QUASILINEAR ELLIPTIC EQUATIONS WITH UNBALANCED GROWTH AND SINGULAR PERTURBATION

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Dedicated to Professor Patrizia Pucci on the occasion of her 70th birthday

Abstract. In this paper, we study parametric quasilinear elliptic equations driven by the double phase operator, where the right-hand side consists of a singular term and a sublinear term. By combining a new Hopf's Lemma with truncation techniques and an abstract critical point theorem, we establish the existence of three bounded positive solutions and provide an explicit upper bound for the parameter.

1. Introduction

The following functional prototype was first introduced and investigated by Zhikov [50] in the context of strongly anisotropic materials:

$$u \mapsto \int_\Omega \left(\frac{|\nabla u|^p}{p} + \mu(x) \frac{|\nabla u|^q}{q} \right) \mathrm{d}x, \tag{1.1}$$

where $1 < p < q$ and $\mu \in L^\infty(\Omega)$ is a nonnegative weight function. The associated Euler-Lagrange operator is the so-called double phase operator, given by

$$\operatorname{div} \left(|\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right). \tag{1.2}$$

According to Marcellini's terminology [32, 33], the functional (1.1) belongs to the class of integral functionals with non-standard growth conditions. Its energy density exhibits ellipticity of order q at points $x \in \Omega$ where $\mu(x) > 0$, and ellipticity of order p at points where $\mu(x) = 0$. Furthermore, the energy density associated with (1.1) can also serve to model the viscosity coefficients of certain non-Newtonian fluids, see Liu–Dai [28] for further details. For a mathematical study of such integral functionals with (p, q)-growth we refer to the works of Baroni–Colombo–Mingione [8, 9, 10], Baroni–Kuusi–Mingione [11], Colombo–Mingione [16, 17], Byun–Oh [15], De Filippis–Palatucci [20], Marcellini [32, 33], Ok [34], Ragusa–Tachikawa [42] and the references therein.

Given a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$ with boundary $\partial\Omega$ of class $C^{1,1}$, in this paper we study quasilinear elliptic equations involving singular terms of the form

$$\begin{aligned} -\operatorname{div} \left(|\nabla u|^{p-2} \nabla u + \mu(x) |\nabla u|^{q-2} \nabla u \right) &= \lambda \left(\xi(x) u^{-\alpha} + f(x, u) \right) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

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where λ is positive parameter and the following conditions are supposed:

(H) (i) $N < p < q$ and $0 \leq \mu(\cdot) \in C^{0,1}(\overline{\Omega})$;

(ii) $0 < \alpha < 1$, $\xi \in C^{0,\beta}(\overline{\Omega})$ for some $0 < \beta \leq 1$ and $\xi(x) > 0$ for all $x \in \overline{\Omega}$;

(iii) there exist $0 \leq g_0(\cdot) \in C_0^1(\overline{\Omega})$ and $\vartheta > N$ such that $\xi(\cdot)g_0(\cdot)^{-\alpha} \in L^{\vartheta}(\Omega)$.

(iv) $f\colon \Omega\times[0,\infty) \to \mathbb{R}$ is a Carathéodory function and there exist constants $s_0 > 0$, $c_0 > 0$ such that $f(x,s) \geq c_0\xi(x)$ for some $s \in [0,s_0]$ and for a.a. $x \in \Omega$;

(v) $\lim\limits_{s\to+\infty} \frac{f(x,s)}{|s|^{p-1}} = 0$ uniformly for a.a. $x \in \Omega$;

(vi) there exists an open ball of radius R_0 centered at y_0, denoted by $B(y_0,R_0)\,(\subset \Omega)$, such that the inequality

$$\begin{aligned}&\int_{B(y_0,R_0)} \int_{C(q\rho)^{\frac{1}{p}}}^{u_1} \big(\xi(x)s^{-\alpha} + f(x,s)\big)\,\mathrm{d}s\,\mathrm{d}x\\ &> \int_\Omega \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} \big(\xi(x)s^{-\alpha} + f(x,s)\big)\,\mathrm{d}s\,\mathrm{d}x\end{aligned}$$

holds, where $\underline{u}$ is a subsolution of (1.3) given in Lemma 3.2 and u_1 is defined by

$$u_1(x) = \begin{cases} C\,(q\rho)^{\frac{1}{p}}, & x \in \Omega\setminus B(y_0,R_0),\\ C\,(q\rho)^{\frac{1}{p}} - \frac{1}{1-\theta}\,(|x|-R_0), & x \in B(y_0,R_0)\setminus B(y_0,\theta R_0),\\ C\,(q\rho)^{\frac{1}{p}} + R_0, & x \in B(y_0,\theta R_0),\end{cases}$$

where $0 \leq \theta < 1$, $\rho > 0$ is defined in (4.4), and C denotes the embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow C(\Omega)$.

The occurrence of the singular term in (1.3) is motivated by various physical models, including the motion of a body through a viscous fluid, the flow field above a moving conveyor belt, shock waves propagating over smooth surfaces, heterogeneous chemical catalysis, and glacial advance. For further discussion and related results, we refer to Ackroyd [1], Aris [2], Crandall–Rabinowitz–Tatar [18], Shi–Yao [45], Sun–Wu–Long [48], Sun–Wu [47], and the references therein.

In the literature, considerable attention has been devoted to singular double phase problems of the form

$$\begin{aligned} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\big) &= \lambda\xi(x)u^{-\alpha} + \tau f(x,u) && \text{in } \Omega,\\ u &> 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega.\end{aligned} \tag{1.4}$$

From a mathematical perspective, the presence of the singular term introduces substantial and intriguing challenges. Liu–Dai–Papageorgiou–Winkert [29] studied problem (1.4) in the case $\lambda = 1$ and $f(x,u) = u^{r-1}$ with $r > q$, and established the existence of two positive solutions whenever $0 < \tau < \tau^*$ for a suitable $\tau^* > 0$. Their result was generalized by Bai–Gasiński–Papageorgiou [6] to the case of a general $(q-1)$-superlinear nonlinearity f while Liu–Papageorgiou [30] considered the setting $\xi(\cdot) = \tau = 1$ and $f(x,u) = \eta(x)u^{r-1}$ with $r > q$, obtaining the same multiplicity result as in [29]. Papageorgiou–Repovš–Vetro [39] investigated the case

$p > q$, $\lambda = 1$, and $f(x,u) = u^{r-1}$ with $r > p$, again deriving analogous conclusions. In a related direction, Liu–Winkert [31] extended the analysis of [29] to the whole space $\mathbb{R}^N$. Further refinements include the work of Bai–Papageorgiou–Zeng [7], who considered $\xi(\cdot) = 1$ and a $(q-1)$-superlinear nonlinearity f. They showed the existence of a bifurcation-type threshold $\tau^* > 0$ depending on λ, where λ has to be sufficiently large, such that problem (1.4) admits at least two bounded positive solutions if $0 < \tau < \tau^*$, at least one positive solution if $\tau = \tau^*$, and no positive solution if $\tau > \tau^*$. Papageorgiou–Rădulescu–Zhang [37] studied (1.4) under variable exponents p, q, α being Lipschitz continuous with $0 < \alpha(x) < 1$, and proved a similar bifurcation phenomenon as in [7]. On the other hand, Failla–Gasiński–Papageorgiou–Skupień [21] considered the case $\lambda = \tau = 1$ with a $(p-1)$-sublinear nonlinearity f, proving the existence of a bounded positive solution and, under an additional monotonicity assumption on f, its uniqueness. Papageorgiou–Rădulescu–Yuan [36] analyzed (1.4) for $\alpha > 1$, $\lambda = \tau = 1$, and $f(x,u) = \eta(x)u^{r-1}$ with $r < p$, establishing the existence of positive solutions while Papageorgiou–Rădulescu–Zhang [38] addressed the case $\tau = 1$, $\alpha > 1$, and $(q-1)$-superlinear nonlinearities, and proved the existence of a positive solution for every $\lambda > 0$. In addition, we point to other related investigations on the double phase operator with singular nonlinearities, in particular the works by Arora–Dwivedi [3], Arora–Fiscella–Mukherjee–Winkert [4], Bahrouni–Rădulescu–Repovš [5], Farkas–Winkert [22], Garain–Mukherjee [23], Guarnotta–Winkert [24], Han–Liu–Papageorgiou [25], Papageorgiou–Peng [35], Pimenta–Winkert [41], Sim–Son [46], see also the references therein. We emphasize that all the above contributions concern the case $1 < p < N$.

Inspired by the aforementioned works, we are led to the following natural questions:

(i) Is it possible to establish the existence of more than two solutions?
(ii) Can one obtain explicit estimates for the parameters ensuring the existence of multiple solutions?

In this paper, for the sake of simplicity, we address the two questions stated above in the context of problem (1.4) under the assumptions $\lambda = \tau$ and $p > N$, that is, for problem (1.3). As our analysis concerns weak solutions, we begin by providing a precise definition of the concept. A function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is said to be a weak solution of problem (1.3), if $\xi(\cdot)u^{-\alpha}v \in L^1(\Omega)$, $u(x) > 0$ for a.a. $x \in \Omega$ and if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v \,\mathrm{d}x = \lambda\int_\Omega \left(\xi(x)u^{-\alpha} + f(x,u)\right) v\,\mathrm{d}x$$

is satisfied for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Here $W_0^{1,\mathcal{H}}(\Omega)$ represents the Musielak-Orlicz Sobolev space which will be introduced in Section 2.

Our main result reads as follows.

Theorem 1.1. *Let hypotheses* (H) *be satisfied. Then there exist an open interval* Λ *and a constant* $M > 0$ *such that for every* $\lambda \in \Lambda$ *problem* (1.3) *has at least three distinct positive solutions in* $W_0^{1,\mathcal{H}}(\Omega)$*, with their* $W_0^{1,\mathcal{H}}(\Omega)$ *norms less than* M*. Furthermore, we have an estimate for the interval* Λ*, which is* $\Lambda \subset [0,b]$*, where*

$$b = \frac{(1+\rho)\rho}{\rho\frac{K(w_1)}{I(w_1)} - \sup\limits_{I(w)<\rho} K(w)} < +\infty$$

with I and K defined in (4.2).

The proof of Theorem 1.1 relies on truncation techniques combined with an abstract critical point theorem (see Theorem 2.4). We begin by establishing a Hopf's Lemma for the double phase problem (3.1), which in turn allows us to construct a subsolution $\underline{u}$ for problem (1.3). Having constructed the subsolution, we truncate the right-hand side of (1.3) to handle the singular term, resulting in the modified problem (4.1), and subsequently apply the abstract critical point theorem to obtain three distinct solutions. A careful analysis of the proof further enables us to determine an explicit upper bound for the parameter λ. We also present an illustrative example in which the computed upper bound of the parameter is approximately 0.00296, a relatively small value. This demonstrates that the multiplicity of solutions for singular double phase problems is highly sensitive to the size of the parameters, particularly when they are sufficiently small. We note that our paper extends the results of Zhao–He–Zhao [49] from the p-Laplacian to the double phase setting. We also emphasize that the abstract critical point theorem employed here has been widely used to study the multiplicity of solutions for a variety of elliptic problems. For instance, we refer to Bonanno–Molica Bisci [13] for Laplace equations, Kristály–Lisei–Varga [26] for p-Laplacian type equations, and Bonanno–Molica Bisci–Rădulescu [14] for Φ-Laplacian type equations.

The rest of the paper is organized as follows. In Section 2 we recall basic definitions and results on Musielak-Orlicz Sobolev spaces and the double phase operator 1.2, and we state the abstract critical point theorem. In Section 3, we prove a Hopf's Lemma for double phase problems and construct a subsolution for problem (1.3). Finally, in Section 4, we provide the proof of Theorem 1.1 and present a nontrivial example illustrating its applicability.

2. Preliminaries

In this section, we recall the main properties of Musielak-Orlicz spaces and the double phase operator (1.2). Most of the results presented here are taken from Crespo-Blanco–Gasiński–Harjulehto–Winkert [19], Liu–Dai [27] and Papageorgiou–Winkert [40]. First, we denote by $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ the standard Lebesgue spaces equipped with the norm $\|\cdot\|_r$ for every $1 \leq r < \infty$. For $1 < r < \infty$, $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ denote the usual Sobolev spaces endowed with the norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0} = \|\nabla \cdot\|_r$, respectively.

Let $M(\Omega)$ the space of all measurable functions $u\colon \Omega \to \mathbb{R}$ and $\mathcal{H}\colon \Omega \times [0,\infty) \to [0,\infty)$ be the function defined by

$$\mathcal{H}(x,t) = t^p + \mu(x)t^q.$$

Then, the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega)\colon \rho_{\mathcal{H}}(u) < +\infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1\right\},$$

where the modular function $\rho_{\mathcal{H}}(\cdot)$ is given by

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^q\right)\mathrm{d}x. \tag{2.1}$$

The Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega)=\left\{u\in L^{\mathcal{H}}(\Omega)\colon |\nabla u|\in L^{\mathcal{H}}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}}=\|\nabla u\|_{\mathcal{H}}+\|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}}=\|\,|\nabla u|\,\|_{\mathcal{H}}$. Moreover, the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$ is denoted by $W_0^{1,\mathcal{H}}(\Omega)$. The spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$, and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive and separable Banach spaces. We equip the space $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\|u\|=\|\nabla u\|_{\mathcal{H}}.$$

We have the following continuous embedding

$$W_0^{1,\mathcal{H}}(\Omega)\hookrightarrow W_0^{1,p}(\Omega). \tag{2.2}$$

The norm $\|\cdot\|_{\mathcal{H}}$ and the modular function $\rho_{\mathcal{H}}$ are related as follows, see Liu–Dai [27, Proposition 2.1].

Proposition 2.1. *Let (H)(i) be satisfied, $y\in L^{\mathcal{H}}(\Omega)$ and $\rho_{\mathcal{H}}$ be defined by (2.1). Then the following hold:*

(i) *If $y\neq 0$, then $\|y\|_{\mathcal{H}}=\lambda$ if and only if $\rho_{\mathcal{H}}(\frac{y}{\lambda})=1$;*
(ii) *$\|y\|_{\mathcal{H}}<1$ (resp. >1, $=1$) if and only if $\rho_{\mathcal{H}}(y)<1$ (resp. >1, $=1$);*
(iii) *If $\|y\|_{\mathcal{H}}<1$, then $\|y\|_{\mathcal{H}}^q\le\rho_{\mathcal{H}}(y)\le\|y\|_{\mathcal{H}}^p$;*
(iv) *If $\|y\|_{\mathcal{H}}>1$, then $\|y\|_{\mathcal{H}}^p\le\rho_{\mathcal{H}}(y)\le\|y\|_{\mathcal{H}}^q$;*
(v) *$\|y\|_{\mathcal{H}}\to 0$ if and only if $\rho_{\mathcal{H}}(y)\to 0$;*
(vi) *$\|y\|_{\mathcal{H}}\to+\infty$ if and only if $\rho_{\mathcal{H}}(y)\to+\infty$.*

Let $A\colon W_0^{1,\mathcal{H}}(\Omega)\to W_0^{1,\mathcal{H}}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u),v\rangle:=\int_\Omega\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x \tag{2.3}$$

for all $u,v\in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}$ is the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. The operator $A\colon W_0^{1,\mathcal{H}}(\Omega)\to W_0^{1,\mathcal{H}}(\Omega)^*$ has the following properties, see Liu–Dai [27].

Proposition 2.2. *Let (H)(i) be satisfied. Then the operator A defined in (2.3) is bounded, continuous, strictly monotone, coercive, a homeomorphism and fulfills the (S_+)-property, that is,*

$$u_n\rightharpoonup u\quad\text{in } W_0^{1,\mathcal{H}}(\Omega)\quad\text{and}\quad\limsup_{n\to\infty}\,\langle A(u_n),u_n-u\rangle\le 0,$$

imply $u_n\to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.

Next, we recall the definition of the (PS)-condition

Definition 2.3. *Let X be a real and reflexive Banach space and $J\in C^1(X)$. The functional J is said to satisfy the $(\mathrm{PS})_c$-condition if, for $c\in\mathbb{R}$, any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that*

$$J(u_n)\to c\quad\text{and}\quad J'(u_n)\to 0, \tag{2.4}$$

admits a convergent subsequence. Any sequence satisfying (2.4) is called a $(\mathrm{PS})_c$-sequence. The functional J is said to satisfy the (PS)-condition if and only if it satisfies the $(\mathrm{PS})_c$-condition for all $c\in\mathbb{R}$.

The main theoretical tool of this paper is the following critical point theorem due to Bonanno [12, Theorem 2.1], see also the works by Ricceri [43, 44],

Theorem 2.4. *Let X be a reflexive and separable real Banach space, and let $I, K\colon X \to \mathbb{R}$ be two Fréchet differentiable functionals satisfying the following conditions:*

(i) *there exists $w_0 \in X$ such that $I(w_0) = K(w_0) = 0$ and $I(w) \geq 0$ for every $w \in X$;*
(ii) *there exists $w_1 \in X$, $\rho > 0$ such that*
$$I(w_1) > \rho \quad \text{and} \quad \frac{K(w_1)}{I(w_1)} > \frac{\sup\limits_{I(w)<\rho} K(w)}{\rho};$$
(iii) *the functional $I - \lambda K$ is sequentially weakly lower semicontinuous and satisfies the (PS)-condition;*
(iv) $\lim\limits_{\|w\|\to+\infty}(I(w) - \lambda K(w)) = +\infty$*, for each $\lambda \in [0, b]$, where*
$$b = \frac{\gamma\rho}{\rho\frac{K(w_1)}{I(w_1)} - \sup\limits_{I(w)<\rho} K(w)} \quad \text{with } \gamma > 1.$$

Then, there exist an open interval $\Lambda \subset [0, b]$ and a number $M > 0$ such that for each $\lambda \in \Lambda$ the equation $I'(w) - \lambda K'(w) = 0$ admits at least three solutions in X having norm less than M.

3. Hopf's Lemma and construction of a subsolution

In this Section we give first a Hopf's Lemma related to our problem and based on this, we construct a subsolution to problem (1.3). To this end, for $0 \leq h \in L^{\vartheta}(\Omega)$ with $\vartheta > N$, we consider the problem
$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) = h(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{3.1}$$
We say that $u \in W^{1,\mathcal{H}}(\Omega)$ is a weak subsolution of problem (3.1) if $u \leq 0$ on $\partial\Omega$, and if
$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v \,\mathrm{d}x \leq \int_\Omega h(x)v\,\mathrm{d}x$$
is fulfilled for all $v \in W^{1,\mathcal{H}}_0(\Omega)$ with $v \geq 0$.

First, we prove the following Hopf's Lemma.

Proposition 3.1. *Let hypothesis (H)(i) be satisfied and u be a solution of (3.1) such that $u \geq 0$ a.e. in Ω and u does not vanish identically on Ω. For $x_0 \in \partial\Omega$, assume $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$. Then*
$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$
where ν is the interior unit normal of $\partial\Omega$ at x_0.

Proof. First, we choose $R > 0$ small enough such that $B(x_1, 2R) \subset \Omega$ and $x_0 \in \partial B(x_1, 2R)$, where $x_1 = x_0 + 2R\nu$. Let $\Omega_1 = \{x \in \Omega\colon R < |x - x_1| < 2R\}$ and $\kappa = \inf\{u(x)\colon |x - x_1| = R\}$. From Theorem 3.3 by Liu–Dai [28] it follows that $\kappa > 0$. Note that if $R \to 0$, then x_1 tends to x_0. Thus, we have
$$\kappa \to 0 \quad \text{and} \quad \frac{\kappa}{R} \to 0.$$

We define

$$M=\sup\{|\nabla\mu(x)|\colon x\in\Omega_1\},\quad \ell=-\ln\left(\frac{\kappa}{R}\right)+\frac{N-1}{R}+M$$

and

$$v(s)=\frac{\kappa\left(e^{\frac{\ell s}{p-1}}-1\right)}{e^{\frac{\ell R}{p-1}}-1},\quad \text{for all } s\in[0,R].$$

We see at once that $v(0)=0$, $v(R)=\kappa$,

$$v'(s)=\frac{\frac{\kappa\ell}{p-1}\cdot e^{\frac{\ell s}{p-1}}}{e^{\frac{\ell R}{p-1}}-1},\quad v''(s)=\frac{\ell}{p-1}v'(s) \tag{3.2}$$

and

$$\ell>0,\quad 0<v'(s)<1,\quad \text{for all } s\in(0,R) \tag{3.3}$$

provided R is sufficiently small.

For simplicity, set $x_1=0$. We write $r=|x|$ and $s=2R-r$. Obviously if $r\in[R,2R]$, then $s\in[0,R]$. Setting

$$w(r)=v(2R-r)=v(s),$$

we can see that

$$w'(r)=-v'(s),\quad w''(r)=v''(s).$$

Now we define

$$w(x)=w(r)\quad \text{for any } x\in\Omega_1 \text{ and } |x|=r.$$

Then it follows from (3.2) and (3.3) that

$$\begin{aligned}
&\operatorname{div}\left(|\nabla w|^{p-2}\nabla w+\mu(x)|\nabla w|^{q-2}\nabla w\right)\\
&=(p-1)\left|w'(r)\right|^{p-2}w''(r)+\frac{N-1}{r}\left|w'(r)\right|^{p-2}w'(r)\\
&\quad+\mu(x)(q-1)\left|w'(r)\right|^{q-2}w''(r)+\mu(x)\frac{N-1}{r}\left|w'(r)\right|^{q-2}w'(r)\\
&\quad+\left|w'(r)\right|^{q-2}w'(r)\sum_{i=1}^{N}\frac{\partial\mu}{\partial x_i}\frac{x_i}{r}\\
&=(p-1)\left(v'(s)\right)^{p-2}v''(s)-\frac{N-1}{r}\left(v'(s)\right)^{p-1}+\mu(x)(q-1)\left(v'(s)\right)^{q-2}v''(s)\\
&\quad-\mu(x)\frac{N-1}{r}\left(v'(s)\right)^{q-1}-\left(v'(s)\right)^{q-1}\sum_{i=1}^{N}\frac{\partial\mu}{\partial x_1}\frac{x_i}{r}\\
&\geq\left(\ell-\frac{N-1}{r}\right)\left(v'(s)\right)^{p-1}+\mu(x)\left(\frac{q-1}{p-1}\ell-\frac{N-1}{r}\right)\left(v'(s)\right)^{q-1}-M\left(v'(s)\right)^{q-1}\\
&\geq\left(\ell-\frac{N-1}{r}-M\right)\left(v'(s)\right)^{q-1}+\mu(x)\left(\frac{q-1}{p-1}\ell-\frac{N-1}{r}\right)\left(v'(s)\right)^{q-1}\\
&\geq\left(-\ln\frac{\kappa}{R}\right)\left(1+\mu(x)\right)\left(v'(s)\right)^{q-1}>0.
\end{aligned}$$

Therefore, we obtain

$$-\operatorname{div}\left(|\nabla w|^{p-2}\nabla w+\mu(x)|\nabla w|^{q-2}\nabla w\right)-h(x)<0,$$

which implies that w is a subsolution of (3.1) in Ω_1 satisfying $w(x_0) = 0$ and $\frac{\partial w}{\partial \nu}(x_0) > 0$. Now we can use Lemma 3.2 of Liu–Dai [28] to conclude that $u \geq w$ in Ω_1. Since $u(x_0) = w(x_0) = 0$, we have

$$\begin{aligned}
\lim_{s\to 0^+} \frac{u(x_0 + s(x_1 - x_0)) - u(x_0)}{s} &\geq \lim_{s\to 0^+} \frac{w(x_0 + s(x_1 - x_0)) - w(x_0)}{s}\\
&= \nabla w(x_0) \cdot (x_1 - x_0)\\
&= \nabla w(x_0) \cdot 2R\nu\\
&= 2R\frac{\partial w}{\partial \nu}(x_0) > 0.
\end{aligned}$$

The left-hand side is equal to $2R\frac{\partial u}{\partial \nu}(x_0)$ and so $\frac{\partial u}{\partial \nu}(x_0) > 0$. This completes the proof of the proposition. □

Lemma 3.2. *Let hypotheses (H)(i)–(iv) be satisfied. Then there exists a subsolution $\underline{u} \in W^{1,\mathcal{H}}(\Omega)$ of (1.3) such that $\underline{u}(x) > 0$ for all $\lambda \in (0, +\infty)$, $\xi\underline{u}^{-\alpha} \in L^{\vartheta}(\Omega)$ with $\vartheta > N$ and $\|\underline{u}\|_\infty \leq s_0$, where $s_0 > 0$ is given by (H)(iv).*

Proof. Using (H)(ii), we can get that the problem

$$-\operatorname{div}\left(|\nabla v|^{p-2}\nabla v + \mu(x)|\nabla v|^{q-2}\nabla v\right) = \lambda\xi(x) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

has a unique positive solution $v \in W_0^{1,\mathcal{H}}(\Omega)$ since $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ is a homeomorphism, see Proposition 2.2. Furthermore, $v \in C^{1,\beta}(\Omega)$ since $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$ for $p > N$ and $\xi \in C^{0,\beta}(\Omega)$, see (2.2). By Proposition 3.1, we know that $\frac{\partial v}{\partial \nu} > 0$ on $\partial\Omega$, where ν is the interior unit normal on $\partial\Omega$.

We claim that there exists a constant C such that

$$Cv(x) \geq g_0(x) \quad \text{for all } x \in \overline{\Omega},$$

where g_0 is given in (H)(iii).

Let $x_0 \in \partial\Omega$ and choose $x \in \Omega$ near x_0 such that $x - x_0$ is in the direction of ν. Since $\frac{\partial v}{\partial \nu} > 0$ on $\partial\Omega$ and $\frac{\partial v}{\partial \nu}$ is a continuous function on $\overline{\Omega}$, there exists a constant $\varepsilon_0 > 0$ such that $\frac{\partial v}{\partial \nu}(x) \geq \varepsilon_0$ for all x near x_0. By (H)(iii), it clear that there is a constant $M > 0$ such that $\frac{\partial g_0}{\partial \nu}(x) \leq M$ for all $x \in \Omega$. Thus we can find a constant $C = (M + 1)/\varepsilon_0$ such that

$$C\frac{\partial v}{\partial \nu}(x) > \frac{\partial g_0}{\partial \nu}(x),$$

for all x near x_0. Combining the fact that $v, g_0 \in W_0^{1,\mathcal{H}}(\Omega)$ and $v(x_0) = g_0(x_0) = 0$ for $x_0 \in \partial\Omega$, we integrate the above inequality from x_0 to x along ν, that is

$$\int_{x_0}^{x} C\frac{\partial v}{\partial \nu}\,\mathrm{d}\nu > \int_{x_0}^{x} \frac{\partial g_0}{\partial \nu}\,\mathrm{d}\nu.$$

This implies

$$Cv(x) > g_0(x), \quad \text{for } x \in \Omega$$

and so

$$Cv(x) \geq g_0(x), \quad \text{for } x \in \overline{\Omega}.$$

By this inequality and hypothesis (H)(iii), we have $\xi v^{-\alpha} \le C^\alpha \xi g_0^{-\alpha} \in L^\vartheta(\Omega)$, so $\xi v^{-\alpha} \in L^\vartheta(\Omega)$. We take $\varepsilon > 0$ small enough such that $\underline{u} = \varepsilon^{\frac{1}{p-1}} v$ satisfies $0 < \underline{u}(x) \le \min\{1, s_0\}$. Thus $\xi \underline{u}^{-\alpha} \in L^\vartheta(\Omega)$. We then get

$$\begin{aligned}&\int_\Omega \left(|\nabla \underline{u}|^{p-2}\nabla \underline{u} + \mu(x)|\nabla \underline{u}|^{q-2}\nabla \underline{u}\right)\cdot \nabla g_0 \,\mathrm{d}x\\ &= \varepsilon\left(\int_\Omega \left(|\nabla v|^{p-2}\nabla v + \varepsilon^{\frac{q-1}{p-1}}\mu(x)|\nabla v|^{q-2}\nabla v\right)\cdot \nabla g_0 \,\mathrm{d}x\right) < \varepsilon\lambda\xi(x).\end{aligned}$$

Combining (H)(iv) with the fact $\|\underline{u}\|_\infty \le 1$, we have

$$\begin{aligned}&-\operatorname{div}\left(|\nabla \underline{u}|^{p-2}\nabla \underline{u} + \mu(x)|\nabla \underline{u}|^{q-2}\nabla \underline{u}\right) - \lambda\xi(x)\underline{u}^{-\alpha}(x) - \lambda f(x,\underline{u}(x))\\ &\le \lambda\xi(x)(\varepsilon - 1 - c_0) \le 0\end{aligned}$$

whenever $\lambda \in [0,\infty)$. Thus $\underline{u}$ is a subsolution of (1.3). ☐

Corollary 3.3. *Let hypotheses (H)(i)–(iv) be satisfied. If u is a solution of (1.3), then $u(x) \ge \underline{u}(x)$ for a.a. $x \in \Omega$ and for all $\lambda \in (0,+\infty)$, where $\underline{u}$ is given in Lemma 3.2.*

4. Three distinct solutions

In this section we are going to prove Theorem 1.1. First, we define

$$g(x,s) = \begin{cases} \xi(x)s^{-\alpha} + f(x,s), & \text{if } s \ge \underline{u}(x),\\ 0, & \text{if } s < \underline{u}(x)\end{cases}$$

and consider the problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) = \lambda g(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega. \tag{4.1}$$

We denote by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \,\mathrm{d}x, \quad K(u) = \int_\Omega G(x,u)\,\mathrm{d}x, \tag{4.2}$$

where $G(x,u) = \int_{\underline{u}}^u g(x,t)\,\mathrm{d}t$, and write

$$J_\lambda(u) = I(u) - \lambda K(u).$$

The definition of $g(\cdot,\cdot)$ and condition (H)(v) imply that for any $\varepsilon > 0$,

$$|g(x,s)| \le \xi\underline{u}^{-\alpha} + c_1 + \varepsilon|s|^{p-1},$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for some $c_1 > 0$. Consequently

$$\begin{aligned}|G(x,u)| &\le \int_{\underline{u}}^u |g(x,s)|\,\mathrm{d}s\\ &\le \int_{\underline{u}}^u (\xi\underline{u}^{-\alpha} + c_1 + \varepsilon|s|^{p-1})\,\mathrm{d}s\\ &= \xi\underline{u}^{-\alpha}(u-\underline{u}) + c_1(u-\underline{u}) + \frac{\varepsilon}{p}(|u|^p - |\underline{u}|^p)\\ &\le \xi\underline{u}^{-\alpha}u + c_1 u + \frac{\varepsilon}{p}|u|^p,\end{aligned} \tag{4.3}$$

since $\underline{u} > 0$. Using Lemma 3.2 and the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ for $1 \le r \le \infty$, we can see that G is integrable over Ω. Therefore J_λ is well-defined and of class C^1. Furthermore, by Corollary 3.3, any critical point of J_λ is a positive weak solution of problem (1.3).

We write

$$\mathcal{S}=\overline{B(y_0,R_0)\setminus B(y_0,\theta R_0)},\quad \mu_0=\min_{x\in\mathcal{S}}\mu(x),\quad \rho=\left(\frac{1}{p}+\frac{\mu_0}{q}\right)R_0^N\omega_N, \tag{4.4}$$

where $y_0\in\Omega$ and R_0 are defined in (H)(vi).

Lemma 4.1. *Let hypothesis (H)(i) be satisfied. Then $\rho/I(u_1)$ is continuous with respect to θ in the interval $[0,1)$. Moreover, there exists $\theta_0\in[0,1)$ such that*

$$\frac{1}{2}<\frac{\rho}{I(u_1)}<1. \tag{4.5}$$

Proof. By the definition of u_1, we can deduce that

$$\begin{aligned} I(u_1)&=\frac{1}{p}\int_\Omega|\nabla u_1|^p\,\mathrm{d}x+\frac{1}{q}\int_\Omega\mu(x)\,|\nabla u_1|^q\,\mathrm{d}x\\ &=\left(\frac{1}{p(1-\theta)^p}+\frac{\mu(y)}{q(1-\theta)^q}\right)\left(1-\theta^N\right)R_0^N\omega_N \end{aligned}$$

for some $y\in B(y_0,R_0)\setminus B(y_0,\theta R_0)$. Thus

$$\frac{\rho}{I(u_1)}=\frac{\frac{1}{p}+\frac{\mu_0}{q}}{\left(\frac{1}{p(1-\theta)^p}+\frac{\mu(y)}{q(1-\theta)^q}\right)\left(1-\theta^N\right)}.$$

This implies that $\rho/I(u_1)$ is continuous with respect to θ in the interval $[0,1)$. Moreover, one has

$$\frac{\rho}{I(u_1)}\to 0\quad\text{as }\theta\to 1\quad\text{and}\quad\frac{\rho}{I(u_1)}\to\frac{\frac{1}{p}+\frac{\mu_0}{q}}{\frac{1}{p}+\frac{\mu(y)}{q}}\le 1\quad\text{as }\theta\to 0.$$

Hence there exists $\theta_0\in[0,1)$ such that

$$\frac{1}{2}<\frac{\rho}{I(u_1)}<1.$$

This completes the proof. □

Lemma 4.2. *Let hypotheses (H) be satisfied. Then the inequality*

$$\sup_{I(u)<\rho}K(u)<\frac{1}{2}K(u_1)<\frac{\rho}{I(u_1)}K(u_1) \tag{4.6}$$

holds true.

Remark 4.3. *Let*

$$b=\frac{(1+\rho)\rho}{\rho\frac{K(w_1)}{I(w_1)}-\sup_{I(u)<\rho}K(u)}.$$

Then it follows from (4.6) that

$$b\le\frac{(1+\rho)\rho}{\rho\frac{K(w_1)}{I(w_1)}-\frac{1}{2}K(u_1)}<+\infty.$$

Proof of Lemma 4.2. By the definition of u_1 in (H)(vi) we know that $u_1(x) > \underline{u}(x)$ for all $x \in \Omega$. Thus by the embedding $W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$, we get that

$$\begin{aligned}
u(x) &\leq \sup_{x\in\Omega} |u(x)| \leq \|u\|_{C^{0,\alpha}} \leq C\,\|u\|_{W_0^{1,p}} = C\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq C\left(\int_\Omega (|\nabla u|^p + \mu(x)|\nabla u|^q)\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq C\left(\frac{q}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{q}{q}\int_\Omega \mu(x)|\nabla u|^q \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&= C\,(qI(u))^{\frac{1}{p}} < C\,(q\rho)^{\frac{1}{p}} \leq u_1(x)
\end{aligned}$$

for all $u \in \{u \in W_0^{1,\mathcal{H}}(\Omega)\colon I(u) < \rho\}$. Moreover, by the definition of g and (H)(iv), we have

$$\sup_{I(u)<\rho} K(u) = \sup_{I(u)<\rho} \int_\Omega \int_{\underline{u}}^{u} g(x,s)\,\mathrm{d}s\,\mathrm{d}x \leq \int_\Omega \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x. \tag{4.7}$$

Again (H)(vi) yields

$$\frac{1}{2}\int_\Omega \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x < \frac{1}{2}\int_{B(y_0,R_0)} \int_{C(q\rho)^{\frac{1}{p}}}^{u_1} g(x,s)\,\mathrm{d}s\,\mathrm{d}x. \tag{4.8}$$

By the additivity of the integral over the domain, we obtain

$$\begin{aligned}
\int_\Omega \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x &= \int_{\Omega\setminus B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x \\
&\quad + \int_{B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&\int_{B(y_0,R_0)} \int_{C(q\rho)^{\frac{1}{p}}}^{u_1} g(x,s)\,\mathrm{d}s\,\mathrm{d}x \\
&= \int_{B(y_0,R_0)} \int_{\underline{u}}^{u_1} g(x,s)\,\mathrm{d}s\,\mathrm{d}x - \int_{B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned} \tag{4.10}$$

From (4.8), (4.9), and (4.10), we get that

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x + \frac{1}{2}\int_{B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x \\
&< \frac{1}{2}\int_{B(y_0,R_0)} \int_{\underline{u}}^{u_1} g(x,s)\,\mathrm{d}s\,\mathrm{d}x - \frac{1}{2}\int_{B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x,
\end{aligned}$$

which implies

$$\frac{1}{2}\int_{\Omega\setminus B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x + \int_{B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x$$

$$< \frac{1}{2} \int_{B(y_0,R_0)} \int_{\underline{u}}^{u_1} g(x,s)\,\mathrm{d}s\,\mathrm{d}x.$$

Hence

$$\begin{aligned}
&\int_{\Omega\setminus B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x + \int_{B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x\\
&< \frac{1}{2} \int_{B(y_0,R_0)} \int_{\underline{u}}^{u_1} g(x,s)\,\mathrm{d}s\,\mathrm{d}x + \frac{1}{2} \int_{\Omega\setminus B(y_0,R_0)} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}$$

Consequently

$$\int_{\Omega} \int_{\underline{u}}^{C(q\rho)^{\frac{1}{p}}} g(x,s)\,\mathrm{d}s\,\mathrm{d}x < \frac{1}{2} \int_{\Omega} \int_{\underline{u}}^{u_1} g(x,s)\,\mathrm{d}s\,\mathrm{d}x. \tag{4.11}$$

From (4.5), (4.7) and (4.11) we see that (4.6) holds true. □

Lemma 4.4. *Let hypotheses* (H) *be satisfied. Then, for all $\lambda > 0$, the functional J_λ is sequentially weakly lower semicontinuous and satisfies the* (PS)*-condition. Furthermore, it is coercive for all $\lambda \in (0,b]$, where b is given in Remark 4.3.*

Proof. The proof is divided into four steps.

Step 1: K is weakly continuous, i.e., if $u_n \rightharpoonup u$, then $K(u_n) \to K(u)$.

Let $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$. The embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ is continuous (see (2.2)) and $W_0^{1,p}(\Omega) \hookrightarrow C(\Omega)$ is compact since $p > N$. Thus $u_n \to u$ in $C(\Omega)$. This implies that u_n converges uniformly to u in Ω as $n \to \infty$.

Using Lemma 3.2 and the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ for $1 \le r \le \infty$, we can see that the right-hand side of (4.3) is integrable over Ω and thus $G(x,u_n)$ has equi-absolutely continuous integrals. From Vitali's convergence theorem it follows that

$$K(u_n) = \int_\Omega G(x,u_n)\,\mathrm{d}x \to \int_\Omega G(x,u)\,\mathrm{d}x = K(u).$$

Step 2: J_λ is sequentially weakly lower semicontinuous.

Clearly, I is sequentially weakly lower semicontinuous by Fatou's lemma. This together with Step 1 implies that J_λ is sequentially weakly lower semicontinuous as well.

Step 3: J_λ satisfies the (PS)-condition.

For every $c \in \mathbb{R}$, let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ be a $(\mathrm{PS})_c$-sequence, see Definition 2.3. We claim that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Indeed, If $\|u_n\| \le 1$, we are done. Let $\|u_n\| > 1$. Then, from (4.3) and Proposition 2.1(iv) we have that

$$\begin{aligned}
c + o(1) &\geq J_\lambda(u_n)\\
&= \frac{1}{p}\int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n|^q \,\mathrm{d}x - \lambda \int_\Omega G(x,u_n)\,\mathrm{d}x\\
&\geq \frac{1}{q}\|u_n\|^p - \lambda\int_\Omega \left(c_1 u_n + \xi \underline{u}^{-\alpha} u_n + \frac{\varepsilon}{p}|u_n|^p\right)\mathrm{d}x\\
&\geq \left(\frac{1}{q} - \frac{c_2\lambda\varepsilon}{p}\right)\|u_n\|^p - c_1\lambda\|u_n\|_1 - \lambda\|\xi\underline{u}^{-\alpha}\|_m\|u_n\|_{m'}\\
&\geq \left(\frac{1}{q} - \frac{c_2\lambda\varepsilon}{p}\right)\|u_n\|^p - c_3\lambda\|u_n\| - c_4\lambda\|u_n\|\\
&= \|u_n\|\left(\left(\frac{1}{q} - \frac{c_2\lambda\varepsilon}{p}\right)\|u_n\|^{p-1} - c_3\lambda - c_4\lambda\right).
\end{aligned} \tag{4.12}$$

Taking $\varepsilon < p/(qc_2\lambda)$ gives us the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}}(\Omega)$. Then, for a subsequence if necessary, not relabeled, we can assume $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$. As in the proof of Step 1, we can show that K' is completely continuous. This means that if $u_n \rightharpoonup u$, then $K'(u_n) \to K'(u)$. Since $J'_\lambda(u_n) = I'(u_n) - K'(u_n) \to 0$, one has that $I'(u_n) \to K'(u)$. Therefore, it follows that $u_n \to u$ because I' is a mapping of type (S_+), see Proposition 2.2.

Step 4: J_λ is coercive for $0 < \lambda \leq b$.

From (4.12) we can easily conclude that J_λ is coercive. □

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. We take $w_0 = 0 \in W_0^{1,\mathcal{H}}(\Omega)$ and the hypothesis (i) in Theorem 2.4 is satisfied. We come to verify hypothesis (ii) in Theorem 2.4 and choose a subset $\Omega_n = \{x \in \Omega\colon \operatorname{dist}(x,\partial\Omega) \geq 1/n\} \subset \Omega$ for $n \geq 1$. Then, for all $\varepsilon > 0$ we can find $n_0 > 0$ such that for $n > n_0$ we have $m(\Omega \setminus \Omega_n) < \varepsilon$, where m denotes the Lebesgue measure on $\mathbb{R}^N$. Let

$$u_1^n(x) = \begin{cases} u_1(x), & \text{if } x \in \Omega_n,\\ 0, & \text{if } x \in \Omega\setminus\Omega_n\end{cases}, \qquad \varphi(x) = \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right), & \text{if } |x| \leq 1,\\ 0, & \text{if } |x| > 1,\end{cases}$$

where

$$c = \frac{1}{\int_{|x|\leq 1}\exp\left(\frac{1}{|x|^2-1}\right)\mathrm{d}x}.$$

We define

$$\varphi_n(x) = n^N\varphi(nx), \quad \text{for } x \in \mathbb{R}^N \quad \text{and} \quad (\varphi_n * u_1^n)(x) = \int_\Omega \varphi_n(x-y)u_1^n(y)\,\mathrm{d}y.$$

Then $\operatorname{supp}(\varphi_n * u_1^n) = \overline{\{x\colon (\varphi_n * u_1^n)(x) \neq 0\}} \subset \Omega$, $(\varphi_n * u_1^n) \in C_0^\infty(\Omega)$ and $(\varphi_n * u_1^n)(x) \to u_1(x)$ as $n \to \infty$ for a.a. $x \in \Omega$. By Lebesgue's dominated convergence theorem and (4.3), we have

$$K(\varphi_n * u_1^n) = \int_\Omega G(x,\varphi_n * u_1^n)\,\mathrm{d}x \to \int_\Omega G(x,u_1)\,\mathrm{d}x = K(u_1). \tag{4.13}$$

By the properties of the mollification and the definition of u_1^n, we can get that

$$\begin{aligned}
&\frac{1}{p}\int_{\Omega\setminus\Omega_n} |\nabla(\varphi_n * u_1^n)(x)|^p \,\mathrm{d}x + \frac{1}{q}\int_{\Omega\setminus\Omega_n} \mu(x)\,|\nabla(\varphi_n * u_1^n)(x)|^q \,\mathrm{d}x\\
&= \frac{1}{p}\int_{\Omega\setminus\Omega_n} |(\varphi_n * \nabla u_1^n)(x)|^p \,\mathrm{d}x + \frac{1}{q}\int_{\Omega\setminus\Omega_n} \mu(x)\,|(\varphi_n * \nabla u_1^n)(x)|^q \,\mathrm{d}x\\
&\leq \int_{\Omega\setminus\Omega_n} \left|\frac{1}{1-\theta}\right|^p \mathrm{d}x + \|\mu\|_\infty \int_{\Omega\setminus\Omega_n} \left|\frac{1}{1-\theta}\right|^q \mathrm{d}x\\
&= (1-\theta)^{-p}\, m(\Omega\setminus\Omega_n) + \|\mu\|_\infty\,(1-\theta)^{-q}\, m(\Omega\setminus\Omega_n)\\
&\leq \Big((1-\theta)^{-p} + \|\mu\|_\infty\,(1-\theta)^{-q}\Big)\,\varepsilon
\end{aligned}$$

whenever $n > n_0$. Therefore, we have

$$I(\varphi_n * u_1^n) \to I(u_1). \tag{4.14}$$

From (4.13) and (4.14), for every $\varepsilon > 0$, we can choose n_0 sufficiently large such that

$$\left|\frac{K(\varphi_n * u_1^n)}{I(\varphi_n * u_1^n)} - \frac{K(u_1)}{I(u_1)}\right| < \frac{\varepsilon}{\rho}, \quad \text{for } n > n_0.$$

So if

$$\varepsilon = \frac{\rho}{I(u_1)}K(u_1) - \sup_{I(u)<\rho} K(u),$$

then

$$\rho\frac{K(u_1)}{I(u_1)} - \rho\frac{K(\varphi_n * u_1^n)}{I(\varphi_n * u_1^n)} < \varepsilon = \frac{\rho}{I(u_1)}K(u_1) - \sup_{I(u)<\rho} K(u).$$

This implies

$$\sup_{I(u)<\rho} K(u) < \rho\frac{K(\varphi_n * u_1^n)}{I(\varphi_n * u_1^n)}, \quad \text{for } n > n_0.$$

In Theorem 2.4 we choose $w_1 = \varphi_n * u_1^n \in W_0^{1,\mathcal{H}}(\Omega)$ for some $n > n_0$. This together with Lemma 4.1, we can deduce that hypothesis (ii) is satisfied.

According to Lemma 4.4, assumptions (iii) and (iv) of Theorem 2.4 also hold. Thus there exist an open interval $\Lambda \subset [0,b]$ and a number $M > 0$ such that for each $\lambda \in \Lambda$, the equation $J_\lambda'(u) = I'(u) - \lambda K'(u) = 0$ admits at least three solutions in $W_0^{1,\mathcal{H}}(\Omega)$ having $W_0^{1,\mathcal{H}}(\Omega)$-norms less than M. From Corollary 3.3, it follows that the three solutions are positive. This concludes the proof. □

Finally, we give an example in order to verify the applicability of Theorem 1.1.

Example 4.5. *Let $\Omega = (-2,2)$, $\xi(x) = 5x^4 + 1$, $p = 2$, $q = 4$, $\mu(x) = x^2$ and let $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be defined as*

$$f(x,s) = \begin{cases} 93e^s, & \text{if } s \leq 6,\\ \sqrt{s} + 93e^6 - \sqrt{6}, & \text{if } s > 6. \end{cases}$$

We consider the problem

$$\begin{aligned} -\left(u'+x^2|u'|^2u'\right)' &= \lambda\left[(5x^4+1)u^{-\alpha}+f(x,u)\right], && x\in(-2,2),\\ u &> 0 && x\in(-2,2),\\ u &= 0, && x\in\{-2,2\}. \end{aligned} \tag{4.15}$$

Let $c_0=1$ and $s_0=1$ be as in condition (H)(iv). From the definition of f, we have

$$f(x,s)=93e^s\geq c_0\xi(x)>0, \quad \textit{for } s\in[0,1]\,.$$

Obviously, f satisfies the condition (H)(v). Next, we will show that f satisfies the condition (H)(vi).

Consider

$$\begin{aligned} -\left(u'+x^2|u'|^2u'\right)' &= 5x^4+1, && x\in(-2,2),\\ u &> 0, && x\in(-2,2),\\ u &= 0, && x\in\{-2,2\}. \end{aligned} \tag{4.16}$$

We can show that $v=\frac{4-x^2}{2}$ is the solution of (4.16) with $\|v\|_\infty=2$. If we choose $\varepsilon\leq 1/4$ and $\underline{u}=\varepsilon v$, we obtain

$$\|\underline{u}\|_\infty=\varepsilon\|v\|_\infty=2\varepsilon\leq\frac{1}{2}<1.$$

So let $\varepsilon=1/4$. From the result of Lemma 3.2, i.e. $\xi(x)\underline{u}^{-\alpha}\in L^\vartheta((-2,2))$ with $\vartheta>1$, and $\xi(x)=5x^4+1$, if we let $\vartheta=2$, we have $\underline{u}^{-\alpha}\in L^2((-2,2))$. Considering this fact and $\underline{u}=\frac{4-x^2}{8}$, we can choose $\alpha<\frac{1}{2}$ such that $\int_{-2}^2|(5x^4+1)\underline{u}^{-\alpha}|^2\,\mathrm{d}x<\infty$. Indeed, this is possible since

$$\int_{-2}^2|(5x^4+1)\underline{u}^{-\alpha}|^2\,\mathrm{d}x\leq 81\int_{-2}^2\frac{8^{2\alpha}}{(4-x^2)^{2\alpha}}\,\mathrm{d}x=81\times 4^{1-2\alpha}8^{2\alpha}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\eta}{(\cos\eta)^{4\alpha-1}}$$

and we can choose $4\alpha-1<1$ such that

$$\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\eta}{(\cos\eta)^{4\alpha-1}}<+\infty.$$

Now we choose $\alpha=1/8$, and denote

$$g(x,s)=\begin{cases}(5x^4+1)s^{-\frac{1}{8}}+93\mathrm{e}^s, & \textit{if } \underline{u}\leq s\leq 6,\\ (5x^4+1)s^{-\frac{1}{8}}+\sqrt{s}+93\mathrm{e}^6-\sqrt{6}, & \textit{if } s>6,\end{cases}$$

$G(x,u)=\int_{\underline{u}}^u g(x,s)\,\mathrm{d}s$, $K(u)=\int_{-2}^2 G(x,u)\,\mathrm{d}x$ and

$$I(u)=\frac{1}{2}\int_{-2}^2|u'|^2\,\mathrm{d}x+\frac{1}{4}\int_{-2}^2x^2|u'|^4\,\mathrm{d}x.$$

Note that

$$\begin{aligned} 2u(x)&=\int_{-2}^x u'(s)\,\mathrm{d}s-\int_x^2 u'(s)\,\mathrm{d}s\leq\int_{-2}^x|u'(s)|\,\mathrm{d}s+\int_x^2|u'(s)|\,\mathrm{d}s\\ &=\int_{-2}^2|u'(s)|\,\mathrm{d}s\leq\left(\int_{-2}^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{-2}^2|u'(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}=2\left(\int_{-2}^2|u'(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}. \end{aligned}$$

Thus we have

$$\|u\|_{C(-2,2)}\leq\|u\|_{1,2}\,.$$

This implies that $C = 1$ in (H)(vi). Taking $R_0 = 3/2$ and $\theta = 1/4$ in (H)(vi), we obtain that

$$\rho = \left(\frac{1}{p} + \frac{\mu_0}{q}\right) R_0^1 \omega_1 = \left(\frac{1}{2} + \frac{\left(\frac{3}{8}\right)^2}{4}\right) \times \frac{3}{2} \times 3 = \frac{1233}{512}.$$

Consequently

$$u_1(x) = \begin{cases} \frac{3}{16}\sqrt{274}, & \text{if } x \in \left(-2, -\frac{3}{2}\right) \cup \left(\frac{3}{2}, 2\right), \\ \frac{4}{3}(x + \frac{3}{2}) + \frac{3}{16}\sqrt{274}, & \text{if } x \in \left(-\frac{3}{2}, -\frac{3}{8}\right), \\ -\frac{4}{3}(x - \frac{3}{2}) + \frac{3}{16}\sqrt{274} & \text{if } x \in \left(\frac{3}{8}, \frac{3}{2}\right), \\ \frac{3}{16}\sqrt{274} + \frac{3}{2}, & \text{if } x \in \left(-\frac{3}{8}, \frac{3}{8}\right) \end{cases}$$

and

$$\begin{aligned} I(u_1) &= \frac{1}{2}\int_\Omega |u_1'|^2 \,\mathrm{d}x + \frac{1}{4}\int_\Omega x^2 |u_1'|^4 \,\mathrm{d}x \\ &= \frac{1}{2}\int_{-\frac{3}{2}}^{-\frac{3}{8}} \left(\frac{4}{3}\right)^2 \mathrm{d}x + \frac{1}{4}\int_{-\frac{3}{2}}^{-\frac{3}{8}} \left(\frac{4}{3}\right)^4 x^2 \,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\frac{3}{8}}^{\frac{3}{2}} \left(-\frac{4}{3}\right)^2 \mathrm{d}x + \frac{1}{4}\int_{\frac{3}{8}}^{\frac{3}{2}} \left(-\frac{4}{3}\right)^4 x^2 \,\mathrm{d}x \\ &= \frac{15}{4}. \end{aligned}$$

Hence

$$1 > \frac{\rho}{I(u_1)} = \frac{411}{640} > \frac{1}{2}.$$

Additionally, we can calculate that

$$\begin{aligned} &\int_{-\frac{3}{2}}^{\frac{3}{2}} \int_{\frac{3}{16}\sqrt{274}}^{u_1} \left[(5x^4+1)s^{-\frac{1}{8}} + 93e^s\right] \mathrm{d}s \,\mathrm{d}x \\ &= \int_{-\frac{3}{2}}^{\frac{3}{2}} \left[\frac{8}{7}(5x^4+1)u_1^{\frac{7}{8}} + 93e^{u_1} - \frac{8}{7}\left(\frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}(5x^4+1) - 93e^{\frac{3}{16}\sqrt{274}}\right] \mathrm{d}x \\ &= \int_{-\frac{3}{2}}^{-\frac{3}{8}} \left[\frac{8}{7}\left(\frac{4}{3}\left(x+\frac{3}{2}\right) + \frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}(5x^4+1) + 93e^{\frac{4}{3}\left(x+\frac{3}{2}\right)+\frac{3}{16}\sqrt{274}}\right] \mathrm{d}x \\ &\quad + \int_{-\frac{3}{8}}^{\frac{3}{8}} \left[\frac{8}{7}\left(\frac{3}{2} + \frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}(5x^4+1) + 93e^{\frac{3}{2}+\frac{3}{16}\sqrt{274}}\right] \mathrm{d}x \\ &\quad + \int_{\frac{3}{8}}^{\frac{3}{2}} \left[\frac{8}{7}\left(-\frac{4}{3}(x-\frac{3}{2}) + \frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}(5x^4+1) + 93e^{-\frac{4}{3}(x-\frac{3}{2})+\frac{3}{16}\sqrt{274}}\right] \mathrm{d}x \\ &\quad - \int_{-\frac{3}{2}}^{\frac{3}{2}} \left[\frac{8}{7}\left(\frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}(5x^4+1) + 93e^{\frac{3}{16}\sqrt{274}}\right] \mathrm{d}x \\ &\approx 11576.45 \end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&\int_{-2}^{2}\int_{\underline{u}}^{\frac{3}{16}\sqrt{274}}\left[(5x^4+1)s^{-\frac{1}{8}}+93\mathrm{e}^{s}\right]\mathrm{d}s\,\mathrm{d}x\\
&=\int_{-2}^{2}\left[\frac{8}{7}\left(\frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}(5x^4+1)+93\mathrm{e}^{\frac{3}{16}\sqrt{274}}-\frac{8}{7}(5x^4+1)\underline{u}^{\frac{7}{8}}-93e^{\underline{u}}\right]\mathrm{d}x \qquad (4.18)\\
&\approx 7958.02.
\end{aligned}$$

From (4.17) *and* (4.18), *we have*

$$\begin{aligned}
&\int_{-\frac{3}{2}}^{\frac{3}{2}}\int_{\frac{3}{16}\sqrt{274}}^{u_1}\left((5x^4+1)s^{-\frac{1}{8}}+f(x,s)\right)\mathrm{d}s\,\mathrm{d}x\\
&>\int_{-2}^{2}\int_{\underline{u}}^{\frac{3}{16}\sqrt{274}}((5x^4+1)s^{-\frac{1}{8}}+f(x,s))\,\mathrm{d}s\,\mathrm{d}x,
\end{aligned}$$

which implies that the condition (H)(vi) *holds. Therefore, according to Theorem 1.1, problem* (4.15) *has at least three bounded positive solutions for* $\lambda < b$. *Furthermore, we can calculate*

$$\begin{aligned}
&K(u_1)\\
&=\int_{-2}^{2}G(x,u_1)\,\mathrm{d}x\\
&=\int_{-2}^{2}\int_{\underline{u}}^{u_1}\left[(5x^4+1)s^{-\frac{1}{8}}+93e^{s}\right]\mathrm{d}s\,\mathrm{d}x\\
&=\int_{-2}^{2}\left[\frac{8}{7}(5x^4+1)s^{\frac{7}{8}}+93e^{s}\right]\Bigg|_{\underline{u}}^{u_1}\mathrm{d}x\\
&=\int_{-2}^{2}\left[\frac{8}{7}(5x^4+1)u_1^{\frac{7}{8}}+93e^{u_1}-\frac{8}{7}(5x^4+1)\left(\frac{4-x^2}{8}\right)^{\frac{7}{8}}-93e^{\frac{4-x^2}{8}}\right]\mathrm{d}x\\
&=\int_{-2}^{-\frac{3}{2}}\left[\frac{8}{7}(5x^4+1)\left(\frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}+93e^{\frac{3}{16}\sqrt{274}}\right]\mathrm{d}x\\
&\quad+\int_{-\frac{3}{2}}^{-\frac{3}{8}}\left[\frac{8}{7}(5x^4+1)\left(\frac{4}{3}\left(x+\frac{3}{2}\right)+\frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}+93e^{\frac{4}{3}(x+\frac{3}{2})+\frac{3}{16}\sqrt{274}}\right]\mathrm{d}x\\
&\quad+\int_{-\frac{3}{8}}^{\frac{3}{8}}\left[\frac{8}{7}(5x^4+1)\left(\frac{3}{2}+\frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}+93e^{\frac{3}{2}+\frac{3}{16}\sqrt{274}}\right]\mathrm{d}x\\
&\quad+\int_{\frac{3}{8}}^{\frac{3}{2}}\left[\frac{8}{7}(5x^4+1)\left(-\frac{4}{3}\left(x-\frac{3}{2}\right)+\frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}+93e^{-\frac{4}{3}(x-\frac{3}{2})+\frac{3}{16}\sqrt{274}}\right]\mathrm{d}x\\
&\quad+\int_{\frac{3}{2}}^{2}\left[\frac{8}{7}(5x^4+1)\left(\frac{3}{16}\sqrt{274}\right)^{\frac{7}{8}}+93e^{\frac{3}{16}\sqrt{274}}\right]\mathrm{d}x\\
&\quad-\int_{-2}^{2}\left[\frac{8}{7}(5x^4+1)\left(\frac{4-x^2}{8}\right)^{\frac{7}{8}}-93e^{\frac{4-x^2}{8}}\right]\mathrm{d}x\\
&\approx 19534.48.
\end{aligned}$$

Consequently

$$b = \frac{(1+\rho)\rho}{\rho\frac{K(u_1)}{I(u_1)} - \sup_{I(u)<\rho} K(u)} < \frac{(1+\rho)\rho}{\rho\frac{K(u_1)}{I(u_1)} - \frac{1}{2}K(u_1)} \approx 0.00296.$$

This means that our results are valid only when λ is sufficiently small.

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