TWO POSITIVE SOLUTIONS FOR NONLINEAR FOURTH-ORDER ELASTIC BEAM EQUATIONS

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Abstract. The aim of this paper is to study the existence of at least two non-trivial solutions to a boundary value problem for fourth-order elastic beam equations given by

$$\begin{aligned} u^{(4)} + Au'' + Bu &= \lambda f(x,u) \quad \text{in } [0,1],\\ u(0) = u(1) &= 0, \quad u''(0) = u''(1) = 0, \end{aligned}$$

under suitable conditions on the nonlinear term on the right hand side. Our approach is based on variational methods, and in particular, on an abstract two critical points theorem given for differentiable functionals defined on a real Banach space.

1. Introduction

This paper deals with the existence of at least two non-trivial solutions for the fourth-order nonlinear differential problem

$$\begin{aligned} u^{(4)} + Au'' + Bu &= \lambda f(x,u) \qquad \text{in } [0,1],\\ u(0) &= u(1) = 0, \\ u''(0) &= u''(1) = 0, \end{aligned} \tag{D_λ}$$

where $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is a L^1-Carathéodory function (see Definition 2.1) that fulfills the Ambrosetti-Rabinowitz condition (the AR-condition for short), A and B are real constants and λ is a positive parameter. The main result of our paper, stated as Theorem 3.1, says that problem (D_λ) has at least two non-trivial, generalized solutions under appropriate, not complicated assumptions on the nonlinear term.

Equation (D_λ) is of fourth-order and equations of this type are usually called elastic beam equations which comes from the fact that they describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported.

A special case of our main result can be given in the following form.

Theorem 1.1. *Let $g\colon\mathbb{R}\to\mathbb{R}$ be a non-negative and continuous function such that*

$$\lim_{s\to 0^+}\frac{g(s)}{s} = +\infty.$$

Moreover, assume that there exist $\nu>2$ and $R>0$ such that

$$0 < \nu\int_0^s g(t)dt \le sg(s) \quad \textit{for all } s\in\mathbb{R} \textit{ with } |s|\ge R.$$

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Then, there exists $\overline{\lambda} > 0$ such that for each $\lambda \in]0, \overline{\lambda}[$, the problem

$$\begin{aligned} u^{(4)} + Au'' + Bu &= \lambda g(u) \qquad \text{in } [0,1], \\ u(0) &= u(1) = 0, \\ u''(0) &= u''(1) = 0, \end{aligned}$$

has at least two non-trivial generalized solutions.

The main novelty of our paper is the fact that we apply a recent critical-points result to equations of fourth-order given in the form (D_λ). Although there exist several existence results to equation (D_λ), our treatment is completely new and gives, in contrast to several other works, multiple solutions in terms of two nontrivial solutions. The assumptions on the nonlinear term are easy to verify and so our results could be applied to several variants of problem (D_λ).

Existence results of at least one solution, or multiple solutions, or even infinitely many solutions have been established by several authors by applying different tools like fixed point theorems, lower and upper solution methods, and critical point theory (see, for instance, [3], [12], [22]). We refer, without any claim to completeness, to the papers of Bai-Wang [2], Bonanno-Di Bella [7, 8, 9], Bonanno-Di Bella-O'Regan [10], Cabada-Cid-Sanchez [13], Franco-O'Regan-Perán [14], Grossinho-Sanchez-Tersian [15], Jiang-Liu-Xu [16], Liu-Li [17, 18], Li-Zhang-Liang [19], Yuan-Jiang-O'Regan [25] and the references therein.

Finally, we also want to point out that the derivation and application of critical point results has been initiated by the works of Ricceri [23, 24] which were the starting point of several generalizations in that direction for smooth and non-smooth functionals. Since it is not possible to state all the published results we refer only to the works of Marano-Motreanu [20, 21], Bonanno-Candito [6] and Bonanno [4, 5] who inspired us in writing this paper.

The paper is organized as follows. In Section 2, we state the main definitions and tools that we are going to need to prove our main results. Especially, we recall the abstract critical point theorem of Bonanno-D'Aguì [11], which is an appropriate combination of the local minimum theorem obtained by Bonanno with the classical and seminal Ambrosetti–Rabinowitz theorem (see [1]), moreover we give a lemma about the relation of our perturbation concerning the AR-condition and the Palais-Smale condition (PS-condition for short). Then, in Section 3, we are going to prove our main result which gives an answer about the existence of solutions to problem (D_λ). To be more precise, we obtain the existence of two non-trivial, generalized solutions of (D_λ), see Theorem 3.1, and the proof is based on the abstract critical points result stated in Section 2. Finally, in Section 4, we consider special cases of problem (D_λ), namely when f has separable variables, and give some corollaries of Theorem 3.1 in order to show the applicability of our results.

2. Basic definitions and preliminary results

In this section, we give the main definitions and tools that we will need later. To this end, let A and B two real constants such that

$$\max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2} - \frac{B}{\pi^4}\right\} < 1. \tag{2.1}$$

For example, condition (2.1) is satisfied if $A \leq 0$ and $B \geq 0$. Moreover, we put

$$\sigma := \max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2} - \frac{B}{\pi^4}, 0\right\} \quad \text{and} \quad \delta := \sqrt{1-\sigma}.$$

The usual Sobolev spaces $H_0^1(0,1)$ and $H^2(0,1)$ are defined by

$$H_0^1(0,1) = \left\{u \in L^2(0,1) \,:\, u' \in L^2(0,1),\, u(0) = u(1) = 0\right\},$$
$$H^2(0,1) = \left\{u \in L^2(0,1) \,:\, u' \in L^2(0,1),\, u'' \in L^2(0,1)\right\}.$$

Furthermore, let $X := H_0^1(0,1) \cap H^2(0,1)$ be the Hilbert space endowed with the following norm

$$\|u\|_X = \left(\int_0^1 \left(|u''|^2 - A|u'|^2 + B|u|^2\right) dx\right)^{1/2}.$$

It is well known that this norm is equivalent to the usual one, see for example Bonanno-Di Bella [7], and, in particular, one has

$$\|u\|_\infty \leq \frac{1}{2\pi\delta}\|u\|_X. \tag{2.2}$$

Definition 2.1. *A function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is said to be a L^1-Carathéodory function if the following is satisfied:*

(a) *$x \to f(x,s)$ is measurable for all $s \in \mathbb{R}$;*
(b) *$s \to f(x,s)$ is continuous for a. a. $x \in [0,1]$;*
(c) *for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that*

$$\sup_{|s|\leq\rho} |f(x,s)| \leq l_\rho(x)$$

for a. a. $x \in [0,1]$.

It is easy to see that if $f(x,s)s < 0$ for every $x \in [0,1]$ and $s \neq 0$, problem (D_λ) has only the trivial solution.

Definition 2.2. *A weak solution of (D_λ) is a function $u \in X$ such that*

$$\int_0^1 [u''(x)v''(x) - Au'(x)v'(x) + Bu(x)v(x)]\, dx - \lambda \int_0^1 f(x,u(x))v(x)\, dx = 0$$

is fulfilled for all $v \in X$. A function $u\colon [0,1] \to \mathbb{R}$ is said to be a generalized solution of problem (D_λ) if $u \in C^3([0,1])$, $u''' \in AC([0,1])$, $u(0) = u(1) = 0$, $u''(0) = u''(1) = 0$, and $u^{(4)} + Au'' + Bu = \lambda f(x,u)$ for a. a. $x \in [0,1]$.

If f is continuous in $[0,1] \times \mathbb{R}$, then each generalized solution u is a classical solution. Moreover, the assumptions on f imply that a weak solution of problem (D_λ) is a generalized one, see Bonanno-Di Bella [7, Proposition 2.2].

Now, we introduce the functional $I_\lambda\colon X \to \mathbb{R}$ defined by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) = \frac{1}{2}\int_0^1 [|u''|^2(x)A|u'|^2(x) + Bu^2(x)]\, dx, \tag{2.3}$$

$$\Psi(u) = \int_0^1 F(x,u(x))\, dx \tag{2.4}$$

for each $u \in X$ and $F(x,s) = \int_0^s f(x,t)\,dt$ for each $(x,s) \in [0,1] \times \mathbb{R}$. We know that problem ($\mathrm{D}_\lambda$) has a variational structure and its weak solutions can be obtained as critical points of the corresponding functional I_λ.

It is well known that these functionals are well-defined on X and one has

$$\Phi'(u)(v) = \int_0^1 \left[u''(x)v''(x) - Au'(x)v'(x) + Bu(x)v(x)\right] dx,$$
$$\Psi'(u)(v) = \int_0^1 f(x,u(x))v(x)\,dx$$

for all $v \in X$.

Our main tool is a two non-zero critical points theorem recently proved by Bonanno-D'Aguì [11, Theorem 2.2]. Let us recall the definition of the PS-condition.

Definition 2.3. *A functional $I\colon X \to \mathbb{R}$ satisfies the PS-condition if any sequence $\{u_k\}_{k\geq 1} \subseteq X$ such that*

- *$\{I(u_k)\}_{k\geq 1}$ is bounded;*
- *$\lim_{k\to+\infty} \|I'(u_k)\|_{X^*} = 0$;*

has a convergent subsequence.

The above mentioned theorem reads as follows, see [11, Theorem 2.2].

Theorem 2.4. *Let X be a real Banach space, $\Phi, \Psi\colon X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals such that $\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $0 < \Phi(\overline{x}) < r$, such that:*

(A1) $\dfrac{\sup_{\Phi(x)\leq r} \Psi(x)}{r} < \dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(A2) *for each* $\lambda \in \left]\dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}, \dfrac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\right[$, *the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the PS-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda_r$, where Λ_r is defined by

$$\Lambda_r := \left]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\right[,$$

the functional I_λ admits at least two non-zero critical points $x_{0,\lambda}$, $x_{1,\lambda}$ such that $I_\lambda(x_{0,\lambda}) < 0 < I_\lambda(x_{1,\lambda})$.

In case when f satisfies the AR-condition we know that the functional I_λ fulfills the classical PS-condition. In particular, the following holds.

Lemma 2.5. *Assume that there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu F(x,s) \leq s f(x,s) \quad \text{for all } x \in [0,1] \text{ and for all } s \in \mathbb{R} \text{ with } |s| \geq R. \tag{2.5}$$

Then, I_λ satisfies the PS-condition and it is unbounded from below.

Proof. In order to verify the PS-condition, we will prove that any PS-sequence is bounded. Let $\{u_k\}_{k\geq 1}$ be a sequence in X such that $\{I_\lambda(u_k)\}_{k\geq 1}$ is bounded and $I'_\lambda(u_k) \to 0$ as $k \to +\infty$. Taking (2.5) into account one has

$$\begin{aligned}\nu I_\lambda(u_k) - I'_\lambda(u_k) &= \left(\frac{\nu}{2} - 1\right)\|u_k\|^2 + \lambda \int_0^1 \left[f(x,u_k(x))u_k(x) - \nu F(x,u_k(x))\right] dx \\ &\geq \left(\frac{\nu}{2} - 1\right)\|u_k\|^2.\end{aligned}$$

Since $\nu > 2$ it follows that $\{u_k\}_{k\geq 1}$ is bounded in X. Therefore, up to a subsequence, $\{u_k(x)\}_{k\geq 1}$ is uniformly convergent to $u_0(x)$ for $x \in [0,1]$ and $\{u_k\}_{k\geq 1}$ is weakly convergent to u_0 in X. The uniform convergent of $\{u_k\}_{k\geq 1}$ and Lebesgue's Dominated Convergence Theorem ensure that we derive from

$$\begin{aligned}
&(I'_\lambda(u_k) - I'_\lambda(u_0))(u_k - u_0)\\
&= \int_0^1 [u''_k(x)(u_k(x) - u_0(x))'' - Au'_k(x)(u_k(x) - u_0(x))']\,dx\\
&\quad + \int_0^1 [Bu_k(x)(u_k(x) - u_0(x))]\,dx - \int_0^1 f(x, u_k(x))(u_k(x) - u_0(x))\,dx\\
&\quad - \int_0^1 [u''_0(x)(u_k(x) - u_0(x))'' - Au'_0(x)(u_k(x) - u_0(x))']\,dx\\
&\quad + \int_0^1 [Bu_0(x)(u_k(x) - u_0(x))]\,dx + \int_0^1 f(x, u_0(x))(u_k(x) - u_0(x))\,dx\\
&= \|u_k - u_0\|^2 - \int_0^1 [f(x, u_k(x)) - f(x, u_0(x))](u_k(x) - u_0(x))\,dx
\end{aligned}$$

that

$$\lim_{k\to\infty} \|u_k - u_0\| = 0.$$

In the other words, $\{u_k\}_{k\geq 1}$ converges strongly to u_0 in X.

Now, observe that, by integrating (2.5), there is a positive constant C such that

$$F(x, s) \geq C|s|^\nu$$

for all $|s| \geq R$ and for all $x \in [0,1]$. So, for any function $u \in X$ such that $|u(x)| > R$ for all $x \in [0,1]$, we obtain

$$I_\lambda(u) \leq \frac{1}{2}\|u\|^2 - \lambda C \int_0^1 |u(x)|^\nu\,dx \leq \frac{1}{2}\|u\|^2 - \lambda C \|u\|^\nu_{L^2}.$$

Recall again that $\nu > 2$, this condition implies that I_λ is unbounded from below. Therefore, I_λ satisfies the PS-condition and the proof is complete. □

3. Main results

In this section, we present the main existence result of our paper. First, we put

$$k = 2\delta^2\pi^2 \left(\frac{2048}{27} - \frac{32}{9}A + \frac{13}{40}B\right)^{-1}.$$

Observe that, by a simple calculation, $0 < k < \frac{1}{2}$.

Our main result is the following.

Theorem 3.1. *Suppose that $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathédory function. Furthermore, assume that there exist two positive constants c, d with $d < c$ such that*

(a_1) *$F(x, s) \geq 0$ for all $(x, s) \in \left(\left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]\right) \times [0, d]$;*

(a_2) $$\frac{\int_0^1 \max_{|s|\leq c} F(x, s)\,dx}{c^2} < k\,\frac{\int_{3/8}^{5/8} F(x, d)\,dx}{d^2}.$$

Moreover, assume that there exist $\nu > 2$ and $R > 0$ such that

$$0 < \nu F(x,s) \leq s f(x,s) \quad \text{for all } x \in [0,1] \text{ and for all } s \in \mathbb{R} \text{ with } |s| \geq R.$$

Then, for every $\lambda \in \Lambda$, where Λ is defined by

$$\Lambda := \left] \frac{2\delta^2\pi^2}{k} \frac{d^2}{\displaystyle\int_{3/8}^{5/8} F(x,d)\,dx}, \; 2\delta^2\pi^2 \frac{c^2}{\displaystyle\int_0^1 \max_{|s|\leq c} F(x,s)\,dx} \right[,$$

problem (D_λ) *admits at least two non-trivial generalized solutions.*

Proof. First, we observe that owing to (a_2) the interval Λ is non-empty. Now, fix λ as in the conclusion. Our goal is to apply Theorem 2.4 to the functionals Φ, Ψ and I_λ as defined in (2.3) and (2.4). All regularity assumptions required on Φ and Ψ are satisfied and from Lemma 2.5 we know that the functional I_λ satisfies the PS-condition and it is unbounded from below. Now, we prove (A1). To this end, we fix $r = 2\delta^2\pi^2c^2$. Taking (2.2) into account, for every $u \in X$ such that $\Phi(u) \leq r$, one has $\max_{x\in[0,1]} |u(x)| \leq c$. Therefore, it follows that

$$\sup_{\Phi(u)\leq r} \Psi(u) \leq \int_0^1 \max_{|s|\leq c} F(x,s)\,dx,$$

that is

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq \frac{1}{2\delta^2\pi^2} \frac{\int_0^1 \max_{|s|\leq c} F(x,s)\,dx}{c^2} < \frac{1}{\lambda}. \tag{3.1}$$

Now, we define a function $\overline{v}$ by

$$\overline{v}(x) = \begin{cases} -\dfrac{64d}{9}\left(x^2 - \dfrac{3}{4}x\right) & \text{if } x \in \left[0, \dfrac{3}{8}\right] \\ d & \text{if } x \in \left]\dfrac{3}{8}, \dfrac{5}{8}\right] \\ -\dfrac{64d}{9}\left(x^2 - \dfrac{5}{4}x + \dfrac{1}{4}\right) & \text{if } x \in \left]\dfrac{5}{8}, 1\right]. \end{cases}$$

It is clear that $\overline{v} \in X$ and

$$\|\overline{v}\|^2 = \left(\frac{4096}{27} - \frac{64}{9}A + \frac{13}{20}B\right) d^2 = \frac{4\delta^2\pi^2}{k} d^2.$$

So, from $d < c$ we have $d < \sqrt{k}c$, hence

$$\Phi(\overline{v}) < r.$$

Now, due to (a_1) one has that

$$\Psi(\overline{v}) \geq \int_{3/8}^{5/8} F(x,d)\,dx.$$

This leads to

$$\frac{\Psi(\overline{v})}{\Phi(\overline{v})} \geq \frac{k}{2\delta^2\pi^2} \frac{\displaystyle\int_{3/8}^{5/8} F(x,d)\,dx}{d^2} > \frac{1}{\lambda}. \tag{3.2}$$

Therefore, from (3.1) and (3.2), condition (A1) of Theorem 2.4 is fulfilled.

Moreover, we observe that

$$\left]\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[\supseteq \left]\frac{2\delta^2\pi^2}{k}\frac{d^2}{\displaystyle\int_{3/8}^{5/8} F(x,d)\,dx}, 2\delta^2\pi^2\frac{c^2}{\displaystyle\int_0^1 \max_{|s|\le c} F(x,s)\,dx}\right[.$$

Finally, the conclusion of Theorem 2.4 can be used. It follows that, for every

$$\lambda \in \left]\frac{2\delta^2\pi^2}{k}\frac{d^2}{\displaystyle\int_{3/8}^{5/8} F(x,d)\,dx}, 2\delta^2\pi^2\frac{c^2}{\displaystyle\int_0^1 \max_{|s|\le c} F(x,s)\,dx}\right[,$$

problem (D_λ) has at least two non-trivial, generalized solutions. □

4. Applications

In this section, we are going to apply the results of Theorem 3.1 when the nonlinear term has separable variables. First, we have the following.

Theorem 4.1. *Let $h\colon [0,1] \to \mathbb{R}$ be a positive and essentially bounded function and let $g\colon \mathbb{R} \to \mathbb{R}$ be a non-negative and continuous function. Put*

$$G(s) = \int_0^s g(x)\,dx \quad \text{for every } s \in \mathbb{R}$$

and

$$h_0 = k\frac{\int_{3/8}^{5/8} h(x)\,dx}{\|h\|_{L^1([0,1])}}.$$

Assume that there exist two positive constants c, d with $d < c$ such that

$$\frac{G(c)}{c^2} < h_0\frac{G(d)}{d^2}.$$

Moreover, assume that there exist $\nu > 2$ and $R > 0$ such that

$$0 < \nu G(s) \le s g(s) \quad \text{for all } s \in \mathbb{R} \text{ with } |s| \ge R.$$

Then, for every

$$\lambda \in \left]\left(\frac{2\pi^2\delta^2}{h_0\|h\|_{L^1([0,1])}}\right)\frac{d^2}{G(d)}, \left(\frac{2\pi^2\delta^2}{\|h\|_{L^1([0,1])}}\right)\frac{c^2}{G(c)}\right[,$$

the problem

$$\begin{aligned} u^{(4)} + Au'' + Bu &= \lambda h(x)g(u) \qquad \text{in } [0,1],\\ u(0) &= u(1) = 0, \\ u''(0) &= u''(1) = 0, \end{aligned} \tag{AD_λ}$$

has at least two non-trivial, generalized solutions.

Proof. The statement of the theorem follows directly by applying Theorem 3.1 to the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,s) = h(x)g(s)$. □

A direct consequence is the following corollary.

Corollary 4.2. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a non-negative and continuous function such that $g(0) \neq 0$. Assume that there exist two positive numbers c, d such that*

$$\frac{G(c)}{c^2} < \frac{k}{4} \frac{G(d)}{d^2}.$$

Moreover, assume that there exist $\nu > 2$ and $R > 0$ such that

$$0 < \nu G(s) \leq s g(s) \quad \text{for all } s \in \mathbb{R} \text{ with } |s| \geq R.$$

Then, for every

$$\lambda \in \left] \frac{8\pi^2\delta^2}{k} \frac{d^2}{G(d))}, (2\pi^2\delta^2) \frac{c^2}{G(c)} \right[,$$

the problem

$$\begin{aligned} u^{(4)} + Au'' + Bu = \lambda g(u) \qquad & \text{in } [0,1],\\ u(0) = u(1) = 0,&\\ u''(0) = u''(1) = 0,& \end{aligned}$$

has at least two non-trivial, non-negative, generalized solutions.

Proof. The proof follows directly from Theorem 4.1. □

Finally, another consequence of Theorem 4.1 can be given through the next theorem.

Theorem 4.3. *Let $h\colon [0,1] \to \mathbb{R}$ be a positive and essentially bounded function and $g\colon \mathbb{R} \to \mathbb{R}$ be a non-negative and continuous function such that*

$$\lim_{s \to 0^+} \frac{g(s)}{s} = +\infty. \tag{4.1}$$

Moreover, assume that there exist $\nu > 2$ and $R > 0$ such that

$$0 < \nu G(s) \leq s g(s) \quad \text{for all } s \in \mathbb{R} \text{ with } |\xi| \geq R.$$

Then, for each $\lambda \in]0, \lambda^*[$, where λ^* is defined by

$$\lambda^* = \left(\frac{2\pi^2\delta^2}{\|h\|_{L^1([0,1])}} \right) \sup_{\bar{c}>0} \frac{\bar{c}^2}{G(\bar{c})},$$

problem (AD$_\lambda$) has at least two non-trivial, generalized solutions.

Proof. For fixed $\lambda \in]0, \lambda^*[$, there exists $c > 0$ such that

$$\lambda < \left(\frac{2\pi^2\delta^2}{\|h\|_{L^1([0,1])}} \right) \frac{c^2}{G(c)}.$$

Condition (4.1) implies that

$$\lim_{s \to 0^+} \frac{G(s)}{s^2} = +\infty.$$

Hence, we find numbers $0 < d < c$ such that

$$\frac{G(d)}{d^2} > \left(\frac{2\pi^2\delta^2}{\|h\|_{L^1([0,1])}} \right) \frac{1}{h_0} \frac{1}{\lambda} > \frac{1}{h_0} \frac{G(c)}{c^2}.$$

Applying Theorem 4.1 yields the assertion of the theorem. □

Remark 4.4. *Theorem 1.1 from the Introduction is a special case of Theorem 4.3 by setting $h(x) \equiv 1$.*

Now, we give an example to illustrate the applicability of our results.

Example 4.5. *Let $A = 2$ and $B = 1$. Then, we see that*

$$\sigma = \frac{2}{\pi^2} \quad \text{and} \quad \delta^2 = \frac{\pi^2 - 2}{\pi^2}.$$

Now we set $f(x,s) = (x+1)g(s)$ for all $(x,s) \in [0,1] \times \mathbb{R}$, where

$$g(s) = \begin{cases} (1+s)e^s & \text{if } s \le 1, \\ 2es^4 & \text{if } s > 1. \end{cases}$$

Obviously, $g : \mathbb{R} \to \mathbb{R}$ is continuous. Note that $\sup_{\overline{c}>0} \frac{\overline{c}^2}{G(\overline{c})}$ is achieved for $c = 1$. Hence from Theorem 4.3, for each $\lambda \in \left]0, \frac{4(\pi^2-2)}{e}\right[$, the problem

$$\begin{gathered} u^{(4)} + 2u'' + u = \lambda(x+1)g(u) \qquad \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{gathered}$$

has at least two non-trivial, generalized solutions.

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