SINGULAR p-LAPLACIAN EQUATIONS WITH SUPERLINEAR PERTURBATION

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Abstract. We consider a nonlinear Dirichlet problem driven by the p-Laplace operator and with a right-hand side which has a singular term and a parametric superlinear perturbation. We are interested in positive solutions and prove a bifurcation-type theorem describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. In addition, we show that for every admissible parameter $\lambda > 0$ the problem has a smallest positive solution $\overline{u}_\lambda$ and we establish the monotonicity and continuity properties of the map $\lambda \to \overline{u}_\lambda$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we deal with the following nonlinear parametric singular problem

$$\begin{aligned} -\Delta_p u &= u^{-\gamma} + \lambda f(x,u) \qquad && \text{in } \Omega,\\ u &> 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{P$_\lambda$}$$

where $1 < p < \infty$, $0 < \gamma < 1$ and Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

In the right-hand side of (P$_\lambda$), $u^{-\gamma}$ is the singular term while λf is the parametric term with $\lambda > 0$ and a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, that is, $x \to f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \to f(x,s)$ is continuous for a.a. $x \in \Omega$. We assume that $f(x,\cdot)$ exhibits $(p-1)$-superlinear growth near $+\infty$ but without satisfying the usual Ambrosetti-Rabinowitz condition, AR-condition for short. We are interested in finding positive solutions and our goal is to determine how the set of positive solutions of (P$_\lambda$) changes as the parameter $\lambda > 0$ varies. We are going to prove a bifurcation-type result which produces a critical parameter value $\lambda^* > 0$ such that

- problem (P$_\lambda$) has at least two positive solutions for all $\lambda \in (0, \lambda^*)$;
- problem (P$_\lambda$) has at least one positive solution for $\lambda = \lambda^*$;
- problem (P$_\lambda$) has no positive solutions for all $\lambda > \lambda^*$.

This result was motivated by the work of Papageorgiou-Smyrlis [15] who proved such a theorem for problem (P$_\lambda$) under the hypotheses that the perturbation term $f(x,\cdot)$ is $(p-1)$-linear near 0^+. This condition removes from consideration nonlinearities with a concave term near 0^+. Our framework removes this restriction and incorporates perturbations which exhibit the competing effects of concave and

2010 *Mathematics Subject Classification.* 35J20, 35J25, 35J67.

Key words and phrases. Singular term, superlinear term, positive solution, nonlinear regularity truncations, comparison principles, minimal positive solutions.

convex terms. This changes the geometry of the problem. Moreover, our growth condition on $f(x,\cdot)$ is more general than that in Papageorgiou-Smyrlis [15].

Nonlinear singular Dirichlet problems were also investigated in the papers of Giacomoni-Schindler-Takáč [5], Papageorgiou-Rădulescu-Repovš [14] and Perera-Zhang [16] for different settings and conditions.

2. Preliminaries and Hypotheses

Let X be a Banach space and let X^* be its topological dual. We denote by $\langle\cdot,\cdot\rangle$ the duality brackets to the pair (X^*,X). Given $\varphi\in C^1(X,\mathbb{R})$ we say that φ satisfies the Cerami condition, C-condition for short, if every sequence $\{u_n\}_{n\geq 1}\subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1}\subseteq\mathbb{R}$ is bounded and such that $(1+\|u_n\|_X)\,\varphi'(u_n)\to 0$ in X^* as $n\to\infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ and leads to following minimax theorem known as the mountain pass theorem.

Theorem 2.1. *Let $\varphi\in C^1(X,\mathbb{R})$ be a functional satisfying the C-condition and let $u_1,u_2\in X, \|u_2-u_1\|_X>\rho>0$,*

$$\max\{\varphi(u_1),\varphi(u_2)\}<\inf\{\varphi(u):\|u-u_1\|_X=\rho\}=:\eta_\rho$$

and $c=\inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1}\varphi(\gamma(t))$ with $\Gamma=\{\gamma\in C([0,1],X):\gamma(0)=u_1,\gamma(1)=u_2\}$. Then $c\geq\eta_\rho$ with c being a critical value of φ, that is, there exists $\hat{u}\in X$ such that $\varphi'(\hat{u})=0$ and $\varphi(\hat{u})=c$.

By $W^{1,p}_0(\Omega)$ we denote the usual Sobolev space with norm $\|\cdot\|$. Thanks to the Poincaré inequality we have

$$\|u\|=\|\nabla u\|_p\quad\text{for all } u\in W^{1,p}_0(\Omega),$$

where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$ and $L^p\left(\Omega;\mathbb{R}^N\right)$, respectively. Furthermore, we need the ordered Banach space $C^1_0(\overline{\Omega})=\{u\in C^1(\overline{\Omega}):u\big|_{\partial\Omega}=0\}$ and its positive cone

$$C^1_0(\overline{\Omega})_+=\left\{u\in C^1_0(\overline{\Omega}):u(x)\geq 0\text{ for all }x\in\overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C^1_0(\overline{\Omega})_+\right)=\left\{u\in C^1_0(\overline{\Omega})_+:u(x)>0\text{ for all }x\in\Omega\text{ and }\frac{\partial u}{\partial n}\bigg|_{\partial\Omega}<0\right\},$$

where n is the outward unit normal on $\partial\Omega$.

The norm of $\mathbb{R}^N$ is denoted by $|\cdot|$ and "$\cdot$" stands for the inner product in $\mathbb{R}^N$. For $s\in\mathbb{R}$, we set $s^\pm=\max\{\pm s,0\}$ and for $u\in W^{1,p}_0(\Omega)$ we define $u^\pm(\cdot)=u(\cdot)^\pm$. It is well known that

$$u^\pm\in W^{1,p}_0(\Omega),\quad |u|=u^++u^-,\quad u=u^+-u^-.$$

For $u,v\in W^{1,p}_0(\Omega)$ with $u(x)\leq v(x)$ for a.a. $x\in\Omega$ we define

$$[u,v]=\left\{y\in W^{1,p}_0(\Omega):u(x)\leq y(x)\leq v(x)\text{ for a.a. }x\in\Omega\right\},$$

$$\operatorname*{int}_{C^1_0(\overline{\Omega})}[u,v]=\text{the interior in }C^1_0(\overline{\Omega})\text{ of }[u,v]\cap C^1_0(\overline{\Omega}),$$

$$[u)=\left\{y\in W^{1,p}_0(\Omega):u(x)\leq y(x)\text{ for a.a. }x\in\Omega\right\}.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. By $p^* > 1$ we denote the Sobolev critical exponent for p defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$$

Finally, if $h_1, h_2 \in L^\infty(\Omega)$, then we write $h_1 \prec h_2$ if and only if for every compact $K \subseteq \Omega$ we have $0 < m_K \leq h_2(x) - h_1(x)$ for a.a. $x \in K$.

Let $A : W_0^{1,p}(\Omega) \to W^{-1.p'}(\Omega) = W_0^{1,p}(\Omega)^*$ with $\frac{1}{p} + \frac{1}{p'} = 1$ be defined by

$$\langle A(u), \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx \quad \text{for all } u, \varphi \in W_0^{1,p}(\Omega). \tag{2.1}$$

The next proposition states the main properties of this map and it can be found in Gasiński-Papageorgiou [4, Problem 2.192, p. 279].

Proposition 2.2. *The map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined in (2.1) is bounded, that is, it maps bounded sets to bounded sets, continuous, strictly monotone, hence maximal monotone and it is of type* $(\mathrm{S})_+$, *that is,*

$$u_n \overset{\mathrm{w}}{\to} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

imply $u_n \to u$ in $W_0^{1,p}(\Omega)$.

Moreover, we denote by $\hat{\lambda}_1$ the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and by $\hat{u}_1 \in W_0^{1,p}(\Omega)$ the corresponding positive, L^p-normalized, that is, $\|\hat{u}_1\|_p = 1$, eigenfunction. We know that $\hat{\lambda}_1 > 0$ and $\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Gasiński-Papageorgiou [3].

Also, for a given $\varphi \in C^1(X, \mathbb{R})$ we denote by K_φ the critical set of φ, that is, $K_\varphi = \{u \in X : \varphi'(u) = 0\}$.

Now we introduce the hypotheses on the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.a. $x \in \Omega$ and

(i) if $a \in L^s(\Omega)$ with $s > N$, then
$$0 < f(x, s) \leq a(x)\left(1 + s^{r-1}\right)$$
for a.a. $x \in \Omega$, for all $s > 0$ and for $p < r < p^*$;

(ii) if $F(x, s) = \int_0^s f(x, t) dt$, then
$$\lim_{s\to+\infty} \frac{F(x, s)}{s^p} = +\infty \quad \text{uniformly for a.a. } x \in \Omega;$$

(iii) if
$$\hat{\eta}_\lambda(x, s) = \left[1 - \frac{p}{1-\gamma}\right] s^{1-\gamma} + \lambda\left[f(x, s)s - pF(x, s)\right]$$
with $\lambda > 0$, then
$$\hat{\eta}_\lambda(x, s_1) \leq \hat{\eta}_\lambda(x, s_2) + \tau_\lambda(x)$$
for a.a. $x \in \Omega$, for all $0 \leq s_1 \leq s_2$ with $\tau_\lambda \in L^1(\Omega)$ and $\lambda \to \tau_\lambda$ is nondecreasing from $(0, +\infty)$ into $L^1(\Omega)$;

(iv) there exist $c_1 > 0$ and $q \leq p$ such that

$$f(x,s) \leq c_1 \left[s^{r-1} + s^{q-1}\right]$$

for a.a. $x \in \Omega$ and for all $s \geq 0$;

(v) for every $\eta > 0$ there exists $m_\eta > 0$ such that

$$f(x,s) \geq m_\eta$$

for a.a. $x \in \Omega$ and for all $s \geq \eta$;

(vi) for every $\rho > 0$ there exists $\hat{\xi}_\rho > 0$ such that the function

$$s \to f(x,s) + \hat{\xi}_\rho s^{p-1}$$

is nondecreasing on $[0,\rho]$ for a.a. $x \in \Omega$.

Remark 2.3. *Since we are interested on positive solutions and the hypotheses above concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality, we may assume that*

$$f(x,s) = 0 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \leq 0. \tag{2.2}$$

Hypotheses H(ii), H(iii) imply that

$$\lim_{s \to +\infty} \frac{f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$

Hence, the perturbation term in (P_λ) is $(p-1)$-superlinear in the second variable. However, we do not employ the usual AR-condition for superlinear problems. Recall that this condition says that there exist $\tau > p$ and $M > 0$ such that

$$0 < \tau F(x,s) \leq f(x,s)s \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M, \tag{2.3}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M). \tag{2.4}$$

In fact this is a unilateral version of the AR-condition on account of (2.2). Integrating (2.3) and using (2.4) we obtain the weaker condition

$$c_2 s^\tau \leq F(x,s) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \geq M \text{ and for some } c_2 > 0.$$

Hence, the AR-condition implies that $f(x,\cdot)$ exhibits at least $(\tau-1)$-polynomial growth. This excludes superlinear nonlinearities with slower growth near $+\infty$ from consideration. Instead we employ the quasimonotonicity condition on $\eta_\lambda(x,\cdot)$ in hypothesis H(iii). This condition is a slight generalization of a hypothesis introduced by Li-Yang [11]. This superlinearity hypothesis is different from the one used by Papageorgiou-Smyrlis [15]. There are easy ways to verify H(iii). For example, condition H(iii) holds if there exists $M > 0$ such that

$$s \to \frac{s^{-\gamma} + \lambda f(x,s)}{s^{p-1}}$$

is nondecreasing on $[M,+\infty)$ for a.a. $x \in \Omega$ or

$$s \to \hat{\eta}_\lambda(x,s)$$

is nondecreasing on $[M,+\infty)$, see Li-Yang [11].

Hypothesis H(iv) allows perturbations which have concave terms. This is excluded from the hypotheses of Papageorgiou-Smyrlis [15]. Hypothesis H(iv) is satisfied if, *for example, $f(x,\cdot)$ is differentiable for a.a. $x \in \Omega$ and for every $\rho > 0$ there exists $c_\rho > 0$ such that*

$$f_s'(x,s) \geq -c_\rho s^{p-1}$$

for a.a. $x \in \Omega$ and for all $0 \leq s \leq \rho$.

Example 2.4. *For the sake of simplicity we drop the x-dependence. The following functions satisfy hypotheses H:*

$$f_1(s) = s^{\tau-1} \text{ with } p < \tau < p^*,$$

$$f_2(s) = \begin{cases} (s^+)^{\vartheta-1} & \text{if } s \leq 1, \\ s^{p-1}[\ln s + 1] & \text{if } 1 < s \end{cases} \quad \text{with } 1 < \vartheta < p < \infty.$$

Note that f_2 fails to satisfy the AR-condition and it is outside the framework of Papageorgiou-Smyrlis [15].

3. Positive Solutions

We introduce the following two sets

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (\mathrm{P}_\lambda) \text{ has a positive solution}\},$$
$$\mathcal{S}_\lambda = \{u : u \text{ is a positive solution of problem } (\mathrm{P}_\lambda)\}.$$

Proposition 3.1. *If hypotheses H hold, then $\mathcal{L} \neq \emptyset$.*

Proof. We consider the following purely singular Dirichlet problem

$$-\Delta_p u = u^{-\gamma} \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0, \quad u > 0. \tag{3.1}$$

From Papageorgiou-Smyrlis [15, Proposition 5] we know that problem (3.1) has a unique positive solution $\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Moreover, we consider the following auxiliary Dirichlet problem

$$-\Delta_p u = 1 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{3.2}$$

Problem (3.2) has a unique solution $e \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ which can be shown easily. For $1 < \tau < +\infty$, we have $e^\tau \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and using Proposition 2.1 of Marano-Papageorgiou [12], see also Gasiński-Papageorgiou [4, Problem 4.180, p. 680], there exists $c_3 > 0$ such that $\hat{u}_1 \leq c_3 e^\tau$ and so

$$\hat{u}_1^{\frac{1}{\tau}} \leq c_3^{\frac{1}{\tau}} e,$$

which implies

$$e^{-\gamma} \leq c_4 \hat{u}_1^{-\frac{\gamma}{\tau}} \tag{3.3}$$

for some $c_4 > 0$. From the Lemma in Lazer-McKenna [9] we know that

$$\hat{u}_1^{-\frac{\gamma}{\tau}} \in L^\tau(\Omega).$$

This fact along with (3.3) gives

$$e^{-\gamma} \in L^\tau(\Omega) \quad \text{and} \quad \left\|e^{-\gamma}\right\|_\tau \leq c_4 \left\|\hat{u}_1^{-\gamma}\right\|_1^{\frac{1}{\tau}}.$$

Hence

$$\limsup_{\tau \to +\infty} \left\|e^{-\gamma}\right\|_\tau \leq c_4. \tag{3.4}$$

On the other hand, from the Chebyshev inequality, we have

$$\eta^\tau \left|\left\{e^{-\gamma} \geq \eta\right\}\right|_N \leq \left\|e^{-\gamma}\right\|_\tau^\tau$$

with $\eta > 0$, or equivalently,

$$\eta \left|\left\{e^{-\gamma} \geq \eta\right\}\right|_N^{\frac{1}{\tau}} \leq \left\|e^{-\gamma}\right\|_\tau .$$

This fact yields

$$\eta \leq \liminf_{\tau \to +\infty} \left\|e^{-\gamma}\right\|_\tau \quad \text{provided} \quad \left|\left\{e^{-\gamma} \geq \eta\right\}\right|_N > 0. \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$e^{-\gamma} \in L^\infty(\Omega) \quad \text{and} \quad \left\|e^{-\gamma}\right\|_\tau \to \left\|e^{-\gamma}\right\|_\infty \quad \text{as } \tau \to +\infty.$$

Now let $c_5 > \|e^{-\gamma}\|_\infty$ and $m_0 = \|e\|_\infty$. For $t > 0$ we consider the function

$$\begin{aligned}
\vartheta(t) &= \frac{t^{p-1} - c_5 t^{-\gamma}}{c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1}\right]} \\
&= \frac{t^{p+\gamma-1} - c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1}\right]} \\
&= \frac{1}{c_1 \left[m_0^{r-1} t^{r-p} + m_0^{q-1} t^{q-p}\right]} - \frac{c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1}\right]} \\
&= \frac{t^{p-q}}{c_1 \left[m_0^{r-1} t^{r-q} + m_0^{q-1}\right]} - \frac{c_5}{c_1 \left[m_0^{r-1} t^{r+\gamma-1} + m_0^{q-1} t^{q+\gamma-1}\right]}.
\end{aligned}$$

Since $q \leq p < r$ we see that

$$\vartheta(t) \to -\infty \text{ as } t \to 0^+ \quad \text{and} \quad \vartheta(t) \to 0^+ \text{ as } t \to +\infty.$$

Therefore, there exists $t_0 > 0$ such that

$$\lambda_0 = \vartheta(t_0) = \max\left[\vartheta(t) : t > 0\right] > 0.$$

Let $\lambda \in (0, \lambda_0)$. We can find $t > 0$ such that $\vartheta(t) \geq \lambda$. Hence

$$t^{p-1} \geq c_5 t^{-\gamma} + \lambda c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1}\right]. \tag{3.6}$$

We set $\overline{u} = te \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Then, because of (3.6), hypothesis H(iv) and the choice of c_5, m_0, we obtain

$$\begin{aligned}
-\Delta_p \overline{u} &= t^{p-1}\left[-\Delta_p e\right] \\
&= t^{p-1} \\
&\geq c_5 t^{-\gamma} + \lambda c_1 \left[m_0^{r-1} t^{r-1} + m_0^{q-1} t^{q-1}\right] \\
&\geq \overline{u}^{-\gamma} + \lambda c_1 \left[\overline{u}^{r-1} + \overline{u}^{q-1}\right] \\
&\geq \overline{u}^{-\gamma} + \lambda f(x, \overline{u}) \quad \text{for a.a. } x \in \Omega.
\end{aligned} \tag{3.7}$$

Since $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, as before, there exists $\vartheta \in (0,1)$ small enough such that $\vartheta \tilde{u} \leq \overline{u}$. If $\tilde{u}_0 = \vartheta \tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, then

$$-\Delta_p \tilde{u}_0 = -\Delta_p\left(\vartheta \tilde{u}\right) = \vartheta^{p-1}\left(-\Delta_p \tilde{u}\right) = \vartheta^{p-1} \tilde{u}^{-\gamma} \leq \left(\vartheta \tilde{u}\right)^{-\gamma} = \tilde{u}_0^{-\gamma} \tag{3.8}$$

since $\vartheta \in (0,1)$. Using the functions $\tilde{u}_0, \overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, we introduce the following truncation of the reaction of problem (P_λ)

$$g_\lambda(x,s) = \begin{cases} \tilde{u}_0(x)^{-\gamma} + \lambda f\left(x, \tilde{u}_0(x)\right) & \text{if } s < \tilde{u}_0(x), \\ s^{-\gamma} + \lambda f(x,s) & \text{if } \tilde{u}_0(x) \leq s \leq \overline{u}(x), \\ \overline{u}(x)^{-\gamma} + \lambda f\left(x, \overline{u}(x)\right) & \text{if } \overline{u}(x) < s, \end{cases} \tag{3.9}$$

with $\lambda \in (0, \lambda_0)$. Evidently, $g_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We set $G_\lambda(x,s) = \int_0^s g_\lambda(x,t)dt$ and consider the functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega G_\lambda(x,u)dx.$$

On account of Proposition 3 of Papageorgiou-Smyrlis [15] we have that $\psi_\lambda \in C^1(W_0^{1,p}(\Omega))$. Moreover, from (3.9) it is clear that ψ_λ is coercive. The Sobolev embedding theorem implies that ψ_λ is sequentially weakly lower semicontinuous. So, by the Weierstraß-Tonelli theorem, there exists $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_\lambda) = \inf\left[\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)\right].$$

Since u_λ is a global minimizer, it fulfills $\psi_\lambda'(u_\lambda) = 0$, which is equivalent to

$$\langle A(u_\lambda), h\rangle = \int_\Omega g_\lambda(x,u_\lambda)hdx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.10}$$

Taking $h = (\tilde{u}_0 - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.10) gives, thanks to (3.9), (3.8) and the fact that $f \geq 0$,

$$\begin{aligned} \left\langle A(u_\lambda), (\tilde{u}_0 - u_\lambda)^+\right\rangle &= \int_\Omega \left[\tilde{u}_0^{-\gamma} + \lambda f\left(x, \tilde{u}_0\right)\right]\left(\tilde{u}_0 - u_\lambda\right)^+ dx \\ &\geq \int_\Omega \tilde{u}_0^{-\gamma}\left(\tilde{u}_0 - u_\lambda\right)^+ dx \\ &\geq \left\langle A\left(\tilde{u}_0\right), \left(\tilde{u}_0 - u_\lambda\right)^+\right\rangle. \end{aligned}$$

Because of the monotonicity of A, see Proposition 2.2, we obtain that $\tilde{u}_0 \leq u_\lambda$. Next, we choose $h = (u_\lambda - \overline{u})^+ \in W_0^{1,p}(\Omega)$ in (3.10). This gives, by applying (3.9) and (3.7), that

$$\left\langle A\left(u_\lambda\right), \left(u_\lambda - \overline{u}\right)^+\right\rangle = \int_\Omega \left[\overline{u}^{-\gamma} + \lambda f\left(x, \overline{u}\right)\right]\left(u_\lambda - \overline{u}\right)^+ dx \leq \left\langle A\left(\overline{u}\right), \left(u_\lambda - \overline{u}\right)^+\right\rangle.$$

As before, by applying Proposition 2.2, it follows that $u_\lambda \leq \overline{u}$. So, we have proved that

$$u_\lambda \in \left[\tilde{u}_0, \overline{u}\right]. \tag{3.11}$$

From (3.9), (3.10), (3.11), it follows that

$$\langle A(u_\lambda), h\rangle = \int_\Omega \left[u_\lambda^{-\gamma} + \lambda f(x,u_\lambda)\right] hdx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.12}$$

Since $\tilde{u}_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, as before, we have that $\tilde{u}_0^{-\gamma} \in L^s(\Omega)$ for $s > N$ and since $0 \leq u_\lambda^{-\gamma} \leq \tilde{u}_0^{-\gamma}$, see (3.11), one has that $u_\lambda^{-\gamma} \in L^s(\Omega)$. From (3.12) it follows that

$$-\Delta_p u_\lambda(x) = u_\lambda(x)^{-\gamma} + \lambda f(x, u_\lambda(x)) \quad \text{for a.a. } x \in \Omega, \quad u_\lambda\big|_{\partial\Omega} = 0. \tag{3.13}$$

From (3.13) and Proposition 1.3 of Guedda-Véron [7] we have that $u_\lambda \in L^\infty(\Omega)$. Let $\xi_\lambda(x) = u_\lambda(x)^{-\gamma} + \lambda f(x, u_\lambda(x))$. Then $\xi_\lambda \in L^s(\Omega)$, see hypothesis H(i). We consider now the following linear Dirichlet problem

$$-\Delta v = \xi_\lambda \quad \text{in } \Omega, \quad v\big|_{\partial\Omega}.$$

This problem has a unique solution v_λ which by the linear regularity theory belongs to $W^{2,s}(\Omega)$, see Gilbarg-Trudinger [6, Theorem 9.15, p. 241]. Then, since $s > N$, the Sobolev embedding theorem implies that

$$v_\lambda \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{with} \quad \alpha = 1 - \frac{N}{s}. \tag{3.14}$$

We set $k_\lambda(x) = \nabla v_\lambda(x)$. Then $k_\lambda \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$, see (3.14). From (3.13) we obtain

$$-\operatorname{div}\left(|\nabla u_\lambda(x)|^{p-2}\nabla u_\lambda(x) - k_\lambda(x)\right) = 0 \quad \text{for a.a. } x \in \Omega, \quad u_\lambda\big|_{\partial\Omega} = 0.$$

Invoking Theorem 1 of Lieberman [10], we infer that $u_\lambda \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Finally from (3.13) and the nonlinear maximum principle, see for example, Gasiński-Papageorgiou [3, Theorem 6.2.8, p. 738] and Pucci-Serrin [17, p. 120], we conclude that $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. It follows that $(0, \lambda_0) \subseteq \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$. $\square$

From the proof above we infer the following corollary.

Corollary 3.2. *If hypotheses H hold and $\lambda \in \mathcal{L}$, then $\mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

In the next proposition we show that $\mathcal{L}$ is in fact an interval.

Proposition 3.3. *If hypotheses H hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$ there exists $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Corollary 3.2. Since $\mu < \lambda$ and $f \geq 0$, we have

$$-\Delta_p u_\lambda(x) = u_\lambda(x)^{-\gamma} + \lambda f(x, u_\lambda(x)) \geq u_\lambda(x)^{-\gamma} + \mu f(x, u_\lambda(x))$$

for a.a. $x \in \Omega$. Recall that $\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique solution of (3.1). Since $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ there exists $t \in (0,1)$ small enough such that $t\tilde{u} \leq u_\lambda$. We set $\tilde{u}_* = t\tilde{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and introduce the following truncation nonlinearity

$$\hat{g}_\mu(x,s) = \begin{cases} \tilde{u}_*(x)^{-\gamma} + \mu f\left(x, \tilde{u}_*(x)\right) & \text{if } s < \tilde{u}_*(x), \\ s^{-\gamma} + \mu f(x,s) & \text{if } \tilde{u}_*(x) \leq s \leq u_\lambda(x), \\ u_\lambda(x)^{-\gamma} + \mu f(x, u_\lambda(x)) & \text{if } u_\lambda(x) < s, \end{cases} \tag{3.15}$$

which is a Carathéodory function. We set $\hat{G}_\mu(x,s) = \int_0^s \hat{g}_\mu(x,t)dt$ and consider the functional $\hat{\psi}_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\mu(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \hat{G}_\mu(x,u)dx.$$

As before, we have $\hat{\psi}_\mu \in C^1(W_0^{1,p}(\Omega))$, see Papageorgiou-Smyrlis [15, Proposition 3]. From (3.15) it is clear that $\hat{\psi}_\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstraß-Tonelli theorem there exists $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_\mu\left(u_\mu\right) = \inf\left[\hat{\psi}_\mu(u) : u \in W_0^{1,p}(\Omega)\right].$$

Hence, $\hat{\psi}'_\mu(u_\mu) = 0$ which is equivalent to

$$\langle A(u_\mu), h\rangle = \int_\Omega \hat{g}_\mu(x, u_\mu) h dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.16}$$

We choose $h = (\tilde{u}_* - u_\mu)^+ \in W_0^{1,p}(\Omega)$ in (3.16). Then, using (3.15), $f \geq 0$, (3.1) and $\tilde{u}_* = t\tilde{u}$ for $0 < t < 1$, we obtain

$$\begin{aligned}\langle A(u_\mu), (\tilde{u}_* - u_\mu)^+\rangle &= \int_\Omega \left[\tilde{u}_*^{-\gamma} + \mu f(x, \tilde{u}_*)\right](\tilde{u}_* - u_\mu)^+ dx \\ &\geq \int_\Omega \tilde{u}_*^{-\gamma} (\tilde{u}_* - u_\mu)^+ dx \\ &\geq \left\langle A(\tilde{u}_*), (\tilde{u}_* - u_\mu)^+ \right\rangle.\end{aligned}$$

Hence, by Proposition 2.2, $\tilde{u}_* \leq u_\mu$. Next, we choose $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ in (3.16). Then, as before, by applying (3.15) and since $f \geq 0$, $\mu < \lambda$ and $u_\lambda \in \mathcal{S}_\lambda$ we obtain

$$\begin{aligned}\left\langle A(u_\mu), (u_\mu - u_\lambda)^+ \right\rangle &= \int_\Omega \left[u_\lambda^{-\gamma} + \mu f(x, u_\mu)\right](u_\mu - u_\lambda)^+ dx \\ &\leq \left[u_\lambda^{-\gamma} + \lambda f(x, u_\lambda)\right](u_\mu - u_\lambda)^+ dx \\ &= \left\langle A(u_\lambda), (u_\mu - u_\lambda)^+ \right\rangle.\end{aligned}$$

Using Proposition 2.2 we see that $u_\mu \leq u_\lambda$.

So, we have proved that

$$u_\mu \in [\tilde{u}_*, u_\lambda]. \tag{3.17}$$

From (3.15), (3.16) and (3.17) we infer that $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so $\mu \in \mathcal{L}$. ☐

A useful byproduct of the proof above is the following corollary.

Corollary 3.4. *If hypotheses H hold,* $0 < \mu < \lambda \in \mathcal{L}$ *and* $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, *then* $\mu \in \mathcal{L}$ *and there exists* $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *such that* $u_\lambda - u_\mu \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$.

In fact using hypotheses H(v), (vi) we can improve the conclusion of the corollary above.

Proposition 3.5. *If hypotheses H hold,* $0 < \mu < \lambda \in \mathcal{L}$ *and if* $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, *then* $\mu \in \mathcal{L}$ *and there exists* $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *such that* $u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Proof. From Corollary 3.4 we already know that $\mu \in \mathcal{L}$ and we can find $u_\lambda \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_\lambda - u_\mu \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(vi). Since $u_\mu \in \mathcal{S}_\mu \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), u_\lambda \in \mathcal{S}_\lambda \subseteq$

int $\left(C_0^1(\overline{\Omega})_+\right)$, $u_\mu \leq u_\lambda$ and because of hypotheses H(v), (vi) we derive

$$\begin{aligned}
&- \Delta_p u_\mu(x) + \lambda \hat{\xi}_\rho u_\mu(x)^{p-1} - u_\mu(x)^{-\gamma} \\
&= \mu f(x, u_\mu(x)) + \lambda \hat{\xi}_\rho u_\mu(x)^{p-1} \\
&= \lambda f(x, u_\mu(x)) + \lambda \hat{\xi}_\rho u_\mu(x)^{p-1} - (\lambda - \mu) f(x, u_\mu(x)) \\
&< \lambda f(x, u_\lambda(x)) + \lambda \hat{\xi}_\rho u_\lambda(x)^{p-1} \\
&= -\Delta_p u_\lambda(x) + \lambda \hat{\xi}_\rho u_\lambda^{p-1} - u_\lambda(x)^{-\gamma}
\end{aligned} \tag{3.18}$$

for a.a. $x \in \Omega$. Let $\hat{h}_0(x) = (\lambda - \mu) f(x, u_\mu(x))$. Since $u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and using hypothesis H(v), we see that $0 \prec \hat{h}_0$. Therefore, from (3.18) and the singular strong comparison principle, see Papageorgiou-Smyrlis [15, Proposition 4], we conclude that $u_\lambda - u_\mu \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

We set $\lambda^* = \sup \mathcal{L}$.

Proposition 3.6. *If hypotheses H hold, then* $\lambda^* < \infty$.

Proof. Recall that

$$\lim_{s \to +\infty} \frac{f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,$$

see hypotheses H(ii), (iii). Therefore, for a given $k > \hat{\lambda}_1$, there exists $M > 0$ such that

$$f(x,s) \geq k s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M. \tag{3.19}$$

On the other hand, we have

$$s^{-\gamma} + \lambda f(x,s) \geq M^{-\gamma} + \lambda f(x,s) \tag{3.20}$$

for a.a. $x \in \Omega$, for all $0 \leq s \leq M$ and for all $\lambda > 0$. Note that, since $f \geq 0$,

$$\lim_{s \to 0^+} \frac{M^{-\gamma} + \lambda f(x,s)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,$$

which implies that there exists $\delta_\lambda > 0$ such that

$$M^{-\gamma} + \lambda f(x,s) \geq \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \delta_\lambda.$$

Combining this with (3.20) we see that

$$s^{-\gamma} + \lambda f(x,s) \geq \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \delta_\lambda. \tag{3.21}$$

Finally, note that on account of hypothesis H(v), there exists $\tilde{\lambda} \geq 1$ large enough such that

$$s^{-\gamma} + \tilde{\lambda} f(x,s) \geq M^{-\gamma} + \tilde{\lambda} m_{\delta_{\tilde{\lambda}}} \geq \hat{\lambda}_1 M^{p-1} \geq \hat{\lambda}_1 s^{p-1} \tag{3.22}$$

for a.a. $x \in \Omega$ and for all $\delta_{\tilde{\lambda}} \leq s \leq M$. Combining (3.19), (3.21), and (3.22) we conclude that

$$s^{-\gamma} + \tilde{\lambda} f(x,s) \geq \hat{\lambda}_1 s^{p-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq 0. \tag{3.23}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. There exists $u_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Let $t > 0$ be such that

$$t \hat{u}_1 \leq u_\lambda. \tag{3.24}$$

Assume that $t > 0$ is the largest positive real number for which (3.24) holds. Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(vi). Applying (3.24), hypothesis H(vi) and (3.23) gives

$$\begin{aligned}
&-\Delta_p u_\lambda(x) + \lambda\hat{\xi}_\rho u_\lambda(x)^{p-1} - u_\lambda(x)^{-\gamma}\\
&\quad= \lambda f(x, u_\lambda(x)) + \lambda\hat{\xi}_\rho u_\lambda(x)^{p-1}\\
&\quad\geq \lambda f(x, t\hat{u}_1(x)) + \lambda\hat{\xi}_\rho (t\hat{u}_1(x))^{p-1}\\
&\quad= \tilde{\lambda} f(x, t\hat{u}_1(x)) + \lambda\hat{\xi}_\rho (t\hat{u}_1(x))^{p-1} + \left(\lambda - \tilde{\lambda}\right) f(x, t\hat{u}_1(x))\\
&\quad\geq \hat{\lambda}_1 (t\hat{u}_1(x))^{p-1} + \lambda\hat{\xi}_\rho (t\hat{u}_1(x))^{p-1}\\
&\quad\geq -\Delta_p (t\hat{u}_1(x)) + \lambda\hat{\xi}_\rho (t\hat{u}_1(x))^{p-1} - (t\hat{u}_1(x))^{-\gamma} \quad \text{for a.a. } x \in \Omega.
\end{aligned} \tag{3.25}$$

We set $\tilde{h}_0(x) = \left(\lambda - \tilde{\lambda}\right) f(x, t\hat{u}_1(x))$. We see that since $\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and because of hypothesis H(v), we have $0 \prec \tilde{h}_0$. Therefore, from (3.25) and Papageorgiou-Smyrlis [15, Proposition 4] we infer that $u_\lambda - t\hat{u}_1 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ which contradicts the maximality of $t > 0$, see (3.24). This shows that $\lambda \notin \mathcal{L}$ and so $\lambda^* \leq \tilde{\lambda} < +\infty$. □

Next we show that the critical parameter $\lambda^* > 0$ is admissible.

Proposition 3.7. *If hypotheses H hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Consider a sequence $\{\lambda_n\}_{n\geq 1} \subseteq (0, \lambda^*) \subseteq \mathcal{L}$ such that $\lambda_n \to (\lambda^*)^-$ as $n \to \infty$. From the proof of Proposition 3.3 we know that there exists $u_n \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for each $n \in \mathbb{N}$ such that

$$\{u_n\}_{n\geq 1} \text{ is increasing and } \tilde{u}_* = t\tilde{u} \leq u_n \quad \text{for all } n \in \mathbb{N}. \tag{3.26}$$

Let $\hat{\psi}_{\lambda_n} \in C^1(W_0^{1,p}(\Omega))$ be as in the proof of Proposition 3.3 resulting from the truncation of the reaction of (P_λ) with λ replaced by λ_n at $\{\tilde{u}_*(x), u_{n+1}(x)\} = \{t\tilde{u}(x), u_{n+1}(x)\}$, see (3.15). We know that $u_n \in [\tilde{u}_*, u_{n+1}]$ is the minimizer of $\hat{\psi}_{\lambda_n}$. Therefore, because of (3.15) with $u_\lambda = u_{n+1}$ and hypothesis H(v), we have

$$\begin{aligned}
\hat{\psi}_{\lambda_n}(u_n) \leq \hat{\psi}_{\lambda_n}(\tilde{u}_*) &= \frac{1}{p}\|\nabla\tilde{u}_*\|_p^p - \int_\Omega \left[\tilde{u}^{1-\gamma} + \lambda_n f(x, \tilde{u}_*)\,\tilde{u}_*\right] dx\\
&= \frac{t^p}{p}\|\nabla\tilde{u}\|_p^p - t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma}dx - \lambda_n \int_\Omega f(x, \tilde{u}_*)\,\tilde{u}_* dx\\
&< \frac{t^p}{p}\|\nabla\tilde{u}\|_p^p - t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma}dx.
\end{aligned} \tag{3.27}$$

We know that

$$\|\nabla\tilde{u}\|_p^p = \int_\Omega \tilde{u}^{1-\gamma}dx,$$

see (3.27). Hence, since $t \in (0, 1)$,

$$t^p\|\nabla\tilde{u}\|_p^p \leq t^{1-\gamma}\int_\Omega \tilde{u}^{1-\gamma}dx.$$

This finally gives

$$\hat{\psi}_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}, \tag{3.28}$$

see (3.27).

Consider now the Carathéodory function $\tilde{g}_{\lambda_n}: \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{g}_{\lambda_n}(x,s) = \begin{cases} \tilde{u}_*(x)^{-\gamma} + \lambda_n f(x, \tilde{u}_*(x)) & \text{if } s \leq \tilde{u}_*(x), \\ s^{-\gamma} + \lambda_n f(x,s) & \text{if } \tilde{u}_*(x) < s. \end{cases} \tag{3.29}$$

We set $\tilde{G}_{\lambda_n}(x,s) = \int_0^s \tilde{g}_{\lambda_n}(x,t)dt$ and consider the C^1-functional $\tilde{\varphi}_{\lambda_n}: W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_{\lambda_n}(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \tilde{G}_{\lambda_n}(x,u)dx.$$

Note that

$$\tilde{\varphi}_{\lambda_n}\big|_{[\tilde{u}_*, u_{n+1}]} = \hat{\psi}_{\lambda_n}\big|_{[\tilde{u}_*, u_{n+1}]}.$$

Then, see (3.28), we have $\tilde{\varphi}_{\lambda_n}(u_n) < 0$ for all $n \in \mathbb{N}$ and so

$$\|\nabla u_n\|_p^p - \int_\Omega p\tilde{G}_{\lambda_n}(x,u_n)dx < 0.$$

Applying (3.29) and the fact that $u_n \in [\tilde{u}_*, u_{n+1}]$ leads to

$$\begin{aligned} &\|\nabla u_n\|_p^p - \int_\Omega p\left[\tilde{u}_*^{1-\gamma} + \lambda_n f(x,\tilde{u}_*)\right]\tilde{u}_* dx \\ &- \frac{p}{1-\gamma}\int_\Omega \left[u_n^{1-\gamma} - u_*^{1-\gamma}\right] - \lambda_n p \int_\Omega \left[F(x,u_n) - F(x,\tilde{u}_*)\right]dx < 0. \end{aligned} \tag{3.30}$$

Moreover, we know that

$$\langle A(u_n), h\rangle = \int_\Omega \tilde{g}_{\lambda_n}(x,u_n)hdx \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ and for all } n \in \mathbb{N}. \tag{3.31}$$

Choosing $h = u_n \in W_0^{1,p}(\Omega)$ in (3.31) and applying (3.29) and the fact that $u_n \in [\tilde{u}_*, u_{n+1}]$ yields

$$-\|\nabla u_n\|_p^p + \int_\Omega \left[u_n^{1-\gamma} + \lambda_n f(x,u_n)u_n\right]dx = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.32}$$

Adding (3.30) and (3.32) we obtain

$$\int_\Omega \hat{\eta}_{\lambda_n}(x,u_n)dx \leq M_1 \quad \text{for some } M_1 > 0 \text{ and for all } n \in \mathbb{N}. \tag{3.33}$$

Suppose that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. By passing to a subsequence if necessary, we may assume that $\|u_n\| \to +\infty$. We set $y_n = \frac{u_n}{\|u_n\|}$ for $n \in \mathbb{N}$. Then we have $\|y_n\| = 1$ and $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \overset{\mathrm{w}}{\to} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^r(\Omega), \quad \text{with } y \geq 0. \tag{3.34}$$

First assume that $y \neq 0$ and set $\Omega^* = \{x \in \Omega : \ y(x) > 0\}$. We have $|\Omega^*|_N > 0$ and $u_n(x) \to +\infty$ for all $x \in \Omega^*$. We have

$$\frac{F(x,u_n(x))}{\|u_n\|^p} = \frac{F(x,u_n(x))}{u_n(x)^p}y_n(x)^p \to +\infty \quad \text{for a.a. } x \in \Omega^*$$

and so, by Fatou's Lemma,

$$\int_{\Omega^*} \frac{F(x,u_n)}{\|u_n\|^p}dx \to +\infty. \tag{3.35}$$

Since $F \geq 0$, we have

$$\int_{\Omega^*} \frac{F(x,u_n)}{\|u_n\|^p} dx \leq \int_\Omega \frac{F(x,u_n)}{\|u_n\|^p} dx$$

and so, by (3.35),

$$\int_\Omega \frac{F(x,u_n)}{\|u_n\|^p} dx \to +\infty. \tag{3.36}$$

Hypothesis H(iii) implies that

$$0 \leq \hat{\eta}_{\lambda_n}(x,u_n(x)) + \tau_{\lambda^*}(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } n \in \mathbb{N}.$$

Then

$$\frac{p}{1-\gamma} u_n(x)^{1-\gamma} + pF(x,u_n(x)) \leq u_n(x)^{1-\gamma} + \lambda_n f(x,u_n(x))u_n(x) + \tau_{\lambda^*}(x) \tag{3.37}$$

for a.a. $x \in \Omega$ and for all $n \in \mathbb{N}$.

From (3.31) with $h = u_n \in W_0^{1,p}(\Omega)$ we obtain by using (3.29) and (3.26)

$$\|\nabla u_n\|_p^p = \int_\Omega \left[u_n^{1-\gamma} + \lambda_n f(x,u_n)u_n\right] dx \quad \text{for all } n \in \mathbb{N}. \tag{3.38}$$

Applying (3.38) in (3.37) gives

$$p\lambda_n \int_\Omega F(x,u_n)dx \leq \|\nabla u_n\|_p^p + \|\tau_{\lambda^*}\|_1 .$$

Hence

$$p\lambda_n \int_\Omega \frac{F(x,u_n)}{\|u_n\|^p} dx \leq \|\nabla y_n\|_p^p + \frac{\|\tau_{\lambda^*}\|_1}{\|u_n\|^p} \quad \text{for all } n \in \mathbb{N}. \tag{3.39}$$

Comparing (3.36) and (3.39) we have a contradiction.

Next suppose that $y = 0$. For $\mu > 0$ we set $v_n = (p\mu)^{\frac{1}{p}} y_n$ for all $n \in \mathbb{N}$. Then $v_n \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $v_n \to 0$ in $L^r(\Omega)$, see (3.34) and recall that $y = 0$. Then, by (3.29), we get

$$\int_\Omega \tilde{G}_{\lambda_n}(x,v_n)dx \to 0 \quad \text{as } n \to \infty. \tag{3.40}$$

Since $\|u_n\| \to +\infty$, there exists a number $n_0 \in \mathbb{N}$ such that

$$(p\mu)^{\frac{1}{p}} \frac{1}{\|u_n\|} \leq 1 \quad \text{for all } n \geq n_0. \tag{3.41}$$

Moreover, let $t_n \in [0,1]$ be such that

$$\tilde{\varphi}_{\lambda_n}(t_n u_n) = \max_{0 \leq t \leq 1} \tilde{\varphi}_{\lambda_n}(t u_n), \quad n \in \mathbb{N}.$$

Applying (3.41), the representation $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and (3.40) leads to

$$\begin{aligned} \tilde{\varphi}_{\lambda_n}(t_n u_n) &\geq \tilde{\varphi}_{\lambda_n}(v_n) \quad \text{for all } n \geq n_0 \\ &= \mu \|\nabla y_n\|_p^p - \int_\Omega \tilde{G}_{\lambda_n}(x,v_n)dx \\ &= \mu - \int_\Omega \tilde{G}(x,v_n)dx \geq \frac{\mu}{2} \quad \text{for all } n \geq n_1 \geq n_0. \end{aligned} \tag{3.42}$$

But recall that $\mu > 0$ is arbitrary. So, from (3.42) we infer that

$$\tilde{\varphi}_{\lambda_n}(t_n u_n) \to +\infty \quad \text{as } n \to \infty. \tag{3.43}$$

We have

$$\tilde{\varphi}_{\lambda_n}(0)=0 \quad \text{and} \quad \tilde{\varphi}_{\lambda_n}(u_n)<0 \quad \text{for all } n\in\mathbb{N}.$$

From this and (3.43) it follows that $t_n\in(0,1)$ for all $n\geq n_2$. Therefore, we obtain

$$\frac{d}{dt}\tilde{\varphi}_{\lambda_n}(tu_n)\big|_{t=t_0}=0 \quad \text{for all } n\geq n_2$$

which means

$$\|\nabla(t_nu_n)\|_p^p=\int_\Omega \tilde{g}_{\lambda_n}(x,t_nu_n)u_n dx$$

and so

$$p\tilde{\varphi}_{\lambda_n}(t_nu_n)+p\int_\Omega \tilde{G}_{\lambda_n}(x,t_nu_n)dx=\int_\Omega \tilde{g}_{\lambda_n}(x,t_nu_n)(t_nu_n)dx.$$

Then we use hypothesis H(iii), (3.29) and recall that $t_n\in(0,1)$ for all $n\geq n_2$ to get

$$p\tilde{\varphi}_{\lambda_n}(t_nu_n)\leq \int_\Omega \hat{\eta}_{\lambda_n}(x,u_n)dx+M_2$$

for some $M_2>0$ and for all $n\geq n_2$. Taking (3.43) into account gives

$$\int_\Omega \hat{\eta}_{\lambda_n}(x,u_n)dx\to+\infty \quad \text{as } n\to\infty.$$

But this last convergence contradicts (3.33).

It follows that $\{u_n\}_{n\geq 1}\subseteq W_0^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n\overset{\text{w}}{\to}u^* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n\to u^* \quad \text{in } L^r(\Omega) \quad \text{with } u^*\geq \tilde{u}_*. \tag{3.44}$$

Choosing $h=u_n-u^*\in W_0^{1,p}(\Omega)$ in (3.31), recalling that $u_n^{-\gamma}\in L^{r'}(\Omega)$ with $\frac{1}{r}+\frac{1}{r'}=1$, passing to the limit as $n\to\infty$ and applying (3.44) results in

$$\lim_{n\to\infty}\langle A(u_n),u_n-u^*\rangle=0.$$

Since A has the $(\mathrm{S})_+$-property, see Proposition 2.2, we infer that

$$u_n\to u^* \quad \text{in } W_0^{1,p}(\Omega). \tag{3.45}$$

So, if we pass to the limit in (3.31) and apply (3.45), then we obtain

$$\langle A(u^*),h\rangle=\int_\Omega \tilde{g}_{\lambda^*}(x,u^*)hdx \quad \text{for all } h\in W_0^{1,p}(\Omega) \text{ with } u^*\geq\tilde{u}_*.$$

Therefore, we have

$$\langle A(u^*),h\rangle=\int_\Omega \left[(u^*)^{-\gamma}+\lambda^*f(x,u^*)\right]hdx \quad \text{for all } h\in W_0^{1,p}(\Omega).$$

Hence, $u^*\in\mathcal{S}_{\lambda^*}\subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\lambda^*\in\mathcal{L}$. □

In summary, we have proved that

$$\mathcal{L}=(0,\lambda^*].$$

Next we show that we have two solutions for all $\lambda\in(0,\lambda^*)$.

Proposition 3.8. *If hypotheses H hold and $0<\lambda<\lambda^*$, then problem* (P_λ) *has two positive solutions $u_0,\hat{u}\in\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

Proof. From Proposition 3.7 we know that $\lambda^* \in \mathcal{L}$. So, there exists $u^* \in \mathcal{S}_{\lambda^*} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, see Corollary 3.2. According to Proposition 3.5 we can find $u_0 \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that

$$u^* - u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.46}$$

Moreover, let $\vartheta \in (0,\lambda) \subseteq \mathcal{L}$ and $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ be such that

$$u_0 - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \tag{3.47}$$

again by Proposition 3.5. From (3.46) and (3.47) it follows that

$$u_0 \in \operatorname*{int}_{C_0^1(\overline{\Omega})} [u_\vartheta, u^*]. \tag{3.48}$$

We consider the Carathéodory functions $k_\lambda, \hat{k}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_\lambda(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} + \lambda f(x, u_\vartheta(x)) & \text{if } s \leq u_\vartheta(x), \\ s^{-\gamma} + \lambda f(x,s) & \text{if } u_\vartheta(x) < s \end{cases} \tag{3.49}$$

and

$$\hat{k}_\lambda(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} + \lambda f(x, u_\vartheta(x)) & \text{if } s < u_\vartheta(x), \\ s^{-\gamma} + \lambda f(x,s) & \text{if } u_\vartheta(x) \leq s \leq u^*(x), \\ u^*(x)^{-\gamma} + \lambda f(x, u^*(x)) & \text{if } u^*(x) < s. \end{cases} \tag{3.50}$$

We set $K_\lambda(x,s) = \int_0^s k_\lambda(x,t)dt$, $\hat{K}_\lambda(x,s) = \int_0^s \hat{k}_\lambda(x,t)dt$ and consider the C^1-functionals $\sigma_\lambda, \hat{\sigma}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \sigma_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega K_\lambda(x,u)dx, \\ \hat{\sigma}_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \hat{K}_\lambda(x,u)dx. \end{aligned}$$

From (3.49) and (3.50) it is clear that

$$\sigma_\lambda\big|_{[u_\vartheta,u^*]} = \hat{\sigma}_\lambda\big|_{[u_\vartheta,u^*]}. \tag{3.51}$$

Moreover, as in the proof of Proposition 3.1, using (3.49) and (3.50), we show that

$$K_{\sigma_\lambda} \subseteq [u_\vartheta) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad K_{\hat{\sigma}_\lambda} \subseteq [u_\vartheta, u_\lambda] \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right). \tag{3.52}$$

From (3.52) we see that we may assume that $K_{\hat{\sigma}_\lambda} = \{u_0\}$, otherwise we already have a second positive solution for problem (P_λ), see (3.50) and (3.52).

From (3.50) and since $u_\vartheta^{-\gamma} \in L^{p'}(\Omega)$ we infer that $\hat{\sigma}_\lambda$ is coercive and from the Sobolev embedding theorem, we know that $\hat{\sigma}_\lambda$ is sequentially weakly lower semicontinuous. Therefore, we can find $u_0^* \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma}_\lambda(u_0^*) = \inf\left[\hat{\sigma}_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.53}$$

That means $u_0^* \in K_{\hat{\sigma}_\lambda}$ and so $u_0^* = u_0$. From (3.48), (3.51) and (3.53) it follows that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of σ_λ and from [5] and [13] we know that

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \sigma_\lambda. \tag{3.54}$$

We assume that K_{σ_λ} is finite or otherwise, on account of (3.49) and (3.52), we already have an infinity of positive smooth solutions for problem (P_λ) and so we are done. From (3.54) we infer that there exists $\rho \in (0,1)$ small enough such that

$$\sigma_\lambda(u_0) < \inf\left[\sigma_\lambda(u) : \|u-u_0\| = \rho\right] = m_\lambda, \tag{3.55}$$

see Aizicovici-Papageorgiou-Staicu [1, Proof of Proposition 29].

Hypothesis H(ii) implies that if $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, then

$$\sigma_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.56}$$

Claim: σ_λ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\sigma_\lambda(u_n)| \leq M_3 \quad \text{for some } M_3 > 0 \text{ and for all } n \in \mathbb{N}, \tag{3.57}$$

$$(1+\|u_n\|)\,\sigma_\lambda'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.58}$$

From (3.58) we have

$$\left|\langle A(u_n), h\rangle - \int_\Omega k_\lambda(x,u_n)h\,dx\right| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{3.59}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. We choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ in (3.59) and use (3.49) to obtain

$$\left\|\nabla u_n^-\right\|_p^p \leq c_6\left\|u_n^-\right\| \quad \text{for some } c_6 > 0 \text{ and for all } n \in \mathbb{N}.$$

Hence

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.60}$$

Then from (3.57) and (3.60) it follows that

$$\left\|\nabla u_n^+\right\|_p^p - \int_\Omega p\hat{K}_\lambda(x,u_n^+)dx \leq M_4 \quad \text{for some } M_4 > 0 \text{ and for all } n \in \mathbb{N}.$$

This implies

$$\begin{aligned}
\left\|\nabla u_n^+\right\|_p^p &- \int_{\left\{u_n^+\leq u_\vartheta\right\}} p\left[u_\vartheta^{-\gamma} + \lambda f(x,u_\vartheta)\right]u_n^+dx\\
&- \frac{p}{1-\gamma}\int_{\left\{u_\vartheta<u_n^+\right\}}\left[\left(u_n^+\right)^{1-\gamma} - u_\vartheta^{1-\gamma}\right]dx\\
&- p\lambda\int_{\left\{u_\vartheta<u_n^+\right\}}\left[F(x,u_n^+) - F(x,u_\vartheta)\right] \leq M_4
\end{aligned}$$

for all $n \in \mathbb{N}$ and so

$$\left\|\nabla u_n^+\right\|_p^p - \frac{p}{1-\gamma}\int_\Omega\left(u_n^+\right)^{1-\gamma}dx - p\lambda\int_\Omega F(x,u_n^+)dx \leq M_5 \tag{3.61}$$

for some $M_5 > 0$ and for all $n \in \mathbb{N}$. Moreover, we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ in (3.59) which gives

$$\begin{aligned}
-\left\|\nabla u_n^+\right\|_p^p &+ \int_{\left\{u_n^+\leq u_\vartheta\right\}}\left[u_\vartheta^{-\gamma} + \lambda f(x,u_\vartheta)\right]u_n^+dx\\
&+ \int_{\left\{u_\vartheta<u_n^+\right\}}\left[\left(u_n^+\right)^{-\gamma} + \lambda f(x,u_n^+)\right]u_n^+dx \leq \varepsilon_n
\end{aligned}$$

for all $n \in \mathbb{N}$. This leads to

$$-\left\|\nabla u_n^+\right\|_p^p + \int_\Omega \left(u_n^+\right)^{1-\gamma} dx + \lambda \int_\Omega f(x, u_n^+) u_n^+ dx \leq M_6 \tag{3.62}$$

for some $M_6 > 0$ and for all $n \in \mathbb{N}$. Adding (3.61) and (3.62) yields

$$\int_\Omega \hat{\eta}_\lambda(x, u_n^+) dx \leq M_7 \quad \text{for some } M_7 > 0 \text{ and for all } n \in \mathbb{N}. \tag{3.63}$$

Applying (3.63) and reasoning as in the proof of Proposition 3.7 (see the part of the proof after (3.33)), we show that $\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded and so, due to (3.60), $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded as well.

So, we may assume that

$$u_n \xrightarrow{\mathrm{w}} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^r(\Omega). \tag{3.64}$$

Choosing $h = u_n - u \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and applying (3.64), we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u \rangle = 0,$$

which by the $(\mathrm{S})_+$-property of A, see Proposition 2.2, results in $u_n \to u$ in $W_0^{1,p}(\Omega)$. Therefore, σ_λ satisfies the C-condition and this proves the Claim.

On account of (3.55), (3.56) and the Claim, we are able to apply the mountain pass theorem stated as Theorem 2.1 and find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\sigma_\lambda} \subseteq [u_\vartheta) \cap \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad m_\lambda \leq \sigma_\lambda(\hat{u}), \tag{3.65}$$

see (3.52). From (3.49), (3.55) and (3.65) we conclude that $\hat{u} \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\hat{u} \neq u_0$. This finishes the proof. ☐

Summarizing the situation for the positive solution of problem (P_λ) as the parameter $\lambda > 0$ varies, we can state the following bifurcation-type theorem.

Theorem 3.9. *If hypotheses H hold, then there exist $\lambda^* > 0$ such that the following is satisfied:*

- *(a) problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for all $\lambda \in (0, \lambda^*)$;*
- *(b) problem (P_λ) has at least one positive solution $u^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for $\lambda = \lambda^*$;*
- *(c) problem (P_λ) has no positive solution for all $\lambda > \lambda^*$.*

4. Minimal Positive Solutions

In this section we show that problem (P_λ) has a smallest positive solution $\overline{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for every $\lambda \in \mathcal{L} = (0, \lambda^*]$ and we prove the monotonicity and continuity properties of the map $\lambda \to \overline{u}_\lambda$.

From Filippakis-Papageorgiou [2] we know that the solution set $\mathcal{S}_\lambda$ is downward directed for every $\lambda \in \mathcal{L} = (0, \lambda^*]$, that is, if $u_1, u_2 \in \mathcal{S}_\lambda$, then there exists $u \in \mathcal{S}_\lambda$ such that $u \leq u_1$ and $u \leq u_2$.

Proposition 4.1. *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem ($\mathrm{P}_\lambda$) has a smallest positive solution $\overline{u}_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, that is, $\overline{u}_\lambda \leq u$ for all $u \in \mathcal{S}_\lambda$.*

Proof. Invoking Lemma 3.10 of Hu-Papageorgiou [8, p. 178] we know that there exists a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq \mathcal{S}_\lambda$ such that $\inf \mathcal{S}_\lambda = \inf_{n\geq 1} u_n$. Recall that $\mathcal{S}_\lambda$ is downward directed.

Claim: $\tilde{u} \leq u_n$ for all $n \in \mathbb{N}$ (see the proof of Proposition 3.1)

Fix $n \in \mathbb{N}$ and let $\vartheta \in (0, \lambda) \subseteq \mathcal{L}$. According to Proposition 3.5 there exists $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $u_n - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. We introduce the Carathéodory function $e_n : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$e_n(x,s) = \begin{cases} u_\vartheta(x)^{-\gamma} & \text{if } s < u_\vartheta(x), \\ s^{-\gamma} & \text{if } u_\vartheta(x) \leq s \leq u_n(x), \\ u_n(x)^{-\gamma} & \text{if } u_n(x) < s. \end{cases} \tag{4.1}$$

We set $E_n(x,s) = \int_0^s e_n(x,t)dt$ and consider the C^1-functional $\gamma_n : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_n(u) = \frac{1}{p}\|\nabla u_n\|_p^p - \int_\Omega E_n(x,u)dx.$$

From (4.1) it is clear that γ_n is coercive and the Sobolev embedding theorem implies that γ_n is sequentially weakly lower semicontinuous. Therefore, we find $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\gamma_n\left(\tilde{u}_0\right) = \inf\left[\gamma_n(u) : u \in W_0^{1,p}(\Omega)\right].$$

In particular, we have $\gamma_n'(\tilde{u}_0) = 0$ which says that

$$\left\langle A\left(\tilde{u}_0\right), h\right\rangle = \int_\Omega e_n\left(x, \tilde{u}_0\right) h dx \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{4.2}$$

We choose $h = (u_\vartheta - \tilde{u}_0)^+ \in W_0^{1,p}(\Omega)$ in (4.2). Then, applying (4.1), the nonnegativity of f and the fact that $u_\vartheta \in \mathcal{S}_\vartheta$ gives

$$\begin{aligned} \left\langle A\left(\tilde{u}_0\right), \left(u_\vartheta - \tilde{u}_0\right)^+\right\rangle &= \int_\Omega u_\vartheta^{-\gamma}\left(u_\vartheta - \tilde{u}_0\right)^+ dx \\ &\leq \int_\Omega \left[u_\vartheta^{-\gamma} + \vartheta f\left(x, u_\vartheta\right)\right]\left(u_\vartheta - \tilde{u}_0\right)^+ dx \\ &= \left\langle A(u_\vartheta), (u_\vartheta - \tilde{u}_0)^+\right\rangle. \end{aligned}$$

Proposition 2.2 then implies $u_\vartheta \leq \tilde{u}_0$. In the same way, choosing $h = (\tilde{u}_0 - u_n)^+ \in W_0^{1,p}(\Omega)$ in (4.2) and applying again (4.1), $f \geq 0$ and $u_n \in \mathcal{S}_\lambda$ results in

$$\begin{aligned} \left\langle A\left(\tilde{u}_0\right), \left(\tilde{u}_0 - u_n\right)^+\right\rangle &= \int_\Omega u_n^{-\gamma}\left(\tilde{u}_0 - u_n\right)^+ dx \\ &\leq \int_\Omega \left[u_n^{-\gamma} + \lambda f(x, u_n)\right]\left(\tilde{u}_0 - u_n\right)^+ dx \\ &= \left\langle A(u_n), (\tilde{u}_0 - u_n)^+\right\rangle. \end{aligned}$$

As before, by Proposition 2.2, we obtain $\tilde{u}_0 \leq u_n$. So, we have proved that

$$\tilde{u}_0 \in [u_\vartheta, u_n]. \tag{4.3}$$

From (4.1) and (4.3) it follows that $\tilde{u}_0$ is a positive solution of the auxiliary problem (3.1). Therefore, $\tilde{u}_0 = \tilde{u}$ which implies $\tilde{u} \leq u_n$ for all $n \in \mathbb{N}$. This proves the Claim.

We have

$$\langle A(u_n), h\rangle = \int_\Omega \left[u_n^{-\gamma} + \lambda f(x,u_n)\right] h dx \tag{4.4}$$

for all $h \in W_0^{1,p}(\Omega)$ and for all $n \in \mathbb{N}$. Since $0 \le u_n \le u_1$ for all $n \ge 1$, from (4.4) with $h = u_n \in W_0^{1,p}(\Omega)$ and using hypothesis H(iv), we infer that

$$\{u_n\}_{n\ge 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \overset{\mathrm{w}}{\to} \overline{u}_\lambda \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to \overline{u}_\lambda \quad \text{in } L^p(\Omega). \tag{4.5}$$

Moreover, we can say that

$$u_n(x)^{-\gamma} \to \overline{u}_\lambda(x)^{-\gamma} \quad \text{for a.a. } x \in \Omega.$$

From the Claim we know that

$$0 \le u_n(x)^{-\gamma} \le \tilde{u}(x)^{-\gamma} \quad \text{for a.a. } x \in \Omega.$$

Since $\tilde{u}(\cdot)^{-\gamma} \in L^{p'}(\Omega)$, see the proof of Proposition 3.1, from Gasiński-Papageorgiou [4, Problem 1.19, p. 38], we have

$$u_n^{-\gamma} \overset{\mathrm{w}}{\to} \overline{u}_\lambda^{-\gamma} \quad \text{in } L^{p'}(\Omega). \tag{4.6}$$

Therefore, if we choose $h = u_n - \overline{u}_\lambda \in W_0^{1,p}(\Omega)$ in (4.4), pass to the limit as $n \to \infty$ and use (4.5) as well as (4.6), then

$$\lim_{n\to\infty} \langle A(u_n), u_n - \overline{u}_\lambda\rangle = 0,$$

which again by Proposition 2.2 leads to

$$u_n \to \overline{u}_\lambda \quad \text{in } W_0^{1,p}(\Omega). \tag{4.7}$$

So, if we pass to the limit in (4.4) as $n \to \infty$ and use (4.5), (4.6), (4.7), we obtain

$$\langle A\left(\overline{u}_\lambda\right), h\rangle = \int_\Omega \left[\overline{u}_\lambda^{-\gamma} + \lambda f\left(x, \overline{u}_\lambda\right)\right] h dx \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

From the Claim it follows that $\tilde{u} \le \overline{u}_\lambda$. Therefore we conclude that

$$\overline{u}_\lambda \in \mathcal{S}_\lambda \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad \overline{u}_\lambda = \inf \mathcal{S}_\lambda.$$

□

In the next proposition we examine the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ and determine the monotonicity and continuity properties of this map.

Proposition 4.2. *If hypotheses H hold, then the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is*

(a) strictly increasing, that is,

$$0 < \vartheta < \lambda \le \lambda^* \quad \textit{implies} \quad \overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right);$$

(b) left continuous.

Proof. (a) From Proposition 3.5 we know that there exists $u_\vartheta \in \mathcal{S}_\vartheta \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ such that $\overline{u}_\lambda - u_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so, since $\overline{u}_\vartheta \leq u_\vartheta$, it follows $\overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. So, the map $\lambda \to \overline{u}_\lambda$ is strictly increasing.

(b) Suppose that $\{\lambda_n, \lambda\}_{n\geq 1} \subseteq \mathcal{L} = (0, \lambda^*]$ and assume that $\lambda_n \to \lambda^-$. We set $\overline{u}_n = \overline{u}_{\lambda_n} \in \mathcal{S}_{\lambda_n} \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ for all $n \in \mathbb{N}$. We have

$$\langle A(\overline{u}_n), h\rangle = \int_\Omega \left[\overline{u}_n^{-\gamma} + \lambda_n f(x, \overline{u}_n)\right] h dx \tag{4.8}$$

for all $h \in W_0^{1,p}(\Omega)$ and for all $n \in \mathbb{N}$. Moreover, by Proposition 4.1,

$$0 \leq \overline{u}_1 \leq \overline{u}_n \leq \overline{u}_{\lambda^*}. \tag{4.9}$$

On account of (4.9) and by the choice $h = \overline{u}_n \in W_0^{1,p}(\Omega)$ in (4.8), we infer that $\{\overline{u}_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. We have

$$\begin{aligned} -\Delta_p \overline{u}_n &= \overline{u}_n^{-\gamma} + \lambda_n f(x, u_n) && \text{in } \Omega,\\ \overline{u}_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for all $n \in \mathbb{N}$. From (4.9) we see that

$$0 \leq \overline{u}_n^{-\gamma} \leq \overline{u}_1^{-\gamma} \in L^s(\Omega) \text{ with } s > N \text{ and for all } n \in \mathbb{N},$$

see also H(i). Similarly, (4.9) and hypothesis H(i) imply that

$$\{f(\cdot, \overline{u}_n(\cdot))\}_{n\geq 1} \subseteq L^s(\Omega) \text{ is bounded.}$$

Then Proposition 1.3 of Guedda-Véron [7] implies that

$$\|\overline{u}_n\|_\infty \leq M_8 \quad \text{for some } M_8 > 0 \text{ and for all } n \in \mathbb{N}.$$

From this as in the proof of Proposition 3.1 and using Theorem 2.1 of Lieberman [10], there exist $\alpha \in (0,1)$ and $M_9 > 0$ such that

$$\overline{u}_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|\overline{u}_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq M_9 \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

Then, (4.10), the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and the monotonicity of the sequence $\{\overline{u}_n\}_{n\geq 1}$ imply that

$$\overline{u}_n \to \tilde{u}_\lambda \quad \text{in } C_0^1(\overline{\Omega}).$$

We claim that $\tilde{u}_\lambda = \overline{u}_\lambda$. If this is not the case, we can find $z_0 \in \Omega$ such that $\overline{u}_\lambda(z_0) < \tilde{u}_\lambda(z_0)$ which implies $\overline{u}_\lambda(z_0) < \overline{u}_n(z_0)$ for all $n \geq n_0$. But this contradicts (a). Therefore, $\tilde{u}_\lambda = \overline{u}_\lambda$ and so $\lambda \to \overline{u}_\lambda$ is left continuous. □

Summarizing the situation concerning the minimal positive solution of problem (P_λ), we can state the following theorem.

Theorem 4.3. *If hypotheses H hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem ($P_\lambda$) has a smallest positive solution $\overline{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C_0^1(\overline{\Omega})$ is*

- *strictly increasing, that is, $0 < \vartheta < \lambda \leq \lambda^*$ implies $\overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$;*
- *left continuous.*

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