SIGN-CHANGING AND EXTREMAL CONSTANT-SIGN SOLUTIONS OF NONLINEAR ELLIPTIC NEUMANN BOUNDARY VALUE PROBLEMS

PATRICK WINKERT

ABSTRACT. Our aim is the study of a class of nonlinear elliptic problems under Neumann conditions involving the p-Laplacian. We prove the existence of at least three nontrivial solutions which means that we get two extremal constant-sign solutions and one sign-changing solution by using truncation techniques and comparison principles for nonlinear elliptic differential inequalities. We also apply the properties of the Fučik Spectrum of the p-Laplacian and in particular, we make use of variational and topological tools, for example, critical point theory, Mountain-Pass Theorem and the Second Deformation Lemma.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the following nonlinear elliptic boundary value problem. Find $u \in W^{1,p}(\Omega) \setminus \{0\}$ and constants $a \in \mathbb{R}, b \in \mathbb{R}$ such that

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u), 1 < p < \infty$, is the negative p-Laplacian, $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of u, and $u^+ = \max\{u, 0\}$ as well as $u^- = \max\{-u, 0\}$ are the positive and negative part of u, respectively. The nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions which are bounded on bounded sets. For reasons of simplification, we drop the notation for the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is used on the functions defined on the boundary $\partial\Omega$.

The motivation of our study is a recent paper of the author [27] in which problem (1.1) was treated in case $a = b$. We extend this approach and prove the existence of multiple solutions for the more general problem (1.1). To be precise, the existence of a smallest positive solution, a greatest negative solution as well as a sign-changing solution of problem (1.1) is proved by using variational and topological tools, for example, critical point theory, Mountain-Pass Theorem and the Second Deformation Lemma. Additionally, the Fučik spectrum for the p-Laplacian takes an important part in our treatments.

Neumann boundary value problems in the form (1.1) arise in different areas of pure and applied mathematics, for example in the theory of quasiregular and

2010 *Mathematics Subject Classification.* 35B38, 35J20, 47J10.
Key words and phrases. Nonlinear elliptic boundary value problem, p-Laplacian, Mountain-Pass Theorem, Extremal constant-sign solutions, Sign-changing solutions, Critical points, Fučik spectrum, Second Deformation Lemma.

quasiconformal mappings in Riemannian manifolds with boundary (see [11],[25]), in the study of optimal constants for the Sobolev trace embedding (see [9], [14], [15], [13]) or at non-Newtonian fluids, flow through porus media, nonlinear elasticity, reaction diffusion problems, glaciology and so on (see [1], [3], [2], [10]).

The existence of multiple solutions for Neumann problems like the form (1.1) has been studied by a number of authors, such as, e.g., [12, 16, 22, 31] and homogeneous Neumann boundary value problems were considered in [19, 30] and [31], respectively. Analogous results for the Dirichlet problem have been recently obtained in [5, 6, 7, 8]. Further references can also be found in the bibliography of [27].

In our consideration, the nonlinearities f and g only need to be Carathéodory functions which are bounded on bounded sets whereby their growth does not need to be necessarily polynomial. The novelty of our paper is the fact that we do not need differentiability, polynomial growth or some integral conditions on the mappings f and g.

First, we have to make an analysis of the associated spectrum of (1.1). The Fučik spectrum for the p-Laplacian with a nonlinear boundary condition is defined as the set $\widetilde{\Sigma}_p$ of $(a,b) \in \mathbb{R}\times\mathbb{R}$ such that

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= a(u^+)^{p-1} - b(u^-)^{p-1} && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

has a nontrivial solution. In view of the identity

$$|u|^{p-2}u = |u|^{p-2}(u^+ - u^-) = (u^+)^{p-1} - (u^-)^{p-1},$$

we see at once that for $a = b = \lambda$ problem (1.2) reduces to the Steklov eigenvalue problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u && \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

We say that λ is an eigenvalue if (1.3) has nontrivial solutions. The first eigenvalue $\lambda_1 > 0$ is isolated, simple and has a first eigenfunction φ_1 which is strictly positive in $\overline{\Omega}$ (see [21]). Furthermore, one can show that φ_1 belongs to $L^\infty(\Omega)$ (cf. [18, Lemma 5.6 and Theorem 4.3] or [28, Theorem 4.1]) and along with the results of Lieberman in [20, Theorem 2] it holds $\varphi_1 \in C^{1,\alpha}(\overline{\Omega})$. This fact combined with $\varphi_1(x) > 0$ in $\overline{\Omega}$ yields $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$, where $\operatorname{int}(C^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) \geq 0, \forall x \in \Omega\}$ in the Banach space $C^1(\overline{\Omega})$, given by

$$\operatorname{int}(C^1(\overline{\Omega})_+) = \left\{u \in C^1(\overline{\Omega}) : u(x) > 0, \forall x \in \overline{\Omega}\right\}.$$

Let us recall some properties of the Fučik spectrum. If λ is an eigenvalue for (1.3) then the point (λ,λ) belongs to $\widetilde{\Sigma}_p$. Since the first eigenfunction of (1.3) is positive, $\widetilde{\Sigma}_p$ clearly contains the two lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$. A first nontrivial curve $\mathcal{C}$ in $\widetilde{\Sigma}_p$ through (λ_2,λ_2) was constructed and variationally characterized by a mountain-pass procedure by Martínez and Rossi [23]. This yields the existence of a continuous path in $\{u \in W^{1,p}(\Omega) : I^{(a,b)}(u) < 0, \|u\|_{L^p(\partial\Omega)} = 1\}$ joining $-\varphi_1$ and φ_1 provided (a,b) is above the curve $\mathcal{C}$. The functional $I^{(a,b)}$ on $W^{1,p}(\Omega)$ is given by

$$I^{(a,b)}(u) = \int_{\Omega} \Big(|\nabla u|^p + |u|^p\Big) dx - \int_{\partial\Omega} \Big(a(u^+)^p + b(u^-)^p\Big) d\sigma.$$

Due to the fact that λ_2 belongs to $\mathcal{C}$, there exists a variational characterization of the second eigenvalue of (1.3) meaning that λ_2 can be represented as

$$\lambda_2 = \inf_{g \in \Pi} \max_{u \in g([-1,1])} \int_{\Omega} \Big(|\nabla u|^p + |u|^p\Big) dx,$$

where

$$\Pi = \{g \in C([-1,1], S) \mid g(-1) = -\varphi_1, g(1) = \varphi_1\},$$

and

$$S = \left\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p d\sigma = 1\right\}.$$

The proof of this result is given in [23].

An important part in our considerations takes the following Neumann boundary value problem defined by

$$\begin{aligned} -\Delta_p u &= -\varsigma |u|^{p-2} u + 1 \qquad &&\text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= 1 &&\text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $\varsigma > 1$ is a constant. As pointed out in [27], there exists a unique solution $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.4) which is required for the construction of sub- and supersolutions of problem (1.1).

2. Notations and Hypotheses

Now, we impose the following conditions on the nonlinearities f and g in problem (1.1). The maps $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions which means that they are measurable in the first argument and continuous in the second one. Furthermore, we suppose the following assumptions.

(H1) (f1) $\displaystyle\lim_{s \to 0} \frac{f(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$.

(f2) $\displaystyle\lim_{|s| \to \infty} \frac{f(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \Omega$.

(f3) f is bounded on bounded sets.

(f4) There exists $\delta_f > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s} \geq 0$ for all $0 < |s| \leq \delta_f$ and for a.a. $x \in \Omega$.

(H2) (g1) $\displaystyle\lim_{s \to 0} \frac{g(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \partial\Omega$.

(g2) $\displaystyle\lim_{|s| \to \infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \partial\Omega$.

(g3) g is bounded on bounded sets.

(g4) g satisfies the condition

$$|g(x_1,s_1) - g(x_2,s_2)| \leq L\Big[|x_1 - x_2|^{\alpha} + |s_1 - s_2|^{\alpha}\Big],$$

for all pairs $(x_1,s_1), (x_2,s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where M_0 is a positive constant and $\alpha \in (0,1]$.

(H3) Let $(a, b) \in \mathbb{R}_+^2$ be above the first nontrivial curve $\mathcal{C}$ of the Fučik spectrum constructed in [23] (see Figure 1).

Note that (H2)(g4) implies that the function $(x, s) \mapsto a|s|^{p-1} - b|s|^{p-1} + g(x, s)$ fulfills a condition as in (H2)(g4), too. Moreover, we see at once that $u = 0$ is a trivial solution of problem (1.1) because of the conditions (H1)(f1) and (H2)(g1) which guarantees that $f(x, 0) = g(x, 0) = 0$. It should be noted that hypothesis (H3) includes that $a, b > \lambda_1$ (see [23] or Figure 1).

Example 2.1. *Let the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x,s) = \begin{cases} |s|^{p-2}s(1 + (s+1)e^{-s}) & \text{if } s \leq -1 \\ \operatorname{sgn}(s)\dfrac{|s|^p}{2}(|(s-1)\cos(s+1)| + s + 1) & \text{if } -1 \leq s \leq 1 \\ s^{p-1}e^{1-s} - |x|(s-1)s^{p-1}e^s & \text{if } s \geq 1, \end{cases}$$

and

$$g(x,s) = \begin{cases} |s|^{p-2}s(s + 1 + e^{s+1}) & \text{if } s \leq -1 \\ |s|^{p-1}se^{(s^2-1)\sqrt{|x|}} & \text{if } -1 \leq s \leq 1 \\ s^{p-1}(\cos(1-s) + (1-s)e^s) & \text{if } s \geq 1. \end{cases}$$

Then all conditions in (H1)(f1)–(f4) and (H2)(g1)-(g4) are fulfilled.

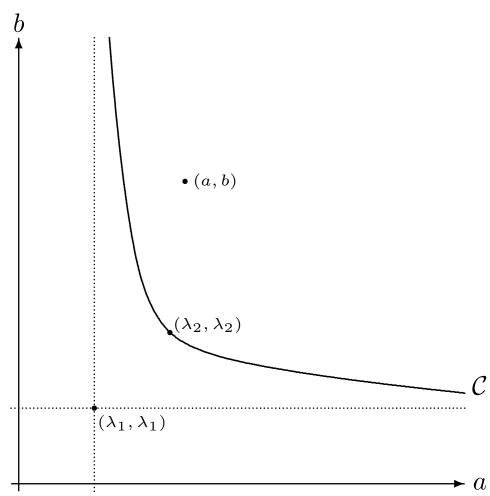

Figure 1. Fučik Spectrum

Definition 2.2. *A function $u \in W^{1,p}(\Omega)$ is called a weak solution of (1.1) if the following holds:*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \int_\Omega (f(x,u) - |u|^{p-2}u)\varphi dx + \int_{\partial\Omega} (a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u))\varphi d\sigma, \forall\varphi \in W^{1,p}(\Omega).$$

Definition 2.3. *A function* $\underline{u} \in W^{1,p}(\Omega)$ *is called a subsolution of (1.1) if the following holds:*

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\varphi dx \le \int_\Omega (f(x,\underline{u}) - |\underline{u}|^{p-2}\underline{u})\varphi dx + \int_{\partial\Omega} (a(\underline{u}^+)^{p-1} - b(\underline{u}^-)^{p-1} + g(x,\underline{u}))\varphi d\sigma, \forall \varphi \in W^{1,p}(\Omega)_+.$$

Definition 2.4. *A function* $\overline{u} \in W^{1,p}(\Omega)$ *is called a supersolution of (1.1) if the following holds:*

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla\varphi dx \ge \int_\Omega (f(x,\overline{u}) - |\overline{u}|^{p-2}\overline{u})\varphi dx + \int_{\partial\Omega} (a(\overline{u}^+)^{p-1} - b(\overline{u}^-)^{p-1} + g(x,\overline{u}))\varphi d\sigma, \forall \varphi \in W^{1,p}(\Omega)_+.$$

We recall that $W^{1,p}(\Omega)_+ := \{\varphi \in W^{1,p}(\Omega) : \varphi \ge 0\}$ denotes all nonnegative functions of $W^{1,p}(\Omega)$. Furthermore, for functions $u, v, w \in W^{1,p}(\Omega)$ satisfying $v \le u \le w$, we have the relation $\gamma(v) \le \gamma(u) \le \gamma(w)$, where $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ stands for the well-known trace operator.

3. Extremal Constant-Sign Solutions

For the rest of the paper we denote by $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ the first eigenfunction of the Steklov eigenvalue problem (1.3) corresponding to its first eigenvalue λ_1. Furthermore, the function $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ stands for the unique solution of the auxiliary Neumann boundary value problem defined in (1.4). Our first lemma reads as follows.

Lemma 3.1. *Let the conditions (H1)–(H2) be satisfied and let* $a, b > \lambda_1$. *Then there exist constants* $\vartheta_a, \vartheta_b > 0$ *such that* $\vartheta_a e$ *and* $-\vartheta_b e$ *are a positive supersolution and a negative subsolution, respectively, of problem (1.1).*

Proof. Setting $\overline{u} = \vartheta_a e$ with a positive constant ϑ_a to be specified and considering the auxiliary problem (1.4), we obtain

$$\begin{aligned}&\int_\Omega |\nabla(\vartheta_a e)|^{p-2}\nabla(\vartheta_a e)\nabla\varphi dx \\ &= -\varsigma\int_\Omega (\vartheta_a e)^{p-1}\varphi dx + \int_\Omega \vartheta_a^{p-1}\varphi dx + \int_{\partial\Omega}\vartheta_a^{p-1}\varphi d\sigma, \quad \forall\varphi \in W^{1,p}(\Omega).\end{aligned}$$

In order to satisfy Definition 2.4 for $\overline{u} = \vartheta_a e$, we have to show that the following inequality holds true meaning

$$\begin{aligned}&\int_\Omega (\vartheta_a^{p-1} - \widetilde{c}(\vartheta_a e)^{p-1} - f(x,\vartheta_a e))\varphi dx \\ &\quad + \int_{\partial\Omega} (\vartheta_a^{p-1} - a(\vartheta_a e)^{p-1} - g(x,\vartheta_a e))\varphi d\sigma \ge 0,\end{aligned} \tag{3.1}$$

where $\widetilde{c} = \varsigma - 1$ with $\widetilde{c} > 0$. Condition (H1)(f2) implies the existence of $s_\varsigma > 0$ such that

$$\frac{f(x,s)}{s^{p-1}} < -\widetilde{c}, \qquad \text{for a.a. } x \in \Omega \text{ and all } s > s_\varsigma,$$

and due to (H1)(f3) we have

$$|-f(x,s)-\widetilde{c}s^{p-1}| \leq |f(x,s)|+\widetilde{c}s^{p-1} \leq c_\varsigma, \quad \text{for a.a. } x\in\Omega \text{ and all } s\in[0,s_\varsigma].$$

Hence, we get

$$f(x,s) \leq -\widetilde{c}s^{p-1}+c_\varsigma, \quad \text{for a.a. } x\in\Omega \text{ and all } s\geq 0. \tag{3.2}$$

Because of hypothesis (H2)(g2) there exists $s_a>0$ such that

$$\frac{g(x,s)}{s^{p-1}} < -a, \quad \text{for a.a. } x\in\partial\Omega \text{ and all } s>s_a,$$

and thanks to condition (H2)(g3) we find a constant $c_a>0$ such that

$$|-g(x,s)-as^{p-1}| \leq |g(x,s)|+as^{p-1} \leq c_a, \quad \text{for a.a. } x\in\partial\Omega \text{ and all } s\in[0,s_a].$$

Finally, we have

$$g(x,s) \leq -as^{p-1}+c_a, \quad \text{for a.a. } x\in\partial\Omega \text{ and all } s\geq 0. \tag{3.3}$$

Using the inequality in (3.2) to the first integral in (3.1) yields

$$\begin{aligned}
&\int_\Omega (\vartheta_a^{p-1}-\widetilde{c}(\vartheta_a e)^{p-1}-f(x,\vartheta_a e))\varphi dx\\
&\geq \int_\Omega (\vartheta_a^{p-1}-\widetilde{c}(\vartheta_a e)^{p-1}+\widetilde{c}(\vartheta_a e)^{p-1}-c_\varsigma)\varphi dx\\
&= \int_\Omega (\vartheta_a^{p-1}-c_\varsigma)\varphi dx,
\end{aligned}$$

which proves its nonnegativity if $\vartheta_a \geq c_\varsigma^{\frac{1}{p-1}}$. Applying (3.3) to the second integral in (3.1) ensures

$$\begin{aligned}
&\int_{\partial\Omega} (\vartheta_a^{p-1}-a(\vartheta_a e)^{p-1}-g(x,\vartheta_a e))\varphi dx\\
&\geq \int_{\partial\Omega} (\vartheta_a^{p-1}-a(\vartheta_a e)^{p-1}+a(\vartheta_a e)^{p-1}-c_a)\varphi dx\\
&\geq \int_{\partial\Omega} (\vartheta_a^{p-1}-c_a)\varphi dx.
\end{aligned}$$

We take $\vartheta_a := \max\left\{c_\varsigma^{\frac{1}{p-1}}, c_a^{\frac{1}{p-1}}\right\}$ to verify that both integrals in (3.1) are nonnegative. Hence, the function $\overline{u}=\vartheta_a e$ is in fact a positive supersolution of problem (1.1). In similar way one proves that $\underline{u}=-\vartheta_b e$ is a negative subsolution, where we apply the following estimates

$$\begin{aligned}
f(x,s) &\geq -\widetilde{c}s^{p-1}-c_\varsigma, \quad \text{for a.a. } x\in\Omega \text{ and all } s\leq 0,\\
g(x,s) &\geq -bs^{p-1}-c_b, \quad \text{for a.a. } x\in\partial\Omega \text{ and all } s\leq 0.
\end{aligned}$$

This completes the proof. □

The next two lemmas show that constant multipliers of φ_1 may be sub- and supersolution of (1.1). More precisely, we have the following result.

Lemma 3.2. *Assume (H1)–(H2) are satisfied. If $a>\lambda_1$, then for $\varepsilon>0$ sufficiently small and any $b\in\mathbb{R}$ the function $\varepsilon\varphi_1$ is a positive subsolution of problem (1.1).*

Proof. The Steklov eigenvalue problem (1.3) implies

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx\\&= -\int_\Omega(\varepsilon\varphi_1)^{p-1}\varphi dx+\int_{\partial\Omega}\lambda_1(\varepsilon\varphi_1)^{p-1}\varphi d\sigma,\quad \forall\varphi\in W^{1,p}(\Omega).\end{aligned}$$

Definition 2.3 is satisfied for $\underline{u}=\varepsilon\varphi_1$ provided the inequality

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx+\int_{\partial\Omega}((\lambda_1-a)(\varepsilon\varphi_1)^{p-1}-g(x,\varepsilon\varphi_1))\varphi d\sigma\le 0,$$

is valid for all $\varphi\in W^{1,p}(\Omega)_+$. With regard to hypothesis (H1)(f4) we obtain, for $\varepsilon\in(0,\delta_f/\|\varphi_1\|_\infty]$,

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx=\int_\Omega -\frac{f(x,\varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}(\varepsilon\varphi_1)^{p-1}\varphi dx\le 0,$$

where $\|\cdot\|_\infty$ denotes the usual supremum norm. Thanks to condition (H2)(g1) there exists a number $\delta_a>0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}}<a-\lambda_1,\qquad \text{for a.a. } x\in\partial\Omega \text{ and all } 0<|s|\le\delta_a.$$

In case $\varepsilon\in\left(0,\frac{\delta_a}{\|\varphi_1\|_\infty}\right]$ we get

$$\begin{aligned}\int_{\partial\Omega}((\lambda_1-a)(\varepsilon\varphi_1)^{p-1}-g(x,\varepsilon\varphi_1))\varphi d\sigma&\le\int_{\partial\Omega}\left(\lambda_1-a+\frac{|g(x,\varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}}\right)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma\\&<\int_{\partial\Omega}(\lambda_1-a+a-\lambda_1)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma\\&=0.\end{aligned}$$

Selecting $0<\varepsilon\le\min\{\delta_f/\|\varphi_1\|_\infty,\delta_\lambda/\|\varphi_1\|_\infty\}$ guarantees that $\underline{u}=\varepsilon\varphi_1$ is a positive subsolution. □

The following lemma on the existence of a negative supersolution can be proved in a similar way.

Lemma 3.3. *Assume (H1)–(H2) are satisfied. If $b>\lambda_1$, then for $\varepsilon>0$ sufficiently small and any $a\in\mathbb{R}$ the function $-\varepsilon\varphi_1$ is a negative supersolution of problem (1.1).*

Concerning Lemma 3.1-3.3, we obtain a positive pair $[\varepsilon\varphi_1,\vartheta_a e]$ and a negative pair $[-\vartheta_b e,-\varepsilon\varphi_1]$ of sub- and supersolutions of problem (1.1) provided $\varepsilon>0$ is sufficiently small.

In the next step we are going to prove the regularity of solutions of problem (1.1) belonging to the order interval $[0,\vartheta_a e]$ and $[-\vartheta_b e,0]$, respectively. We also point out that $\underline{u}=\overline{u}=0$ is both, a subsolution and a supersolution because of the hypotheses (H1)(f1) and (H2)(g1).

Lemma 3.4. *Assume (H1)–(H2) and let $a,b>\lambda_1$. If $u\in[0,\vartheta_a e]$ (respectively, $u\in[-\vartheta_b e,0]$) is a solution of problem (1.1) satisfying $u\not\equiv 0$ in Ω, then it holds $u\in\operatorname{int}(C^1(\overline{\Omega})_+)$ (respectively, $u\in-\operatorname{int}(C^1(\overline{\Omega})_+)$).*

Proof. We just show the first case, the other case acts in the same way. Let u be a solution of (1.1) satisfying $0 \leq u \leq \vartheta_a e$. We directly obtain the L^∞-boundedness, and hence, the regularity results of Lieberman in [20, Theorem 2] imply $u \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$. Due to the assumptions (H1)(f1),(H1)(f3),(H2)(g1) and (H2)(g3), we obtain the existence of constants $c_f, c_g > 0$ satisfying

$$\begin{aligned} |f(x,s)| &\leq c_f s^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } 0 \leq s \leq \vartheta_a \|e\|_\infty, \\ |g(x,s)| &\leq c_g s^{p-1}, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 \leq s \leq \vartheta_a \|e\|_\infty. \end{aligned} \tag{3.4}$$

Applying (3.4) to (1.1) provides

$$\Delta_p u \leq \widetilde{c} u^{p-1}, \quad \text{a.e. in } \Omega,$$

where $\widetilde{c}$ is a positive constant. We set $\beta(s) = \widetilde{c} s^{p-1}$ for all $s > 0$ and use Vázquez's strong maximum principle (cf. [26]) which is possible because $\int_{0^+} \frac{1}{(s\beta(s))^{\frac{1}{p}}} ds = +\infty$. Hence, it holds $u > 0$ in Ω. Finally, we suppose the existence of $x_0 \in \partial\Omega$ satisfying $u(x_0) = 0$. Applying again the maximum principle yields $\frac{\partial u}{\partial \nu}(x_0) < 0$. However, because of $g(x_0, u(x_0)) = g(x_0, 0) = 0$ in combination with the Neumann condition in (1.1) we get $\frac{\partial u}{\partial \nu}(x_0) = 0$. This is a contradiction and hence, $u > 0$ in $\overline{\Omega}$ which proves $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. □

The main result in this section about the existence of extremal constant-sign solutions is given in the following theorem.

Theorem 3.5. *Assume (H1)–(H2). For every $a > \lambda_1$ and $b \in \mathbb{R}$ there exists a smallest positive solution $u_+ = u_+(a) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of (1.1) in the order interval $[0, \vartheta_a e]$ with the constant ϑ_a as in Lemma 3.1. For every $b > \lambda_1$ and $a \in \mathbb{R}$ there exists a greatest solution $u_- = u_-(b) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ in the order interval $[-\vartheta_b e, 0]$ with the constant ϑ_b as in Lemma 3.1.*

Proof. Let $a > \lambda_1$. Lemma 3.1 and Lemma 3.2 guarantee that $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is a subsolution of problem (1.1) and $\overline{u} = \vartheta_a e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is a supersolution of problem (1.1). Moreover, we choose $\varepsilon > 0$ sufficiently small such that $\varepsilon\varphi_1 \leq \vartheta_a e$. Applying the method of sub- and supersolution (see [4]) corresponding to the order interval $[\varepsilon\varphi_1, \vartheta_a e]$ provides the existence of a smallest positive solution $u_\varepsilon = u_\varepsilon(\lambda)$ of problem (1.1) fulfilling $\varepsilon\varphi_1 \leq u_\varepsilon \leq \vartheta_a e$. In view of Lemma 3.4 we have $u_\varepsilon \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Hence, for every positive integer n sufficiently large there exists a smallest solution $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.1) in the order interval $[\frac{1}{n}\varphi_1, \vartheta_a e]$. We obtain

$$u_n \downarrow u_+ \text{ pointwise }, \tag{3.5}$$

with some function $u_+ : \Omega \to \mathbb{R}$ satisfying $0 \leq u_+ \leq \vartheta_a e$.

Claim 1: u_+ is a solution of problem (1.1).

As $u_n \in [\frac{1}{n}\varphi_1, \vartheta_a e]$ and $\gamma(u_n) \in [\gamma(\frac{1}{n}\varphi_1), \gamma(\vartheta_a e)]$, we obtain the boundedness of u_n in $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. Definition 2.2 holds, in particular, for $u = u_n$ and $\varphi = u_n$ which results in

$$\begin{aligned} \|\nabla u_n\|_{L^p(\Omega)}^p &\leq \int_\Omega |f(x,u_n)| u_n dx + \|u_n\|_{L^p(\Omega)}^p + a\|u_n\|_{L^p(\partial\Omega)}^p + \int_\Omega |g(x,u_n)| u_n d\sigma \\ &\leq a_1 \|u_n\|_{L^p(\Omega)} + \|u_n\|_{L^p(\Omega)}^p + a\|u_n\|_{L^p(\partial\Omega)}^p + a_2 \|u_n\|_{L^p(\partial\Omega)} \\ &\leq a_3, \end{aligned}$$

with some positive constants $a_i, i = 1, \dots, 3$ independent of n. Consequently, u_n is bounded in $W^{1,p}(\Omega)$ and due to the reflexivity of $W^{1,p}(\Omega), 1 < p < \infty$, we obtain the existence of a weakly convergent subsequence of u_n . Because of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, the monotony of u_n and the compactness of the trace operator γ, we get for the entire sequence u_n

$$\begin{aligned} &u_n \rightharpoonup u_+ \ \text{ in } W^{1,p}(\Omega),\\ &u_n \to u_+ \ \text{ in } L^p(\Omega) \text{ and for a.a. } x \in \Omega,\\ &u_n \to u_+ \ \text{ in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega. \end{aligned} \tag{3.6}$$

Since u_n solves problem (1.1), one obtains, for all $\varphi \in W^{1,p}(\Omega)$,

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx\\ &= \int_\Omega (f(x,u_n) - u_n^{p-1})\varphi dx + \int_{\partial\Omega} (au_n^{p-1} + g(x,u_n))\varphi d\sigma. \end{aligned} \tag{3.7}$$

Setting $\varphi = u_n - u_+ \in W^{1,p}(\Omega)$ in (3.7) results in

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+)dx\\ &= \int_\Omega (f(x,u_n) - u_n^{p-1})(u_n - u_+)dx + \int_{\partial\Omega} (au_n^{p-1} + g(x,u_n))(u_n - u_+)d\sigma. \end{aligned}$$

Using (3.6) and the hypotheses (H1)(f3) as well as (H2)(g3) yields

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+)dx \le 0,$$

which provides by the (S_+)-property of $-\Delta_p$ on $W^{1,p}(\Omega)$ along with (3.6)

$$u_n \to u_+ \text{ in } W^{1,p}(\Omega). \tag{3.8}$$

The uniform boundedness of the sequence (u_n) in conjunction with the strong convergence in (3.8) and the conditions (H1)(f3) as well as (H2)(g3) admit us to pass to the limit in (3.7). This shows that u_+ is a solution of problem (1.1).

Claim 2: $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

In order to apply Lemma 3.4, we have to prove that $u_+ \not\equiv 0$. Let us assume this assertion is not valid meaning $u_+ \equiv 0$. From (3.5) it follows

$$u_n(x) \downarrow 0 \ \text{ for all } x \in \Omega. \tag{3.9}$$

We set

$$\widetilde{u}_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \ \text{ for all } n.$$

It is clear that the sequence $(\widetilde{u}_n)$ is bounded in $W^{1,p}(\Omega)$ which ensures the existence of a weakly convergent subsequence of $\widetilde{u}_n$, denoted again by $\widetilde{u}_n$, such that

$$\begin{aligned} &\widetilde{u}_n \rightharpoonup \widetilde{u} \ \text{ in } W^{1,p}(\Omega),\\ &\widetilde{u}_n \to \widetilde{u} \ \text{ in } L^p(\Omega) \text{ and for a.a. } x \in \Omega,\\ &\widetilde{u}_n \to \widetilde{u} \ \text{ in } L^p(\partial\Omega) \text{ and for a.a. } x \in \partial\Omega, \end{aligned} \tag{3.10}$$

with some function $\widetilde{u}:\Omega\to\mathbb{R}$ belonging to $W^{1,p}(\Omega)$. In addition, we may suppose there are functions $z_1\in L^p(\Omega)_+, z_2\in L^p(\partial\Omega)_+$ such that

$$\begin{aligned} |\widetilde{u}_n(x)| &\le z_1(x) \quad \text{for a.a. all } x\in\Omega,\\ |\widetilde{u}_n(x)| &\le z_2(x) \quad \text{for a.a. all } x\in\partial\Omega. \end{aligned} \tag{3.11}$$

With the aid of (3.7), we obtain for $\widetilde{u}_n$ the following variational equation

$$\begin{aligned} \int_\Omega |\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n\nabla\varphi dx &= \int_\Omega \left(\frac{f(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}-\widetilde{u}_n^{p-1}\right)\varphi dx+\int_{\partial\Omega} a\widetilde{u}_n^{p-1}\varphi d\sigma\\ &\quad+\int_{\partial\Omega}\frac{g(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}\varphi d\sigma,\quad \forall\varphi\in W^{1,p}(\Omega). \end{aligned} \tag{3.12}$$

We select $\varphi=\widetilde{u}_n-\widetilde{u}\in W^{1,p}(\Omega)$ in the last equality to get

$$\begin{aligned} &\int_\Omega |\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n\nabla(\widetilde{u}_n-\widetilde{u})dx\\ &=\int_\Omega\left(\frac{f(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}-\widetilde{u}_n^{p-1}\right)(\widetilde{u}_n-\widetilde{u})dx+\int_{\partial\Omega}a\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})d\sigma\\ &\quad+\int_{\partial\Omega}\frac{g(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})d\sigma. \end{aligned} \tag{3.13}$$

Making use of (3.4) in combination with (3.11) results in

$$\frac{|f(x,u_n(x))|}{u_n^{p-1}(x)}\widetilde{u}_n^{p-1}(x)|\widetilde{u}_n(x)-\widetilde{u}(x)|\le c_f z_1(x)^{p-1}(z_1(x)+|\widetilde{u}(x)|), \tag{3.14}$$

respectively,

$$\frac{|g(x,u_n(x))|}{u_n^{p-1}(x)}\widetilde{u}_n^{p-1}(x)|\widetilde{u}_n(x)-\widetilde{u}(x)|\le c_g z_2(x)^{p-1}(z_2(x)+|\widetilde{u}(x)|). \tag{3.15}$$

We see at once that the right-hand sides of (3.14) and (3.15) belong to $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, which allows us to apply Lebesgue's dominated convergence theorem. This fact and the convergence properties in (3.10) show

$$\begin{aligned} \lim_{n\to\infty}\int_\Omega \frac{f(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})dx&=0,\\ \lim_{n\to\infty}\int_{\partial\Omega}\frac{g(x,u_n)}{u_n^{p-1}}\widetilde{u}_n^{p-1}(\widetilde{u}_n-\widetilde{u})d\sigma&=0. \end{aligned} \tag{3.16}$$

From (3.10), (3.13), (3.16) we infer

$$\limsup_{n\to\infty}\int_\Omega|\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n\nabla(\widetilde{u}_n-\widetilde{u})dx=0,$$

and the (S_+)-property of $-\Delta_p$ corresponding to $W^{1,p}(\Omega)$ implies

$$\widetilde{u}_n\to\widetilde{u}\quad\text{in } W^{1,p}(\Omega). \tag{3.17}$$

Remark that $\|\widetilde{u}\|_{W^{1,p}(\Omega)}=1$ which means $\widetilde{u}\not\equiv 0$. Applying (3.9) and (3.17) along with the conditions (H1)(f1),(H2)(g1) to (3.12) provides

$$\int_\Omega|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u}\nabla\varphi dx=-\int_\Omega\widetilde{u}^{p-1}\varphi dx+\int_{\partial\Omega}a\widetilde{u}^{p-1}\varphi d\sigma,\quad\forall\varphi\in W^{1,p}(\Omega).$$

The equation above is the weak formulation of the Steklov eigenvalue problem in (1.3) where $\widetilde{u}\ge 0$ is the eigenfunction with respect to the eigenvalue $a>\lambda_1$. As $\widetilde{u}\ge 0$ is nonnegative in $\overline{\Omega}$, we get a contradiction to the results of Martínez et al.

in [21, Lemma 2.4] because $\widetilde{u}$ must change sign on $\partial\Omega$. Hence, $u_+ \not\equiv 0$. Applying Lemma 3.4 yields $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

Claim 3: $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ is the smallest positive solution of (1.1) in $[0, \vartheta_a e]$.

Let $u \in W^{1,p}(\Omega)$ be a positive solution of (1.1) satisfying $0 \le u \le \vartheta_a e$. Lemma 3.4 immediately implies $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Then there exists an integer n sufficiently large such that $u \in [\frac{1}{n}\varphi_1, \vartheta_a e]$. However, we already know that u_n is the smallest solution of (1.1) in $[\frac{1}{n}\varphi_1, \vartheta_a e]$ which yields $u_n \le u$. Passing to the limit proves $u_+ \le u$. Hence, u_+ must be the smallest positive solution of (1.1). The existence of the greatest negative solution of (1.1) within $[-\vartheta_b e, 0]$ can be proved similarly. This completes the proof of the theorem. □

4. Variational Characterization of Extremal Solutions

Theorem 3.5 ensures the existence of extremal positive and negative solutions of (1.1) for all $a > \lambda_1$ and $b > \lambda_1$ denoted by $u_+ = u_+(a) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $u_- = u_-(b) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$, respectively. Now, we introduce truncation functions $T_+, T_-, T_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $T_+^{\partial\Omega}, T_-^{\partial\Omega}, T_0^{\partial\Omega} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ as follows.

$$T_+(x,s) = \begin{cases} 0 & \text{if } s \le 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}, \quad T_+^{\partial\Omega}(x,s) \begin{cases} 0 & \text{if } s \le 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}$$

$$T_-(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \ge 0 \end{cases}, \quad T_-^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \ge 0 \end{cases}$$

$$T_0(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases},$$

$$T_0^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \le u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \ge u_+(x) \end{cases}$$

For $u \in W^{1,p}(\Omega)$ the truncation operators on $\partial\Omega$ apply to the corresponding traces $\gamma(u)$. We just write for simplification $T_+^{\partial\Omega}(x,u), T_+^{\partial\Omega}(x,u), T_+^{\partial\Omega}(x,u)$ without γ. Furthermore, the truncation operators are continuous, uniformly bounded, and Lipschitz continuous with respect to the second argument. By means of these truncations, we define the following associated functionals given by

$$\begin{aligned} E_+(u) =& \frac{1}{p}[\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{u(x)} f(x, T_+(x,s)) ds dx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[a T_+^{\partial\Omega}(x,s)^{p-1} + g(x, T_+^{\partial\Omega}(x,s)) \right] ds d\sigma, \end{aligned}$$

$$\begin{aligned} E_-(u) =&\frac{1}{p}[\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{u(x)} f(x, T_-(x,s)) ds dx \\ &+ \int_{\partial\Omega}\int_0^{u(x)} \left[b|T_-^{\partial\Omega}(x,s)|^{p-1} - g(x, T_-^{\partial\Omega}(x,s))\right] ds d\sigma, \end{aligned}$$

$$\begin{aligned} E_0(u) =&\frac{1}{p}[\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] - \int_\Omega \int_0^{u(x)} f(x, T_0(x,s)) ds dx \\ &- \int_{\partial\Omega}\int_0^{u(x)} \left[aT_+^{\partial\Omega}(x,s)^{p-1} - b|T_-^{\partial\Omega}(x,s)|^{p-1} + g(x, T_0^{\partial\Omega}(x,s))\right] ds d\sigma, \end{aligned}$$

which are well-defined and belong to $C^1(W^{1,p}(\Omega))$. Due to the truncations, one can easily show that these functionals are coercive and weakly lower semicontinuous which implies that their global minimizers exist. Moreover, they also satisfy the Palais-Smale condition.

Lemma 4.1. *Let u_+ and u_- be the extremal constant-sign solutions of (1.1). Then the following holds:*

(i) *A critical point $v \in W^{1,p}(\Omega)$ of E_+ is a nonnegative solution of (1.1) satisfying $0 \leq v \leq u_+$.*
(ii) *A critical point $v \in W^{1,p}(\Omega)$ of E_- is a nonpositive solution of (1.1) satisfying $u_- \leq v \leq 0$.*
(iii) *A critical point $v \in W^{1,p}(\Omega)$ of E_0 is a solution of (1.1) satisfying $u_- \leq v \leq u_+$.*

Proof. Let v be a critical point of E_0 meaning $E_0'(v) = 0$. We have

$$\begin{aligned} &\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi dx \\ &= \int_\Omega [f(x, T_0(x,v)) - |v|^{p-2}v]\varphi dx + \int_{\partial\Omega} aT_+^{\partial\Omega}(x,v)^{p-1}\varphi d\sigma \\ &\quad + \int_{\partial\Omega}[-b|T_-^{\partial\Omega}(x,v)|^{p-1} + g(x, T_0^{\partial\Omega}(x,v))]\varphi d\sigma, \quad \forall\varphi\in W^{1,p}(\Omega). \end{aligned} \tag{4.1}$$

As u_+ is a positive solution of (1.1) it satisfies

$$\begin{aligned} \int_\Omega |\nabla u_+|^{p-2}\nabla u_+\nabla\varphi dx = &\int_\Omega [f(x,u_+) - u_+^{p-1}]\varphi dx \\ &+ \int_{\partial\Omega}[au_+^{p-1} + g(x,u_+)]\varphi d\sigma, \quad \forall\varphi\in W^{1,p}(\Omega). \end{aligned} \tag{4.2}$$

Subtracting (4.2) from (4.1) and setting $\varphi = (v-u_+)^+ \in W^{1,p}(\Omega)$ provides

$$\begin{aligned} &\int_\Omega [|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+]\nabla(v-u_+)^+ dx + \int_\Omega [|v|^{p-2}v - u_+^{p-1}](v-u_+)^+ dx \\ &= \int_\Omega [f(x, T_0(x,v)) - f(x,u_+)](v-u_+)^+ dx \\ &\quad + \int_{\partial\Omega}[aT_+^{\partial\Omega}(x,v)^{p-1} - b|T_-^{\partial\Omega}(x,v)|^{p-1} - au_+^{p-1}](v-u_+)^+ d\sigma \\ &\quad + \int_{\partial\Omega}[g(x, T_0^{\partial\Omega}(x,v)) - g(x,u_+)](v-u_+)^+ d\sigma. \end{aligned}$$

Based on the definition of the truncation operators, we see that the right-hand side of the equality above is equal to zero. On the other hand the integrals on the left-hand side are strictly positive in case $v > z_+$ which is a contradiction. Thus, we get $(v-u_+)^+ = 0$ and hence, $v \le u_+$. The proof for $v \ge u_-$ acts in a similar way which shows that $T_0(x,v) = v, T_+^{\partial\Omega}(x,v) = v^+, T_-^{\partial\Omega}(x,v) = v^-$ and therefore, v is a solution of (1.1) satisfying $u_- \le v \le u_+$. The statements in (i) and (ii) can be shown in the same way. □

An important tool in our considerations is the relation between local $C^1(\overline{\Omega})$-minimizers and local $W^{1,p}(\Omega)$-minimizers for C^1-functionals. Fact is that every local C^1-minimizer of E_0 is a local $W^{1,p}(\Omega)$-minimizer of E_0 which was proved in similar form in [27, Proposition 5.3]. This result reads as follows.

Proposition 4.2. *If $z_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})-$minimizer of E_0 meaning that there exists $r_1 > 0$ such that*

$$E_0(z_0) \le E_0(z_0+h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \le r_1,$$

then z_0 is a local minimizer of E_0 in $W^{1,p}(\Omega)$ meaning that there exists $r_2 > 0$ such that

$$E_0(z_0) \le E_0(z_0+h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\|_{W^{1,p}(\Omega)} \le r_2.$$

We also refer to a recent paper (see [29]) in which the proposition above was extended to the more general case of nonsmooth functionals. With the aid of Proposition 4.2, we can formulate the next lemma about the existence of local and global minimizers with respect to the functionals E_+, E_- and E_0.

Lemma 4.3. *Let $a > \lambda_1$ and $b > \lambda_1$. Then the extremal positive solution u_+ of (1.1) is the unique global minimizer of the functional E_+ and the extremal negative solution u_- of (1.1) is the unique global minimizer of the functional E_-. In addition, both u_+ and u_- are local minimizers of the functional E_0.*

Proof. As $E_+ : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous, its global minimizer $v_+ \in W^{1,p}(\Omega)$ exists meaning that v_+ is a critical point of E_+. Concerning Lemma 4.1 we know that v_+ is a nonnegative solution of (1.1) satisfying $0 \le v_+ \le u_+$. Due to condition (H2)(g1) there exists a number $\delta_a > 0$ such that

$$|g(x,s)| \le (a-\lambda_1)s^{p-1}, \quad \forall s : 0 < s \le \delta_a. \tag{4.3}$$

Choosing $\varepsilon < \min\left\{\frac{\delta_f}{\|\varphi_1\|_\infty}, \frac{\delta_a}{\|\varphi_1\|_\infty}\right\}$ and applying assumption (H1)(f4), inequality (4.3) along with the Steklov eigenvalue problem in (1.3) implies

$$\begin{aligned} E_+(\varepsilon\varphi_1) = &- \int_\Omega \int_0^{\varepsilon\varphi_1(x)} f(x,s)dsdx + \frac{\lambda_1 - a}{p}\varepsilon^p \|\varphi_1\|^p_{L^p(\partial\Omega)} \\ &- \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} g(x,s)dsd\sigma \\ &< \frac{\lambda_1 - a}{p}\varepsilon^p \|\varphi_1\|_{L^p(\partial\Omega)} + \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} (a-\lambda_1)s^{p-1}dsd\sigma \\ &= 0. \end{aligned}$$

From the calculations above, we see at once that $E_+(v_+) < 0$ which means that $v_+ \neq 0$. This allows us to apply Lemma 3.4 getting $v_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Since u_+ is the smallest positive solution of (1.1) in $[0, \vartheta_a e]$ fulfilling $0 \leq v_+ \leq u_+$, it must hold $v_+ = u_+$ which proves that u_+ is the unique global minimizer of E_+. The same considerations show that u_- is the unique global minimizer of E_-. In order to complete the proof, we are going to show that u_+ and u_- are local minimizers of the functional E_0 as well. The extremal positive solution u_+ belongs to $\operatorname{int}(C^1(\overline{\Omega})_+)$ which means that there is a neighborhood V_{u_+} of u_+ in the space $C^1(\overline{\Omega})$ satisfying $V_{u_+} \subset C^1(\overline{\Omega})_+$. Therefore $E_+ = E_0$ on V_{u_+} proves that u_+ is a local minimizer of E_0 on $C^1(\overline{\Omega})$. Applying Proposition 4.2 yields that u_+ is also a local $W^{1,p}(\Omega)$-minimizer of E_0. Similarly we see that u_- is a local minimizer of E_0 which completes the proof. □

Lemma 4.4. *The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ has a global minimizer v_0 which is a nontrivial solution of (1.1) satisfying $u_- \leq v_0 \leq u_+$.*

Proof. As we know, the functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. Hence, it has a global minimizer v_0. More precisely, v_0 is a critical point of E_0 which is a solution of (1.1) satisfying $u_- \leq v_0 \leq u_+$ (see Lemma 4.1). The fact that $E_0(u_+) = E_+(u_+) < 0$ (see the proof of Lemma 4.3) proves that v_0 is nontrivial meaning $v_0 \neq 0$. □

5. Existence of Sign-Changing Solutions

The main result in this section about the existence of a nontrivial solution of problem (1.1) reads as follows.

Theorem 5.1. *Under hypotheses (H1)–(H3) problem (1.1) has a nontrivial sign-changing solution $u_0 \in C^1(\overline{\Omega})$.*

Proof. In view of Lemma 4.4 the existence of a global minimizer $v_0 \in W^{1,p}(\Omega)$ of E_0 satisfying $v_0 \neq 0$ has been proved. This means that v_0 is a nontrivial solution of (1.1) belonging to $[u_-, u_+]$. If $v_0 \neq u_-$ and $v_0 \neq u_+$, then $u_0 := v_0$ must be a sign-changing solution because u_- is the greatest negative solution and u_+ is the smallest positive solution of (1.1) which proves the theorem in this case. We still have to show the theorem in case that either $v_0 = u_-$ or $v_0 = u_+$. Let us only consider the case $v_0 = u_+$ because the case $v_0 = u_-$ can be proved similarly. The function u_- is a local minimizer of E_0. Without loss of generality we suppose that u_- is a strict local minimizer, otherwise we would obtain infinitely many critical points v of E_0 which are sign-changing solutions due to $u_- \leq v \leq u_+$ and the extremality of the solutions u_-, u_+. Under these assumptions, there exists a $\rho \in (0, \|u_+ - u_-\|_{W^{1,p}(\Omega)})$ such that

$$E_0(u_+) \leq E_0(u_-) < \inf\{E_0(u) : u \in \partial B_\rho(u_-)\}, \tag{5.1}$$

where $\partial B_\rho = \{u \in W^{1,p}(\Omega) : \|u - u_-\|_{W^{1,p}(\Omega)} = \rho\}$. Now, we may apply the Mountain-Pass Theorem to E_0 (cf. [24]) thanks to (5.1) along with the fact that E_0 satisfies the Palais-Smale condition. This yields the existence of $u_0 \in W^{1,p}(\Omega)$ satisfying $E_0'(u_0) = 0$ and

$$\inf\{E_0(u) : u \in \partial B_\rho(u_-)\} \leq E_0(u_0) = \inf_{\pi \in \Pi} \max_{t \in [-1,1]} E_0(\pi(t)), \tag{5.2}$$

where

$$\Pi = \{\pi \in C([-1,1], W^{1,p}(\Omega)) : \pi(-1) = u_-, \pi(1) = u_+\}.$$

It is clear that (5.1) and (5.2) imply $u_0 \neq u_-$ and $u_0 \neq u_+$. Hence, u_0 is a sign-changing solution provided $u_0 \neq 0$. We have to show that $E_0(u_0) \neq 0$ which is fulfilled if there exists a path $\widetilde{\pi} \in \Pi$ such that

$$E_0(\widetilde{\pi}(t)) \neq 0, \ \forall t \in [-1,1].$$

Let $S = W^{1,p}(\Omega) \cap \partial B_1^{L^p(\partial\Omega)}$, where $\partial B_1^{L^p(\partial\Omega)} = \{u \in L^p(\partial\Omega) : \|u\|_{L^p(\partial\Omega)} = 1\}$, and $S_C = S \cap C^1(\overline{\Omega})$ be equipped with the topologies induced by $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$, respectively. Furthermore, we set

$$\Pi_0 = \{\pi \in C([-1,1], S) : \pi(-1) = -\varphi_1, \pi(1) = \varphi_1\},$$
$$\Pi_{0,C} = \{\pi \in C([-1,1], S_C) : \pi(-1) = -\varphi_1, \pi(1) = \varphi_1\}.$$

Because of the results of Martínez and Rossi in [23] there exists a continuous path $\pi \in \Pi_0$ satisfying $t \mapsto \pi(t) \in \{u \in W^{1,p}(\Omega) : I^{(a,b)}(u) < 0, \|u\|_{L^p(\partial\Omega)} = 1\}$ provided (a,b) is above the curve $\mathcal{C}$ of hypothesis (H3). Recall that the functional $I^{(a,b)}$ is given by

$$I^{(a,b)}(u) = \int_\Omega (|\nabla u|^p + |u|^p)dx - \int_{\partial\Omega} (a(u^+)^p + b(u^-)^p)d\sigma.$$

This implies the existence of $\mu > 0$ such that

$$I^{(a,b)}(\pi(t)) \leq -\mu < 0, \quad \forall t \in [-1,1].$$

It is well known that S_C is dense in S which implies the density of $\Pi_{0,C}$ in Π_0. Thus, a continuous path $\pi_0 \in \Pi_{0,C}$ exists such that

$$|I^{(a,b)}(\pi(t)) - I^{(a,b)}(\pi_0(t))| < \frac{\mu}{2}, \quad \forall t \in [-1,1].$$

The boundedness of the set $\pi_0([-1,1])(\overline{\Omega})$ in $\mathbb{R}$ ensures the existence of $M > 0$ such that

$$|\pi_0(t)(x)| \leq M \qquad \text{for all } x \in \overline{\Omega} \text{ and for all } t \in [-1,1].$$

Lemma 3.5 yields that $u_+, -u_- \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Thus, for every $u \in \pi_0([-1,1])$ and any bounded neighborhood V_u of u in $C^1(\overline{\Omega})$ there exist positive numbers h_u and j_u satisfying

$$u_+ - hv \in \operatorname{int}(C^1(\overline{\Omega})_+) \ \text{ and } \ -u_- + jv \in \operatorname{int}(C^1(\overline{\Omega})_+), \tag{5.3}$$

for all $h : 0 \leq h \leq h_u$, for all $j : 0 \leq j \leq j_u$, and for all $v \in V_u$. Using (5.3) along with a compactness argument implies the existence of $\varepsilon_0 > 0$ such that

$$u_-(x) \leq \varepsilon\pi_0(t)(x) \leq u_+(x), \tag{5.4}$$

for all $x \in \Omega$, for all $t \in [-1,1]$, and for all $\varepsilon \leq \varepsilon_0$. Representing E_0 in terms of $I^{(a,b)}$, we obtain

$$
\begin{aligned}
E_0(u) =& \frac{1}{p} I^{(a,b)}(u) + \int_{\partial\Omega} (a(u^+)^p + b(u^-)^p) d\sigma - \int_\Omega \int_0^{u(x)} f(x, T_0(x,s)) ds dx \\
& - \int_{\partial\Omega} \int_0^{u(x)} (a T_+^{\partial\Omega}(x,s)^{p-1} - b |T_-^{\partial\Omega}(x,s)|^{p-1}) ds d\sigma \\
& - \int_{\partial\Omega} \int_0^{u(x)} g(x, T_0^{\partial\Omega}(x,s)) ds d\sigma.
\end{aligned}
$$

In view of (5.4) we get for all $\varepsilon \leq \varepsilon_0$ and all $t \in [-1,1]$

$$
\begin{aligned}
& E_0(\varepsilon\pi_0(t)) \\
& = \frac{1}{p} I^{(a,b)}(\varepsilon\pi_0(t)) - \int_\Omega \int_0^{\varepsilon\pi_0(t)(x)} f(x,s) ds dx - \int_{\partial\Omega} \int_0^{\varepsilon\pi_0(t)(x)} g(x,s) ds d\sigma. \\
& = \varepsilon^p \Bigg[\frac{1}{p} I^{(a,b)}(\pi_0(t)) - \frac{1}{\varepsilon^p} \int_\Omega \int_0^{\varepsilon\pi_0(t)(x)} f(x,s) ds dx \\
& \qquad - \frac{1}{\varepsilon^p} \int_{\partial\Omega} \int_0^{\varepsilon\pi_0(t)(x)} g(x,s) ds d\sigma \Bigg] \\
& < \varepsilon^p \Bigg[-\frac{\mu}{2p} + \frac{1}{\varepsilon^p} \int_\Omega \left| \int_0^{\varepsilon\pi_0(t)(x)} f(x,s) ds \right| dx \\
& \qquad + \frac{1}{\varepsilon^p} \int_{\partial\Omega} \left| \int_0^{\varepsilon\pi_0(t)(x)} g(x,s) ds \right| d\sigma \Bigg]
\end{aligned} \tag{5.5}
$$

Due to hypotheses (H1)(f1) and (H2)(g1) there exist positive constants δ_1, δ_2 such that

$$
\begin{aligned}
|f(x,s)| &\leq \frac{\mu}{5M^p} |s|^{p-1}, \quad \text{for a.a. } x \in \Omega \text{ and all } s : |s| \leq \delta_1, \\
|g(x,s)| &\leq \frac{\mu}{5M^p} |s|^{p-1}, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s : |s| \leq \delta_2.
\end{aligned} \tag{5.6}
$$

Choosing $\varepsilon > 0$ such that $\varepsilon < \min\{\varepsilon_0, \frac{\delta_1}{M}, \frac{\delta_2}{M}\}$ and using (5.6) provides

$$
\begin{aligned}
\frac{1}{\varepsilon^p} \int_\Omega \left| \int_0^{\varepsilon\pi_0(t)(x)} f(x,s) ds \right| dx &\leq \frac{\mu}{5p}, \\
\frac{1}{\varepsilon^p} \int_{\partial\Omega} \left| \int_0^{\varepsilon\pi_0(t)(x)} g(x,s) ds \right| d\sigma &\leq \frac{\mu}{5p}.
\end{aligned} \tag{5.7}
$$

Applying (5.7) to (5.5) yields

$$
E_0(\varepsilon\pi_0(t)) \leq \varepsilon^p \left(-\frac{\mu}{2p} + \frac{\mu}{5p} + \frac{\mu}{5p}\right) < 0, \qquad \text{for all } t \in [-1,1]. \tag{5.8}
$$

We have constructed a continuous path $\varepsilon\pi_0$ joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. In order to construct continuous paths π_+, π_- connecting $\varepsilon\varphi_1$ and u_+, respectively, u_- and $-\varepsilon\varphi_1$, we first denote

$$
c_+ = E_+(\varepsilon\varphi_1), \; m_+ = E_+(u_+), \; E_+^{c_+} = \{u \in W^{1,p}(\Omega) : E_+(u) \leq c_+\}.
$$

It holds $m_+ < c_+$ because u_+ is a global minimizer of E_+. By Lemma 4.1 the functional E_+ has no critical values in the interval $(m_+, c_+]$. The coercivity of E_+ along with its property to satisfy the Palais-Smale condition allows us to apply the Second Deformation Lemma (see, e.g. [17, p. 366]) to E_+. This ensures the existence of a continuous mapping $\eta \in C([0,1] \times E_+^{c_+}, E_+^{c_+})$ satisfying the following properties.

(i) $\eta(0,u) = u, \quad$ for all $u \in E_+^{c_+}$,
(ii) $\eta(1,u) = u_+, \quad$ for all $u \in E_+^{c_+}$,
(iii) $E_+(\eta(t,u)) \le E_+(u), \quad$ for all $t \in [0,1]$ and for all $u \in E_+^{c_+}$.

Next, we introduce the path $\pi_+ : [0,1] \to W^{1,p}(\Omega)$ given by $\pi_+(t) = \eta(t, \varepsilon\varphi_1)^+ = \max\{\eta(t,\varepsilon\varphi_1), 0\}$ for all $t \in [0,1]$ which is obviously continuous in $W^{1,p}(\Omega)$ joining $\varepsilon\varphi_1$ and u_+. Additionally, one has

$$E_0(\pi_+(t)) = E_+(\pi_+(t)) \le E_+(\eta(t,\varepsilon\varphi_1)) \le E_+(\varepsilon\varphi_1) < 0, \qquad \text{for all } t \in [0,1]. \tag{5.9}$$

Similarly, the Second Deformation Lemma can be applied to the functional E_-. We get a continuous path $\pi_- : [0,1] \to W^{1,p}(\Omega)$ connecting $-\varepsilon\varphi_1$ and u_- such that

$$E_0(\pi_-(t)) < 0, \qquad \text{for all } t \in [0,1]. \tag{5.10}$$

In the end, we combine the curves $\pi_-, \varepsilon\pi_0$ and π_+ to obtain a continuous path $\widetilde{\pi} \in \Pi$ joining u_- and u_+. Taking into account (5.8), (5.9), and (5.10), we get $u_0 \neq 0$. This yields the existence of a nontrivial sign-changing solution u_0 of problem (1.1) satisfying $u_- \le u_0 \le u_+$ which completes the proof. □

References

[1] D. Arcoya, J. I. Diaz, and L. Tello. S-shaped bifurcation branch in a quasilinear multivalued model arising in climatology. *J. Differential Equations*, 150(1):215–225, 1998.

[2] C. Atkinson and C. R. Champion. On some boundary value problems for the equation $\nabla \cdot (F(|\nabla w|)\nabla w) = 0$. *Proc. Roy. Soc. London Ser. A*, 448(1933):269–279, 1995.

[3] C. Atkinson and K. El Kalli. Some boundary value problems for the bingham model. *J. Differential Equations*, 41:339–363, 1992.

[4] S. Carl. Existence and comparison results for noncoercive and nonmonotone multivalued elliptic problems. *Nonlinear Anal.*, 65(8):1532–1546, 2006.

[5] S. Carl and D. Motreanu. Constant-sign and sign-changing solutions of a nonlinear eigenvalue problem involving the p-Laplacian. *Differential Integral Equations*, 20(3):309–324, 2007.

[6] S. Carl and D. Motreanu. Sign-changing and extremal constant-sign solutions of nonlinear elliptic problems with supercritical nonlinearities. *Comm. Appl. Nonlinear Anal.*, 14(4):85–100, 2007.

[7] S. Carl and D. Motreanu. Constant-sign and sign-changing solutions for nonlinear eigenvalue problems. *Nonlinear Anal.*, 68(9):2668–2676, 2008.

[8] S. Carl and K. Perera. Sign-changing and multiple solutions for the p-Laplacian. *Abstr. Appl. Anal.*, 7(12):613–625, 2002.

[9] M. del Pino and C. Flores. Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains. *Comm. Partial Differential Equations*, 26(11-12):2189–2210, 2001.

[10] J. I. Díaz. *Nonlinear partial differential equations and free boundaries. Vol. I*, volume 106 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985. Elliptic equations.

[11] J. F. Escobar. Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate. *Comm. Pure Appl. Math.*, 43(7):857–883, 1990.

[12] J. Fernández Bonder. Multiple positive solutions for quasilinear elliptic problems with sign-changing nonlinearities. *Abstr. Appl. Anal.*, vol. 2004(12):1047–1055, 2004.

[13] J. Fernández Bonder. Multiple solutions for the p-Laplace equation with nonlinear boundary conditions. *Electron. J. Differential Equations*, pages No. 37, 7 pp. (electronic), 2006.

[14] J. Fernández Bonder, E. Lami Dozo, and J. D. Rossi. Symmetry properties for the extremals of the Sobolev trace embedding. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(6):795–805, 2004.

[15] J. Fernández Bonder, S. R. Martínez, and J. D. Rossi. The behavior of the best Sobolev trace constant and extremals in thin domains. *J. Differential Equations*, 198(1):129–148, 2004.

[16] J. Fernández Bonder and J. D. Rossi. Existence results for the p-Laplacian with nonlinear boundary conditions. *J. Math. Anal. Appl.*, 263(1):195–223, 2001.

[17] L. Gasiński and N. S. Papageorgiou. *Nonsmooth critical point theory and nonlinear boundary value problems*, volume 8 of *Series in Mathematical Analysis and Applications*. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[18] A. Lê. Eigenvalue problems for the p-Laplacian. *Nonlinear Anal.*, 64(5):1057–1099, 2006.

[19] C. Li and S. Li. Multiple solutions and sign-changing solutions of a class of nonlinear elliptic equations with Neumann boundary condition. *J. Math. Anal. Appl.*, 298(1):14–32, 2004.

[20] G. M. Lieberman. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.*, 12(11):1203–1219, 1988.

[21] S. R. Martínez and J. D. Rossi. Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition. *Abstr. Appl. Anal.*, 7(5):287–293, 2002.

[22] S. R. Martínez and J. D. Rossi. Weak solutions for the p-Laplacian with a nonlinear boundary condition at resonance. *Electron. J. Differential Equations*, pages No. 27, 14 pp. (electronic), 2003.

[23] S. R. Martínez and J. D. Rossi. On the Fučik spectrum and a resonance problem for the p-Laplacian with a nonlinear boundary condition. *Nonlinear Anal.*, 59(6):813–848, 2004.

[24] P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

[25] P. Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *J. Differential Equations*, 51(1):126–150, 1984.

[26] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3):191–202, 1984.

[27] P. Winkert. Constant-sign and sign-changing solutions for nonlinear elliptic equations with neumann boundary values. *Adv. Differential Equations*, 15(5–6):561–599, 2010.

[28] P. Winkert. L^∞-estimates for nonlinear elliptic neumann boundary value problems. *NoDEA Nonlinear Differential Equations Appl.*, 17(3):289–302, 2010.

[29] P. Winkert. Local $C^1(\overline{\Omega})$-minimizers versus local $W^{1,p}(\Omega)$-minimizers of nonsmooth functionals. *Nonlinear Anal.*, 72(11):4298–4303, 2010.

[30] X. Wu and K.-K. Tan. On existence and multiplicity of solutions of Neumann boundary value problems for quasi-linear elliptic equations. *Nonlinear Anal.*, 65(7):1334–1347, 2006.

[31] J.-H. Zhao and P.-H. Zhao. Existence of infinitely many weak solutions for the p-Laplacian with nonlinear boundary conditions. *Nonlinear Anal.*, 69(4):1343–1355, 2008.

Technische Universität Berlin, Institut für Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany

E-mail address: `winkert@math.tu-berlin.de`