} \subseteq (0, \lambda^*) \subseteq \mathcal{L}$ such that $\lambda_n \to (\lambda^*)^-$ as $n \to \infty$. From the proof of Proposition 3.3 we know that there exists $u_n \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for each $n \in \mathbb{N}$ such that

$$\{u_n\}_{n\geq 1} \text{ is increasing and } \tilde{u}_* = t\tilde{u} \leq u_n \quad \text{for all } n \in \mathbb{N}. \tag{3.26}$$

Let $\hat{\psi}_{\lambda_n} \in C^1(W_0^{1,p}(\Omega))$ be as in the proof of Proposition 3.3 resulting from the truncation of the reaction of (P_λ) with λ replaced by λ_n at $\{\tilde{u}_*(x), u_{n+1}(x)\} = \{t\tilde{u}(x), u_{n+1}(x)\}$, see (3.15). We know that $u_n \in [\tilde{u}_*, u_{n+1}]$ is the minimizer of $\hat{\psi}_{\lambda_n}$. Therefore, because of (3.15) with $u_\lambda = u_{n+1}$ and hypothesis H(v), we have

$$\begin{aligned}
\hat{\psi}_{\lambda_n}(u_n) \leq \hat{\psi}_{\lambda_n}\left(\tilde{u}_*\right) &= \frac{1}{p}\left\|\nabla\tilde{u}_*\right\|_p^p - \int_\Omega \left[\tilde{u}^{1-\gamma} + \lambda_n f\left(x, \tilde{u}_*\right)\tilde{u}_*\right] dx\\
&= \frac{t^p}{p}\left\|\nabla\tilde{u}\right\|_p^p - t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma} dx - \lambda_n \int_\Omega f\left(x, \tilde{u}_*\right)\tilde{u}_* dx\\
&< \frac{t^p}{p}\left\|\nabla\tilde{u}\right\|_p^p - t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma} dx.
\end{aligned} \tag{3.27}$$

We know that

$$\left\|\nabla\tilde{u}\right\|_p^p = \int_\Omega \tilde{u}^{1-\gamma} dx,$$

see (3.27). Hence, since $t \in (0,1)$,

$$t^p \left\|\nabla\tilde{u}\right\|_p^p \leq t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma} dx.$$

This finally gives

$$\hat{\psi}_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}, \tag{3.28}$$

see (3.27).

Consider now the Carathéodory function $\tilde{g}_{\lambda_n} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{g}_{\lambda_n}(x,s) = \begin{cases} \tilde{u}_*(x)^{-\gamma} + \lambda_n f(x, \tilde{u}_*(x)) & \text{if } s \leq \tilde{u}_*(x), \\ s^{-\gamma} + \lambda_n f(x,s) & \text{if } \tilde{u}_*(x) < s. \end{cases} \tag{3.29}$$

We set $\tilde{G}_{\lambda_n}(x,s) = \int_0^s \tilde{g}_{\lambda_n}(x,t)dt$ and consider the C^1-functional $\tilde{\varphi}_{\lambda_n} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_{\lambda_n}(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \tilde{G}_{\lambda_n}(x,u)dx.$$

Note that

$$\tilde{\varphi}_{\lambda_n}\big|_{[\tilde{u}_*,u_{n+1}]} = \hat{\psi}_{\lambda_n}\big|_{[\tilde{u}_*,u_{n+1}]}.$$

Then, see (3.28), we have $\tilde{\varphi}_{\lambda_n}(u_n) < 0$ for all $n \in \mathbb{N}$ and so

$$\|\nabla u_n\|_p^p - \int_\Omega p\tilde{G}_{\lambda_n}(x,u_n)dx < 0.$$

Applying (3.29) and the fact that $u_n \in [\tilde{u}_*, u_{n+1}]$ leads to

$$\begin{aligned} &\|\nabla u_n\|_p^p - \int_\Omega p\left[\tilde{u}_*^{1-\gamma} + \lambda_n f(x,\tilde{u}_*)\right]\tilde{u}_* dx \\ &- \frac{p}{1-\gamma}\int_\Omega \left[u_n^{1-\gamma} - u_*^{1-\gamma}\right] - \lambda_n p \int_\Omega \left[F(x,u_n) - F(x,\tilde{u}_*)\right]dx < 0. \end{aligned} \tag{3.30}$$

Moreover, we know that

$$\langle A(u_n), h\rangle = \int_\Omega \tilde{g}_{\lambda_n}(x,u_n)hdx \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ and for all } n \in \mathbb{N}. \tag{3.31}$$

Choosing $h = u_n \in W_0^{1,p}(\Omega)$ in (3.31) and applying (3.29) and the fact that $u_n \in [\tilde{u}_*, u_{n+1}]$ yields

$$-\|\nabla u_n\|_p^p + \int_\Omega \left[u_n^{1-\gamma} + \lambda_n f(x,u_n)u_n\right]dx = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.32}$$

Adding (3.30) and (3.32) we obtain

$$\int_\Omega \hat{\eta}_{\lambda_n}(x,u_n)dx \leq M_1 \quad \text{for some } M_1 > 0 \text{ and for all } n \in \mathbb{N}. \tag{3.33}$$

Suppose that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. By passing to a subsequence if necessary, we may assume that $\|u_n\| \to +\infty$. We set $y_n = \frac{u_n}{\|u_n\|}$ for $n \in \mathbb{N}$. Then we have $\|y_n\| = 1$ and $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \overset{\text{w}}{\to} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^r(\Omega), \quad \text{with } y \geq 0. \tag{3.34}$$

First assume that $y \neq 0$ and set $\Omega^* = \{x \in \Omega : \ y(x) > 0\}$. We have $|\Omega^*|_N > 0$ and $u_n(x) \to +\infty$ for all $x \in \Omega^*$. We have

$$\frac{F(x,u_n(x))}{\|u_n\|^p} = \frac{F(x,u_n(x))}{u_n(x)^p} y_n(x)^p \to +\infty \quad \text{for a.a. } x \in \Omega^*$$

and so, by Fatou's Lemma,

$$\int_{\Omega^*} \frac{F(x,u_n)}{\|u_n\|^p}dx \to +\infty. \tag{3.35}$$

Since $F \geq 0$, we have

$$\int_{\Omega^*} \frac{F(x,u_n)}{\|u_n\|^p} dx \leq \int_{\Omega} \frac{F(x,u_n)}{\|u_n\|^p} dx$$

and so, by (3.35),

$$\int_{\Omega} \frac{F(x,u_n)}{\|u_n\|^p} dx \to +\infty. \tag{3.36}$$

Hypothesis H(iii) implies that

$$0 \leq \hat{\eta}_{\lambda_n}(x,u_n(x)) + \tau_{\lambda^*}(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } n \in \mathbb{N}.$$

Then

$$\frac{p}{1-\gamma} u_n(x)^{1-\gamma} + pF(x,u_n(x)) \leq u_n(x)^{1-\gamma} + \lambda_n f(x,u_n(x))u_n(x) + \tau_{\lambda^*}(x) \tag{3.37}$$

for a.a. $x \in \Omega$ and for all $n \in \mathbb{N}$.

From (3.31) with $h = u_n \in W_0^{1,p}(\Omega)$ we obtain by using (3.29) and (3.26)

$$\|\nabla u_n\|_p^p = \int_\Omega \left[u_n^{1-\gamma} + \lambda_n f(x,u_n)u_n\right] dx \quad \text{for all } n \in \mathbb{N}. \tag{3.38}$$

Applying (3.38) in (3.37) gives

$$p\lambda_n \int_\Omega F(x,u_n)dx \leq \|\nabla u_n\|_p^p + \|\tau_{\lambda^*}\|_1 .$$

Hence

$$p\lambda_n \int_\Omega \frac{F(x,u_n)}{\|u_n\|^p} dx \leq \|\nabla y_n\|_p^p + \frac{\|\tau_{\lambda^*}\|_1}{\|u_n\|^p} \quad \text{for all } n \in \mathbb{N}. \tag{3.39}$$

Comparing (3.36) and (3.39) we have a contradiction.

Next suppose that $y = 0$. For $\mu > 0$ we set $v_n = (p\mu)^{\frac{1}{p}} y_n$ for all $n \in \mathbb{N}$. Then $v_n \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $v_n \to 0$ in $L^r(\Omega)$, see (3.34) and recall that $y = 0$. Then, by (3.29), we get

$$\int_\Omega \tilde{G}_{\lambda_n}(x,v_n)dx \to 0 \quad \text{as } n \to \infty. \tag{3.40}$$

Since $\|u_n\| \to +\infty$, there exists a number $n_0 \in \mathbb{N}$ such that

$$(p\mu)^{\frac{1}{p}} \frac{1}{\|u_n\|} \leq 1 \quad \text{for all } n \geq n_0. \tag{3.41}$$

Moreover, let $t_n \in [0,1]$ be such that

$$\tilde{\varphi}_{\lambda_n}(t_n u_n) = \max_{0 \leq t \leq 1} \tilde{\varphi}_{\lambda_n}(tu_n), \quad n \in \mathbb{N}.$$

Applying (3.41), the representation $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and (3.40) leads to

$$\begin{aligned} \tilde{\varphi}_{\lambda_n}(t_n u_n) &\geq \tilde{\varphi}_{\lambda_n}(v_n) \quad \text{for all } n \geq n_0 \\ &= \mu \|\nabla y_n\|_p^p - \int_\Omega \tilde{G}_{\lambda_n}(x,v_n)dx \\ &= \mu - \int_\Omega \tilde{G}(x,v_n)dx \geq \frac{\mu}{2} \quad \text{for all } n \geq n_1 \geq n_0. \end{aligned} \tag{3.42}$$

But recall that $\mu > 0$ is arbitrary. So, from (3.42) we infer that

$$\tilde{\varphi}_{\lambda_n}(t_n u_n) \to +\infty \quad \text{as } n \to \infty. \tag{3.43}$$

We have

$$\tilde{\varphi}_{\lambda_n}(0) = 0 \quad \text{and} \quad \tilde{\varphi}_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}.$$

From this and (3.43) it follows that $t_n \in (0,1)$ for all $n \geq n_2$. Therefore, we obtain

$$\frac{d}{dt}\tilde{\varphi}_{\lambda_n}(tu_n)\big|_{t=t_0} = 0 \quad \text{for all } n \geq n_2$$

which means

$$\|\nabla(t_n u_n)\|_p^p = \int_\Omega \tilde{g}_{\lambda_n}(x, t_n u_n) u_n dx$$

and so

$$p\tilde{\varphi}_{\lambda_n}(t_n u_n) + p\int_\Omega \tilde{G}_{\lambda_n}(x, t_n u_n)dx = \int_\Omega \tilde{g}_{\lambda_n}(x, t_n u_n)(t_n u_n)dx.$$

Then we use hypothesis H(iii), (3.29) and recall that $t_n \in (0,1)$ for all $n \geq n_2$ to get

$$p\tilde{\varphi}_{\lambda_n}(t_n u_n) \leq \int_\Omega \hat{\eta}_{\lambda_n}(x, u_n)dx + M_2$$

for some $M_2 > 0$ and for all $n \geq n_2$. Taking (3.43) into account gives

$$\int_\Omega \hat{\eta}_{\lambda_n}(x, u_n)dx \to +\infty \quad \text{as } n \to \infty.$$

But this last convergence contradicts (3.33).

It follows that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n \overset{\text{w}}{\to} u^* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u^* \quad \text{in } L^r(\Omega) \quad \text{with } u^* \geq \tilde{u}_*. \tag{3.44}$$

Choosing $h = u_n - u^* \in W_0^{1,p}(\Omega)$ in (3.31), recalling that $u_n^{-\gamma} \in L^{r'}(\Omega)$ with $\frac{1}{r} + \frac{1}{r'} = 1$, passing to the limit as $n \to \infty$ and applying (3.44) results in

$$\lim_{n\to\infty} \langle A(u_n), u_n - u^* \rangle = 0.$$

Since A has the $(\mathrm{S})_+$-property, see Proposition 2.2, we infer that

$$u_n \to u^* \quad \text{in } W_0^{1,p}(\Omega). \tag{3.45}$$

So, if we pass to the limit in (3.31) and apply (3.45), then we obtain

$$\langle A(u^*), h\rangle = \int_\Omega \tilde{g}_{\lambda^*}(x, u^*) h dx \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } u^* \geq \tilde{u}_*.$$

Therefore, we have

$$\langle A(u^*), h\rangle = \int_\Omega \left[(u^*)^{-\gamma} + \lambda^* f(x, u^*)\right] h dx \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Hence, $u^* \in \mathcal{S}_{\lambda^*} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\lambda^* \in \mathcal{L}$. □

In summary, we have proved that

$$\mathcal{L} = (0, \lambda^*].$$

Next we show that we have two solutions for all $\lambda \in (0, \lambda^*)$.

Proposition 3.8. *If hypotheses H hold and $0 < \lambda < \lambda^*$, then problem (P_λ) has two positive solutions $u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

Proof. From Proposition 3.7 we know that $\lambda^* \in \mathcal{L}$. So, there exists $u^* \in \mathcal{S}_{\lambda^*} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Corollary 3.2. According to Proposition 3.5 we can find $u_0 \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that

$$u^* - u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.46}$$

Moreover, let $\vartheta \in (0,\lambda) \subseteq \mathcal{L}$ and $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ be such that

$$u_0 - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \tag{3.47}$$

again by Proposition 3.5. From (3.46) and (3.47) it follows that

$$u_0 \in \operatorname*{int}_{C_0^1(\overline{\Omega})} [u_\vartheta, u^*]. \tag{3.48}$$

We consider the Carathéodory functions $k_\lambda, \hat{k}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_\lambda(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} + \lambda f(x, u_\vartheta(x)) & \text{if } s \leq u_\vartheta(x), \\ s^{-\gamma} + \lambda f(x,s) & \text{if } u_\vartheta(x) < s \end{cases} \tag{3.49}$$

and

$$\hat{k}_\lambda(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} + \lambda f(x, u_\vartheta(x)) & \text{if } s < u_\vartheta(x), \\ s^{-\gamma} + \lambda f(x,s) & \text{if } u_\vartheta(x) \leq s \leq u^*(x), \\ u^*(x)^{-\gamma} + \lambda f(x, u^*(x)) & \text{if } u^*(x) < s. \end{cases} \tag{3.50}$$

We set $K_\lambda(x,s) = \int_0^s k_\lambda(x,t)dt$, $\hat{K}_\lambda(x,s) = \int_0^s \hat{k}_\lambda(x,t)dt$ and consider the C^1-functionals $\sigma_\lambda, \hat{\sigma}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \sigma_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega K_\lambda(x,u)dx, \\ \hat{\sigma}_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \hat{K}_\lambda(x,u)dx. \end{aligned}$$

From (3.49) and (3.50) it is clear that

$$\sigma_\lambda\big|_{[u_\vartheta,u^*]} = \hat{\sigma}_\lambda\big|_{[u_\vartheta,u^*]}. \tag{3.51}$$

Moreover, as in the proof of Proposition 3.1, using (3.49) and (3.50), we show that

$$K_{\sigma_\lambda} \subseteq [u_\vartheta) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad K_{\hat{\sigma}_\lambda} \subseteq [u_\vartheta, u_\lambda] \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.52}$$

From (3.52) we see that we may assume that $K_{\hat{\sigma}_\lambda} = \{u_0\}$, otherwise we already have a second positive solution for problem (P_λ), see (3.50) and (3.52).

From (3.50) and since $u_\vartheta^{-\gamma} \in L^{p'}(\Omega)$ we infer that $\hat{\sigma}_\lambda$ is coercive and from the Sobolev embedding theorem, we know that $\hat{\sigma}_\lambda$ is sequentially weakly lower semicontinuous. Therefore, we can find $u_0^* \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma}_\lambda(u_0^*) = \inf\left[\hat{\sigma}_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.53}$$

That means $u_0^* \in K_{\hat{\sigma}_\lambda}$ and so $u_0^* = u_0$. From (3.48), (3.51) and (3.53) it follows that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of σ_λ and from [5] and [13] we know that

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \sigma_\lambda. \tag{3.54}$$

We assume that K_{σ_λ} is finite or otherwise, on account of (3.49) and (3.52), we already have an infinity of positive smooth solutions for problem (P_λ) and so we are done. From (3.54) we infer that there exists $\rho \in (0,1)$ small enough such that

$$\sigma_\lambda(u_0) < \inf\left[\sigma_\lambda(u) : \|u-u_0\| = \rho\right] = m_\lambda, \tag{3.55}$$

see Aizicovici-Papageorgiou-Staicu [1, Proof of Proposition 29].

Hypothesis H(ii) implies that if $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, then

$$\sigma_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.56}$$

Claim: σ_λ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\sigma_\lambda(u_n)| \leq M_3 \quad \text{for some } M_3 > 0 \text{ and for all } n \in \mathbb{N}, \tag{3.57}$$

$$(1+\|u_n\|)\,\sigma_\lambda'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.58}$$

From (3.58) we have

$$\left|\langle A(u_n), h\rangle - \int_\Omega k_\lambda(x,u_n)h\,dx\right| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{3.59}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. We choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ in (3.59) and use (3.49) to obtain

$$\left\|\nabla u_n^-\right\|_p^p \leq c_6\left\|u_n^-\right\| \quad \text{for some } c_6 > 0 \text{ and for all } n \in \mathbb{N}.$$

Hence

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.60}$$

Then from (3.57) and (3.60) it follows that

$$\left\|\nabla u_n^+\right\|_p^p - \int_\Omega p\hat{K}_\lambda(x,u_n^+)dx \leq M_4 \quad \text{for some } M_4 > 0 \text{ and for all } n \in \mathbb{N}.$$

This implies

$$\begin{aligned}
&\left\|\nabla u_n^+\right\|_p^p - \int_{\left\{u_n^+ \leq u_\vartheta\right\}} p\left[u_\vartheta^{-\gamma} + \lambda f(x,u_\vartheta)\right]u_n^+ dx\\
&\quad - \frac{p}{1-\gamma}\int_{\left\{u_\vartheta < u_n^+\right\}}\left[\left(u_n^+\right)^{1-\gamma} - u_\vartheta^{1-\gamma}\right]dx\\
&\quad - p\lambda\int_{\left\{u_\vartheta < u_n^+\right\}}\left[F(x,u_n^+) - F(x,u_\vartheta)\right] \leq M_4
\end{aligned}$$

for all $n \in \mathbb{N}$ and so

$$\left\|\nabla u_n^+\right\|_p^p - \frac{p}{1-\gamma}\int_\Omega\left(u_n^+\right)^{1-\gamma}dx - p\lambda\int_\Omega F(x,u_n^+)dx \leq M_5 \tag{3.61}$$

for some $M_5 > 0$ and for all $n \in \mathbb{N}$. Moreover, we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ in (3.59) which gives

$$\begin{aligned}
&-\left\|\nabla u_n^+\right\|_p^p + \int_{\left\{u_n^+ \leq u_\vartheta\right\}}\left[u_\vartheta^{-\gamma} + \lambda f(x,u_\vartheta)\right]u_n^+ dx\\
&\quad + \int_{\left\{u_\vartheta < u_n^+\right\}}\left[\left(u_n^+\right)^{-\gamma} + \lambda f(x,u_n^+)\right]u_n^+ dx \leq \varepsilon_n
\end{aligned}$$

for all $n \in \mathbb{N}$. This leads to

$$-\left\|\nabla u_n^+\right\|_p^p + \int_\Omega \left(u_n^+\right)^{1-\gamma} dx + \lambda \int_\Omega f(x, u_n^+) u_n^+ dx \leq M_6 \tag{3.62}$$

for some $M_6 > 0$ and for all $n \in \mathbb{N}$. Adding (3.61) and (3.62) yields

$$\int_\Omega \hat{\eta}_\lambda(x, u_n^+) dx \leq M_7 \quad \text{for some } M_7 > 0 \text{ and for all } n \in \mathbb{N}. \tag{3.63}$$

Applying (3.63) and reasoning as in the proof of Proposition 3.7 (see the part of the proof after (3.33)), we show that $\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded and so, due to (3.60), $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded as well.

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^r(\Omega). \tag{3.64}$$

Choosing $h = u_n - u \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and applying (3.64), we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

which by the $(\mathrm{S})_+$-property of A, see Proposition 2.2, results in $u_n \to u$ in $W_0^{1,p}(\Omega)$. Therefore, σ_λ satisfies the C-condition and this proves the Claim.

On account of (3.55), (3.56) and the Claim, we are able to apply the mountain pass theorem stated as Theorem 2.1 and find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\sigma_\lambda} \subseteq [u_\vartheta) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad m_\lambda \leq \sigma_\lambda(\hat{u}), \tag{3.65}$$

see (3.52). From (3.49), (3.55) and (3.65) we conclude that $\hat{u} \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\hat{u} \neq u_0$. This finishes the proof. □

Summarizing the situation for the positive solution of problem (P_λ) as the parameter $\lambda > 0$ varies, we can state the following bifurcation-type theorem.

Theorem 3.9. *If hypotheses H hold, then there exist $\lambda^* > 0$ such that the following is satisfied:*

- *(a) problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for all $\lambda \in (0, \lambda^*)$;*
- *(b) problem (P_λ) has at least one positive solution $u^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for $\lambda = \lambda^*$;*
- *(c) problem (P_λ) has no positive solution for all $\lambda > \lambda^*$.*

4. Minimal Positive Solutions

In this section we show that problem (P_λ) has a smallest positive solution $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for every $\lambda \in \mathcal{L} = (0, \lambda^*]$ and we prove the monotonicity and continuity properties of the map $\lambda \to \overline{u}_\lambda$.

From Filippakis-Papageorgiou [2] we know that the solution set $\mathcal{S}_\lambda$ is downward directed for every $\lambda \in \mathcal{L} = (0, \lambda^*]$, that is, if $u_1, u_2 \in \mathcal{S}_\lambda$, then there exists $u \in \mathcal{S}_\lambda$ such that $u \leq u_1$ and $u \leq u_2$.

Proposition 4.1. *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem ($\mathrm{P}_\lambda$) has a smallest positive solution $\overline{u}_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, that is, $\overline{u}_\lambda \leq u$ for all $u \in \mathcal{S}_\lambda$.*

Proof. Invoking Lemma 3.10 of Hu-Papageorgiou [8, p. 178] we know that there exists a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq \mathcal{S}_\lambda$ such that $\inf \mathcal{S}_\lambda = \inf_{n\geq 1} u_n$. Recall that $\mathcal{S}_\lambda$ is downward directed.

Claim: $\tilde{u} \leq u_n$ for all $n \in \mathbb{N}$ (see the proof of Proposition 3.1)

Fix $n \in \mathbb{N}$ and let $\vartheta \in (0, \lambda) \subseteq \mathcal{L}$. According to Proposition 3.5 there exists $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_n - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. We introduce the Carathéodory function $e_n : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$e_n(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} & \text{if } s < u_\vartheta(x), \\ s^{-\gamma} & \text{if } u_\vartheta(x) \leq s \leq u_n(x), \\ u_n(x)^{-\gamma} & \text{if } u_n(x) < s. \end{cases} \tag{4.1}$$

We set $E_n(x,s) = \int_0^s e_n(x,t)dt$ and consider the C^1-functional $\gamma_n : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_n(u) = \frac{1}{p}\|\nabla u_n\|_p^p - \int_\Omega E_n(x,u)dx.$$

From (4.1) it is clear that γ_n is coercive and the Sobolev embedding theorem implies that γ_n is sequentially weakly lower semicontinuous. Therefore, we find $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\gamma_n\left(\tilde{u}_0\right) = \inf\left[\gamma_n(u) : u \in W_0^{1,p}(\Omega)\right].$$

In particular, we have $\gamma_n'(\tilde{u}_0) = 0$ which says that

$$\left\langle A\left(\tilde{u}_0\right), h\right\rangle = \int_\Omega e_n\left(x, \tilde{u}_0\right) h dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{4.2}$$

We choose $h = (u_\vartheta - \tilde{u}_0)^+ \in W_0^{1,p}(\Omega)$ in (4.2). Then, applying (4.1), the nonnegativity of f and the fact that $u_\vartheta \in \mathcal{S}_\vartheta$ gives

$$\begin{aligned} \left\langle A\left(\tilde{u}_0\right), \left(u_\vartheta - \tilde{u}_0\right)^+\right\rangle &= \int_\Omega u_\vartheta^{-\gamma}\left(u_\vartheta - \tilde{u}_0\right)^+ dx \\ &\leq \int_\Omega \left[u_\vartheta^{-\gamma} + \vartheta f\left(x, u_\vartheta\right)\right]\left(u_\vartheta - \tilde{u}_0\right)^+ dx \\ &= \left\langle A(u_\vartheta), (u_\vartheta - \tilde{u}_0)^+\right\rangle. \end{aligned}$$

Proposition 2.2 then implies $u_\vartheta \leq \tilde{u}_0$. In the same way, choosing $h = (\tilde{u}_0 - u_n)^+ \in W_0^{1,p}(\Omega)$ in (4.2) and applying again (4.1), $f \geq 0$ and $u_n \in \mathcal{S}_\lambda$ results in

$$\begin{aligned} \left\langle A\left(\tilde{u}_0\right), \left(\tilde{u}_0 - u_n\right)^+\right\rangle &= \int_\Omega u_n^{-\gamma}\left(\tilde{u}_0 - u_n\right)^+ dx \\ &\leq \int_\Omega \left[u_n^{-\gamma} + \lambda f(x, u_n)\right]\left(\tilde{u}_0 - u_n\right)^+ dx \\ &= \left\langle A(u_n), (\tilde{u}_0 - u_n)^+\right\rangle. \end{aligned}$$

As before, by Proposition 2.2, we obtain $\tilde{u}_0 \leq u_n$. So, we have proved that

$$\tilde{u}_0 \in [u_\vartheta, u_n]. \tag{4.3}$$

From (4.1) and (4.3) it follows that $\tilde{u}_0$ is a positive solution of the auxiliary problem (3.1). Therefore, $\tilde{u}_0 = \tilde{u}$ which implies $\tilde{u} \leq u_n$ for all $n \in \mathbb{N}$. This proves the Claim.

We have

$$\langle A(u_n), h\rangle = \int_\Omega \left[u_n^{-\gamma} + \lambda f(x, u_n)\right] h dx \tag{4.4}$$

for all $h \in W_0^{1,p}(\Omega)$ and for all $n \in \mathbb{N}$. Since $0 \leq u_n \leq u_1$ for all $n \geq 1$, from (4.4) with $h = u_n \in W_0^{1,p}(\Omega)$ and using hypothesis H(iv), we infer that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{\text{w}} \overline{u}_\lambda \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to \overline{u}_\lambda \quad \text{in } L^p(\Omega). \tag{4.5}$$

Moreover, we can say that

$$u_n(x)^{-\gamma} \to \overline{u}_\lambda(x)^{-\gamma} \quad \text{for a.a. } x \in \Omega.$$

From the Claim we know that

$$0 \leq u_n(x)^{-\gamma} \leq \tilde{u}(x)^{-\gamma} \quad \text{for a.a. } x \in \Omega.$$

Since $\tilde{u}(\cdot)^{-\gamma} \in L^{p'}(\Omega)$, see the proof of Proposition 3.1, from Gasiński-Papageorgiou [4, Problem 1.19, p. 38], we have

$$u_n^{-\gamma} \xrightarrow{\text{w}} \overline{u}_\lambda^{-\gamma} \quad \text{in } L^{p'}(\Omega). \tag{4.6}$$

Therefore, if we choose $h = u_n - \overline{u}_\lambda \in W_0^{1,p}(\Omega)$ in (4.4), pass to the limit as $n \to \infty$ and use (4.5) as well as (4.6), then

$$\lim_{n\to\infty} \langle A(u_n), u_n - \overline{u}_\lambda\rangle = 0,$$

which again by Proposition 2.2 leads to

$$u_n \to \overline{u}_\lambda \quad \text{in } W_0^{1,p}(\Omega). \tag{4.7}$$

So, if we pass to the limit in (4.4) as $n \to \infty$ and use (4.5), (4.6), (4.7), we obtain

$$\langle A\left(\overline{u}_\lambda\right), h\rangle = \int_\Omega \left[\overline{u}_\lambda^{-\gamma} + \lambda f\left(x, \overline{u}_\lambda\right)\right] h dx \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

From the Claim it follows that $\tilde{u} \leq \overline{u}_\lambda$. Therefore we conclude that

$$\overline{u}_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad \overline{u}_\lambda = \inf \mathcal{S}_\lambda.$$

□

In the next proposition we examine the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ and determine the monotonicity and continuity properties of this map.

Proposition 4.2. *If hypotheses H hold, then the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is*

(a) strictly increasing, that is,

$$0 < \vartheta < \lambda \leq \lambda^* \quad \textit{implies} \quad \overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right);$$

(b) left continuous.

Proof. (a) From Proposition 3.5 we know that there exists $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $\overline{u}_\lambda - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so, since $\overline{u}_\vartheta \leq u_\vartheta$, it follows $\overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. So, the map $\lambda \to \overline{u}_\lambda$ is strictly increasing.

(b) Suppose that $\{\lambda_n, \lambda\}_{n\geq 1} \subseteq \mathcal{L} = (0, \lambda^*]$ and assume that $\lambda_n \to \lambda^-$. We set $\overline{u}_n = \overline{u}_{\lambda_n} \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for all $n \in \mathbb{N}$. We have

$$\langle A(\overline{u}_n), h\rangle = \int_\Omega \left[\overline{u}_n^{-\gamma} + \lambda_n f(x, \overline{u}_n)\right] h dx \tag{4.8}$$

for all $h \in W_0^{1,p}(\Omega)$ and for all $n \in \mathbb{N}$. Moreover, by Proposition 4.1,

$$0 \leq \overline{u}_1 \leq \overline{u}_n \leq \overline{u}_{\lambda^*}. \tag{4.9}$$

On account of (4.9) and by the choice $h = \overline{u}_n \in W_0^{1,p}(\Omega)$ in (4.8), we infer that $\{\overline{u}_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. We have

$$\begin{aligned} -\Delta_p \overline{u}_n &= \overline{u}_n^{-\gamma} + \lambda_n f(x, u_n) \qquad && \text{in } \Omega,\\ \overline{u}_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for all $n \in \mathbb{N}$. From (4.9) we see that

$$0 \leq \overline{u}_n^{-\gamma} \leq \overline{u}_1^{-\gamma} \in L^s(\Omega) \text{ with } s > N \text{ and for all } n \in \mathbb{N},$$

see also H(i). Similarly, (4.9) and hypothesis H(i) imply that

$$\{f(\cdot, \overline{u}_n(\cdot))\}_{n\geq 1} \subseteq L^s(\Omega) \text{ is bounded.}$$

Then Proposition 1.3 of Guedda-Véron [7] implies that

$$\|\overline{u}_n\|_\infty \leq M_8 \quad \text{for some } M_8 > 0 \text{ and for all } n \in \mathbb{N}.$$

From this as in the proof of Proposition 3.1 and using Theorem 2.1 of Lieberman [10], there exist $\alpha \in (0,1)$ and $M_9 > 0$ such that

$$\overline{u}_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|\overline{u}_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq M_9 \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

Then, (4.10), the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and the monotonicity of the sequence $\{\overline{u}_n\}_{n\geq 1}$ imply that

$$\overline{u}_n \to \tilde{u}_\lambda \quad \text{in } C_0^1(\overline{\Omega}).$$

We claim that $\tilde{u}_\lambda = \overline{u}_\lambda$. If this is not the case, we can find $z_0 \in \Omega$ such that $\overline{u}_\lambda(z_0) < \tilde{u}_\lambda(z_0)$ which implies $\overline{u}_\lambda(z_0) < \overline{u}_n(z_0)$ for all $n \geq n_0$. But this contradicts (a). Therefore, $\tilde{u}_\lambda = \overline{u}_\lambda$ and so $\lambda \to \overline{u}_\lambda$ is left continuous. □

Summarizing the situation concerning the minimal positive solution of problem (P_λ), we can state the following theorem.

Theorem 4.3. *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem ($P_\lambda$) has a smallest positive solution $\overline{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is*

- *strictly increasing, that is, $0 < \vartheta < \lambda \leq \lambda^*$ implies $\overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$;*
- *left continuous.*

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(N. S. Papageorgiou) National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

Email address: `npapg@math.ntua.gr`

(P. Winkert) Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany

Email address: `winkert@math.tu-berlin.de`