MULTI-BUMP TYPE NODAL SOLUTIONS FOR A FRACTIONAL p-LAPLACIAN LOGARITHMIC SCHRÖDINGER EQUATION WITH DEEPENING POTENTIAL WELL

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Abstract. This article concerns the existence and multiplicity of multi-bump type nodal solutions for a class of fractional p-Laplacian Schrödinger equations involving logarithmic nonlinearity and deepening potential well. We apply suitable variational arguments to show that the equation has at least $2^k - 1$ multi-bump type nodal solutions as the parameter becomes large enough.

1. Introduction

This paper is devoted to the existence of multi-bump type nodal solutions for fractional p-Laplacian logarithmic Schrödinger equations of the form

$$\begin{cases} (-\Delta)_p^s u + \lambda V(x)|u|^{p-2}u = |u|^{p-2}u \log|u|^p & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $p \in [2,\infty)$, $N > sp$ and $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous potential satisfying the following conditions:

(V_1) $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$;
(V_2) $\Omega := \operatorname{int} V^{-1}(0)$ is a non-empty bounded open subset with smooth boundary and $\overline{\Omega} = V^{-1}(0)$, where $\operatorname{int} V^{-1}(0)$ denotes the set of the interior points of $V^{-1}(0)$;
(V_3) Ω consists of k components

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$$

and $\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$ for all $i \neq j$.

Here, $(-\Delta)_p^s$ is the fractional p-Laplacian operator which is defined for any $u\colon \mathbb{R}^N \to \mathbb{R}$ belonging to the Schwartz class by

$$(-\Delta)_p^s u(x) = 2 \lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus B_\delta(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,\mathrm{d}y \quad (x \in \mathbb{R}^N),$$

for any $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\delta(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius δ.

When the logarithmic nonlinearity is replaced by a power-type nonlinearity, problem (1.1) is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modeled by Lévy processes, see, for example, Di Nezza-Palatucci-Valdinoci [14] for a physical background. We also recall that the analysis of fractional and nonlocal operators is strongly motivated by the fact that these operators play a fundamental role in describing various physical phenomena such as, among others, phase transitions, crystal dislocations, anomalous diffusions, conservation laws, flame propagation and chemical reactions of liquids. For more details and applications, we refer the interested reader to the works by Applebaum [11], Bahrouni-Rădulescu-Winkert [12], Di Nezza-Palatucci-Valdinoci [14], Molica Bisci-Rădulescu-Servadei [18], see also the references therein.

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In these last years, many intriguing existence and multiplicity results have been established for fractional p-Laplacian Schrödinger equations given by

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N,$$

see for instance Alves-Miyagaki [3], Ambrosio [7, 8], Ji [15], Qu-He [21] for the case $p = 2$ and Alves-Ambrosio [1], Ambrosio-Figueiredo-Isernia [9], Ambrosio-Isernia [10], Pucci-Xiang-Zhang [19, 20] whenever $p \in (1,\infty)$. In particular, Alves-Ambrosio [1] obtained an existence and concentration result when f is a logarithmic nonlinearity and V verifies the following local conditions:

(V_1') $V(x) \in C(\mathbb{R}^N,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > -1$;

(V_2') There exists a bounded open set $\Omega \subset \mathbb{R}^N$ such that

$$-1 < V_0 = \inf_{x\in\Omega} V(x) < \min_{\partial\Omega} V \quad \text{and} \quad M = \{x \in \Omega\colon V(x) = V_0\} \neq \emptyset.$$

They employed the penalization method to demonstrate the existence of positive solutions, as well as the concentration behavior under conditions (V_1') and (V_2').

Recently, the following time-dependent logarithmic Schrödinger equation given by

$$i\varepsilon\frac{\partial\Phi}{\partial t} = -\varepsilon^2\Delta\Phi + W(x)\Phi - \Phi\log|\Phi|^2, \quad N \geq 3 \tag{1.2}$$

where $\Phi\colon [0,+\infty)\times\mathbb{R}^N \to \mathbb{C}$, has also obtained special attention due to its physical influence, such as quantum mechanics, quantum optics, nuclear physics, effective quantum and Bose-Einstein condensation. Standing wave solutions for (1.2) have the form $\Phi(t,x) = u(x)e^{-i\omega t/\varepsilon}$, where $\omega \in \mathbb{R}$, which leads to a system of the shape

$$-\varepsilon^2\Delta u + V(x)u = u\log u^2 \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x)-\omega$. From the mathematical point of view, (1.3) is very interesting because many difficulties arise when using variational methods to find solutions. Alves-de Morais Filho [2] considered semiclassical state solutions for the logarithmic elliptic equation (1.3) when V satisfies the following global condition

(V_{global}) $V \in C(\mathbb{R}^N,\mathbb{R})$ and $V_\infty = \lim_{|x|\to\infty} V(x) > V_* = \inf_{x\in\mathbb{R}^N} V(x) > -1$.

They obtained the existence of solutions of (1.3) as well as the concentration behavior of solutions as $\varepsilon \to 0$. Alves-Ji [4] continued to study (1.3) where V satisfies the local conditions (V_1') and (V_2'). Moreover, Alves-Ji [5] studied the existence of multi-bump positive solutions for the following Schrödinger equation with logarithmic nonlinearity and deepening potential well

$$\begin{cases} -\Delta u + \lambda V(x)u = u\log u^2 & \text{in } \mathbb{R}^N,\\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.4}$$

Then, Ji [16] was concerned with the existence and multiplicity of multi-bump type nodal solutions for problem (1.4). We also refer to the works by Alves-Ambrosio [1], Alves-Ji [6], d'Avenia-Montefusco-Squassina [13], Ji-Szulkin [17], Tanaka-Zhang [24] and the references therein.

Motivated by the above papers, in this work we obtain the existence of multi-bump type nodal solutions for problem (1.1). More precisely, our main results are as follows.

Theorem 1.1. *Suppose that V satisfies (V_1)–(V_3). Then, for any non-empty subset Γ of $\{1,2,\ldots,k\}$, there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, problem (1.1) has a nodal solution u_λ. Moreover, the family $\{u_\lambda\}_{\lambda\geq\lambda^*}$ has the following properties: for any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that $u_{\lambda_{n_i}}$ converges strongly in $W^{s,p}(\mathbb{R}^N)$ to a function u which satisfies $u(x) = 0$ for $x \notin \Omega_\Gamma$ and the restriction $u\big|_{\Omega_j}$ is a nodal solution with least energy of*

$$\begin{cases} (-\Delta)_p^s u = |u|^{p-2}u\log|u|^p & \text{in } \Omega_\Gamma,\\ u = 0 & \text{on } \partial\Omega_\Gamma, \end{cases}$$

where $\Omega_\Gamma = \bigcup_{j\in\Gamma}\Omega_j$.

From this result, we obtain the following direct consequence.

Corollary 1.2. *Under the assumptions of Theorem 1.1, there exists $\lambda_* > 0$ such that for all $\lambda \geq \lambda_*$, the problem (1.1) has at least $2^k - 1$ nodal solutions.*

Corollary 1.2 can be directly obtained from Theorem 1.1. Our approach is mainly based on variational methods. First note that the associated energy functional of problem (1.1) may take the value $+\infty$, since there is a function $u \in W^{s,p}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |u|^p \log |u|^p \,\mathrm{d}x = -\infty$. Thus, the energy functional is not well defined on $W^{s,p}(\mathbb{R}^N)$ and the classical variational methods cannot be applied here. To find solutions of equation (1.1), we will perform a technical decomposition to obtain a functional which is a sum of a lower semicontinuous convex functional and a C^1-functional. Here, we have made great use of the fact that the energy functional is of class C^1 in $W^{s,p}(\mathcal{D})$, when $\mathcal{D} \subset \mathbb{R}^N$ is a bounded domain. Based on this observation, for each $R > 0$ and $\lambda > 0$ large enough, we find a nodal solution $u_{\lambda,R} \in W_0^{s,p}(B_R(0))$ by penalization arguments, and after taking the limit of $R \to +\infty$, we get a nodal solution for the original problem.

In fact, by the method presented in this paper, we can also demonstrate the existence of multi-bump solutions that join positive, negative, and nodal least energy solutions. For this purpose, we need to make some modifications. For example, if we want to get a positive solution ω_1 on Ω_1 and a negative solution ω_2 on Ω_2, we need to change $\omega_1^{\pm}$ and $\omega_2^{\pm}$ by ω_1 and ω_2, respectively. We also need to make some modifications for the definition of $b_{\lambda,R,\Gamma}$ and the set $\mathcal{A}_{\mu,R}^{\lambda}$, which are defined in Section 4 and 5. In addition, we need to replace d_1 and d_2 with mountain pass levels c_1 and c_2 associated with the energy functionals $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. From this, we have the following theorem.

Theorem 1.3. *Suppose that V satisfies* (V_1)–(V_3). *Then, for any non-empty subset Γ_1, Γ_2 and Γ_3 of $\{1, 2, \ldots, k\}$ with $\Gamma_i \cap \Gamma_j = \emptyset$, for $i \neq j$, there is $\lambda^* > 0$ such that, for all $\lambda \geq \lambda^*$, problem (1.1) has a nontrivial solution u_λ. Moreover, the family $\{u_\lambda\}_{\lambda \geq \lambda^*}$ has the following properties: for any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that $u_{\lambda_{n_i}}$ converges strongly in $W^{s,p}\left(\mathbb{R}^N\right)$ to a function u which satisfies $u(x) = 0$ for $x \notin \Omega_\Gamma (= \cup_{j\in\Gamma}\Omega_j)$ where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, and the restriction $u|_{\Omega_j}$ is a positive solution if $j \in \Gamma_1$, a negative solution if $j \in \Gamma_2$ and a nodal solution with least energy of*

$$\begin{cases} (-\Delta)_p^s u = |u|^{p-2} u \log |u|^p & \text{in } \Omega_j, \\ u = 0 & \text{on } \partial\Omega_j, \end{cases}$$

where $j \in \Gamma_3$.

The paper is organized as follows. In Section 2, we recall some lemmas which we will use in the paper. In Section 3–5, we establish an auxiliary problem and prove the existence of multi-bump nodal solutions for the auxiliary problem in the ball $B_R(0)$ for $R > 0$. In Section 6, we provide the proof of Theorem 1.1.

2. Preliminaries

In this section, we present the main tools and notions that will occur in Sections 3–6. If $A \subset \mathbb{R}^N$, we denote by $|u|_{L^q(A)}$ the $L^q(A)$-norm of a function $u \colon \mathbb{R}^N \to \mathbb{R}$, and by $|u|_q$ its $L^q(\mathbb{R}^N)$-norm. With $B_r(x_0)$ we indicate the ball in $\mathbb{R}^N$ centered at $x_0 \in \mathbb{R}^N$ with radius $r > 0$. When $x_0 = 0$, we simply write B_r instead of $B_r(0)$.

Let $s \in (0, 1)$, $p \in (1, \infty)$ and $N > sp$. We define $D^{s,p}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to

$$[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}x \,\mathrm{d}y,$$

or equivalently

$$D^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{p_s^*}(\mathbb{R}^N) \colon [u]_{s,p} < \infty \right\},$$

where $p_s^* = \frac{Np}{N-sp}$ is the fractional critical Sobolev exponent. The fractional Sobolev space $W_{s,p}(\mathbb{R}^N)$ is given by

$$W^{s,p}(\mathbb{R}^N) = \left\{u \in L^p(\mathbb{R}^N) \colon [u]_{s,p} < \infty\right\},$$

endowed with the norm

$$\|u\|^p_{W^{s,p}(\mathbb{R}^N)} = [u]^p_{s,p} + |u|^p_p.$$

We know that there exists a constant $S_* = S(N,s,p) > 0$ such that $S_*\|u\|^p_{L^{p_s^*}(\mathbb{R}^N)} \leq [u]^p_{s,p}$ for all $u \in D^{s,p}(\mathbb{R}^N)$. Now, we recall the following main embeddings for fractional Sobolev spaces, see Di Nezza-Palatucci-Valdinoci [14].

Lemma 2.1. *Let $s \in (0,1)$, $p \in (1,\infty)$ and $N > sp$. Then $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $q \in [p, p_s^*)$ and compactly in $L^q_{\text{loc}}(\mathbb{R}^N)$ for all $q \in [1, p_s^*)$, and $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$.*

We also recall the following vanishing Lions-type result for $W^{s,p}(\mathbb{R}^N)$, see Ambrosio-Isernia [10].

Lemma 2.2. *If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^p \, \mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (p, p_s^*)$.

From now on, we suppose $p \in [2,\infty)$ and we shall work on the following function space

$$E_\lambda := \left\{u \in W^{s,p}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} V(x)|u|^p \, \mathrm{d}x < \infty\right\}$$

endowed with the norm

$$\|u\|^p_\lambda := [u]^p_{s,p} + \int_{\mathbb{R}^N} (\lambda V(x) + 1)\, |u|^p \, \mathrm{d}x.$$

Obviously, E_λ is a uniformly convex Banach space, the duality pairing associated with the norm is given by

$$(u,v)_\lambda = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \, \mathrm{d}x\, \mathrm{d}y + \int_{\mathbb{R}^N} (\lambda V(x) + 1)\, |u|^{p-2}uv \, \mathrm{d}x.$$

Since $V(x) \geq 0$ for all $x \in \mathbb{R}^N$, the embedding $E_\lambda \hookrightarrow W^{s,p}(\mathbb{R}^N)$ is continuous, and so the embedding $E_\lambda \hookrightarrow L^q(\mathbb{R}^3)$ is also continuous for all $q \in [p, p_s^*]$.

For each $R > 0$, we define a norm $\|\cdot\|_{\lambda,R}$ on $W^{s,p}(B_R(0))$ by

$$\|u\|^p_{\lambda,R} := \int_{B_R(0)} \int_{B_R(0)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}x\, \mathrm{d}y + \int_{B_R(0)} (\lambda V(x) + 1)\, |u|^p \, \mathrm{d}x,$$

which is equivalent to the usual norm in that space for all λ, $R > 0$. In what follows, we will denote by $E_{\lambda,R}$ the space E_λ endowed with the norm $\|\cdot\|_{\lambda,R}$.

Note that a weak solution of (1.1) in $W^{s,p}(\mathbb{R}^N)$ is a critical point of the associated energy functional

$$\mathcal{I}_\lambda(u) := \frac{1}{p}\|u\|^p_\lambda - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \log |u|^p \, \mathrm{d}x. \tag{2.1}$$

Definition 2.3. *A solution of problem (1.1) is a function $u \in W^{s,p}(\mathbb{R}^N)$ such that $|u|^p \log |u|^p \in L^1(\mathbb{R}^N)$ and*

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \, \mathrm{d}x\, \mathrm{d}y + \int_{\mathbb{R}^N} \lambda V(x)|u|^{p-2}uv \, \mathrm{d}x$$
$$= \int_{\mathbb{R}^N} |u|^{p-2}uv \log |u|^p \, \mathrm{d}x$$

for all $v \in C_c^\infty(\mathbb{R}^N)$.

Due to the lack of smoothness of $\mathcal{I}_\lambda$, we shall use the approach explored by Ji-Szulkin [17] and Squassina-Szulkin [22, 23]. For this purpose, we decompose $\mathcal{I}_\lambda$ into a sum of a C^1 functional plus a convex lower semicontinuous functional, respectively. For $\delta > 0$, we define the functions

$$F_1(\varsigma) = \begin{cases} 0, & \text{if } \varsigma = 0, \\ -\frac{1}{p}|\varsigma|^p \log|\varsigma|^p & \text{if } 0 < |\varsigma| < \delta, \\ -\frac{1}{p}|\varsigma|^p \left(\log\delta^p + \frac{p+1}{p-1}\right) + \frac{2}{p-1}\delta^{p-1}|\varsigma| - \frac{1}{p}\delta^p, & \text{if } |\varsigma| \geq \delta, \end{cases}$$

and

$$F_2(\varsigma) = \begin{cases} 0, & \text{if } |\varsigma| < \delta, \\ \frac{1}{p}|\varsigma|^p \log\left(\frac{|\varsigma|^p}{\delta^p}\right) + \frac{2}{p-1}\delta^{p-1}|\varsigma| - \frac{p+1}{p(p-1)}|\varsigma|^p - \frac{1}{p}\delta^p, & \text{if } |\varsigma| \geq \delta. \end{cases}$$

Then,

$$F_2(\varsigma) - F_1(\varsigma) = \frac{1}{p}|\varsigma|^p \log|\varsigma|^p \quad \text{for all } \varsigma \in \mathbb{R},$$

and the functional $\mathcal{I}_\lambda \colon E_\lambda \to (-\infty, +\infty]$ may be rewritten as

$$\mathcal{I}_\lambda(u) = \Phi_\lambda(u) + \Psi(u), \quad u \in E_\lambda,$$

where

$$\Phi_\lambda(u) = \frac{1}{p}\|u\|_\lambda^p - \int_{\mathbb{R}^N} F_2(u)\,\mathrm{d}x,$$

and

$$\Psi(u) = \int_{\mathbb{R}^N} F_1(u)\,\mathrm{d}x.$$

As proven in Ji-Szulkin [17] and Squassina-Szulkin [22, 23], $F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R})$. If $\delta > 0$ is small enough, F_1 is convex, even,

$$F_1(\varsigma) \geq 0 \quad \text{and} \quad 0 \leq \frac{1}{p}F_1'(\varsigma)\varsigma \leq F_1(\varsigma) \leq F_1'(\varsigma)\varsigma \quad \text{for all } \varsigma \in \mathbb{R}. \tag{2.2}$$

For each fixed $q \in (p, p_s^*)$, there exists $C > 0$ such that

$$|F_2'(\varsigma)| \leq C|\varsigma|^{q-1} \quad \text{for all } \varsigma \in \mathbb{R}. \tag{2.3}$$

Note that $\Phi_\lambda \in C^1\left(W^{s,p}(\mathbb{R}^N), \mathbb{R}\right)$, Ψ is convex and lower semicontinuous in $W^{s,p}(\mathbb{R}^N)$, but Ψ is not a C^1-functional due to the unboundedness of $\mathbb{R}^N$.

3. The auxiliary problem

For each $j \in \{1, \ldots, k\}$, we fix a bounded open subset Ω_j' with smooth boundary such that

$$\overline{\Omega_j} \subset \Omega_j',$$

and

$$\overline{\Omega_j'} \cap \overline{\Omega_l'} = \emptyset \quad \text{for all } j \neq l.$$

From now on, we fix a non-empty subset $\Gamma \subset \{1, \ldots, k\}$ and $R > 0$ such that $\Omega_\Gamma' \subset B_R(0)$ and

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j, \quad \Omega_\Gamma' = \bigcup_{j\in\Gamma} \Omega_j'.$$

To prove our main theorem, we modify problem (1.1) and then consider the existence of solutions to the auxiliary problem.

By a simple observation, it is easy to verify that $\frac{F_2'(\varsigma)}{\varsigma^{p-1}}$ is nondecreasing for $\varsigma > 0$ and $\frac{F_2'(\varsigma)}{\varsigma^{p-1}}$ is strictly increasing for $\varsigma > \delta$,

$$\lim_{\varsigma \to +\infty} \frac{F_2'(\varsigma)}{\varsigma^{p-1}} = +\infty$$

and

$$F_2'(\varsigma) \geq 0 \quad \text{for } \varsigma > 0 \quad \text{and} \quad F_2'(\varsigma) > 0 \quad \text{for } \varsigma > \delta.$$

Moreover, $\frac{F_2'(\varsigma)}{\varsigma^{p-1}}$ is nonincreasing for $\varsigma < 0$ and $\frac{F_2'(\varsigma)}{\varsigma^{p-1}}$ is strictly decreasing for $\varsigma < -\delta$,

$$\lim_{\varsigma \to -\infty} \frac{F_2'(\varsigma)}{\varsigma^{p-1}} = +\infty,$$

and

$$F_2'(\varsigma) \leq 0 \quad \text{for } \varsigma < 0 \quad \text{and} \quad F_2'(\varsigma) < 0 \quad \text{for } \varsigma < -\delta.$$

Let $\ell > 0$ be small and $a_0 > 0$ such that

$$\max\left\{\frac{F_2'(a_0)}{a_0^{p-1}}, \frac{F_2'(-a_0)}{(-a_0)^{p-1}}\right\} = \ell.$$

It is clear that $a_0 > \delta$. We define

$$\tilde{F}_2'(\varsigma) = \begin{cases} \dfrac{F_2'(-a_0)}{(-a_0)^{p-1}}\varsigma^{p-1} & \text{if } \varsigma < -a_0, \\ F_2'(\varsigma) & \text{if } |\varsigma| \leq a_0, \\ \dfrac{F_2'(a_0)}{a_0^{p-1}}\varsigma^{p-1} & \text{if } \varsigma > a_0, \end{cases}$$

$$\tilde{F}_2'(\varsigma) \leq F_2'(\varsigma) \quad \text{for } \varsigma \geq 0, \quad \tilde{F}_2'(\varsigma) \geq F_2'(\varsigma) \quad \text{for } \varsigma \leq 0$$

and

$$G_2'(x,u) = \chi_\Gamma(x) F_2'(u) + (1 - \chi_\Gamma(x))\, \tilde{F}_2'(u),$$

where

$$\chi_\Gamma(x) := \begin{cases} 1, & x \in \Omega_\Gamma', \\ 0, & x \in B_R(0) \setminus \Omega_\Gamma'. \end{cases}$$

Then, we define the auxiliary problem given by

$$\begin{cases} (-\Delta)_p^s u + (\lambda V(x) + 1)\, |u|^{p-2}u = G_2'(x,u) - F_1'(u), & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0). \end{cases} \tag{3.1}$$

Remark 3.1. *Note that, if $u_{\lambda,R}$ is a nodal solution of (3.1) satisfying $|u_{\lambda,R}| \leq a_0$ for each $x \in B_R(0) \setminus \Omega_\Gamma'$, then $G_2'(x, u_{\lambda,R}) = F_2'(u_{\lambda,R})$ and consequently, $u_{\lambda,R}$ is also a nodal solution of*

$$\begin{cases} (-\Delta)_p^s u + \lambda V(x) |u|^{p-2}u = |u|^{p-2}u \log |u|^p & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0). \end{cases} \tag{3.2}$$

It is clear that weak solutions of (3.1) are nontrivial critical points of the following energy functional

$$\mathcal{I}_{\lambda,R}(u) := \frac{1}{p}\|u\|_{\lambda,R}^p + \int_{B_R(0)} F_1(u)\,\mathrm{d}x - \int_{B_R(0)} G_2(x,u)\,\mathrm{d}x,$$

in the sub-differential sense, and $G_2(x,t) = \int_0^t G_2'(x,\zeta)\,\mathrm{d}\zeta$ for all $(x,t) \in B_R(0) \times \mathbb{R}$. It is standard to verify that $\mathcal{I}_{\lambda,R} \in C^1(E_{\lambda,R}, \mathbb{R})$.

The next lemma implies that $\mathcal{I}_{\lambda,R}$ possesses the Mountain Pass geometry.

Lemma 3.2. *For all $\lambda > 0$, the functional $\mathcal{I}_{\lambda,R}$ satisfies the following conditions:*

(i) *There exist $\alpha, \rho > 0$ such that $\mathcal{I}_{\lambda,R}(u) \geq \rho$ with $\|u\|_{\lambda,R} = \alpha$;*
(ii) *There exists $e \in E_{\lambda,R}$ such that $\|u\|_{\lambda,R} > \alpha$ and $\mathcal{I}_{\lambda,R}(e) < 0$.*

Proof. First, note that

$$\mathcal{I}_{\lambda,R}(u) \geq \frac{1}{p}\|u\|^p_{\lambda,R} - \int_{B_R(0)} F_2(u)\,\mathrm{d}x,$$

which follows from (2.3) for $q \in (2, 2^*_s)$ such that

$$\mathcal{I}_{\lambda,R}(u) \geq \frac{1}{p}\|u\|^p_{\lambda,R} - C_1\|u\|^q_{\lambda,R}.$$

The claim follows if we choose ρ and $\|u\|_{\lambda,R} = \alpha$ small enough.

On the other hand, fixing $\varphi \in C_0^\infty(\Omega_\Gamma) \setminus \{0\}$, by (2.2), we have

$$\begin{aligned}\mathcal{I}_{\lambda,R}(\tau\varphi) &= \frac{\tau^p}{p}\|\varphi\|^p_{\lambda,R} - \frac{1}{p}\int_{B_R(0)} \tau^p\varphi^p\log(|\tau\varphi|^p)\,\mathrm{d}x \\ &\leq \tau^p\left(\mathcal{I}_{\lambda,R}(\varphi) - \log(\tau)\int_{\Omega'_\Gamma}\varphi^p\,\mathrm{d}x\right).\end{aligned}$$

As $\tau \to +\infty$, then

$$\mathcal{I}_{\lambda,R}(\tau\varphi) \to -\infty,$$

and the proof of the lemma is complete. □

By Lemma 3.2 and Willem [25], there exists a (PS)-sequence $\{u_n\}_{n\in\mathbb{N}} \subset E_{\lambda,R}$ of $\mathcal{I}_{\lambda,R}$ at the level $c_{\lambda,R} > 0$, where

$$c_{\lambda,R} = \inf_{\gamma\in\Gamma_{\lambda,R}}\max_{t\in[0,1]}\mathcal{I}_{\lambda,R}(\gamma(t)),$$

and $\Gamma_{\lambda,R} := \left\{\gamma \in C^1([0,1], E_{\lambda,R})\colon \gamma(0) = 0, \mathcal{I}_{\lambda,R}(\gamma(1)) < 0\right\}$. Moreover, by Lemma 3.2, we have

$$c_{\lambda,R} \geq \alpha > 0 \quad \text{for all } \lambda > 0 \text{ and } R > 0 \text{ large enough.}$$

Now, we will prove some results that will be useful in the proof of Theorem 1.1.

Lemma 3.3. *For any $\lambda > 0$, all* (PS)*-sequences of $\mathcal{I}_{\lambda,R}$ are bounded in $E_{\lambda,R}$.*

Proof. Since $\{u_n\}_{n\in\mathbb{N}} \subset E_{\lambda,R}$ is a $(\mathrm{PS})_{c_{\lambda,R}}$-sequence, one gets

$$p\mathcal{I}_{\lambda,R}(u_n) - \mathcal{I}'_{\lambda,R}(u_n)u_n = pc_{\lambda,R} + 1 + o_n(1)\,\|u_n\|_{\lambda,R}\,, \tag{3.3}$$

for n large enough. Note that,

$$\int_{B_R(0)}[(pF_1(u_n) - F_1'(u_n)u_n) + (F_2'(u_n)u_n - pF_2(u_n))]\,\mathrm{d}x = \int_{B_R(0)}|u_n|^p\,\mathrm{d}x.$$

From this, one has

$$\begin{aligned}&p\mathcal{I}_{\lambda,R}(u_n) - \mathcal{I}'_{\lambda,R}(u_n)u_n \\ &= \int_{B_R(0)}[(pF_1(u_n) - F_1'(u_n)u_n) + (G_2'(x,u_n)u_n - pG_2(x,u_n))]\,\mathrm{d}x \\ &= \int_{B_R(0)}|u_n|^p\,\mathrm{d}x + \int_{B_R(0)}(pF_2(u_n) - F_2'(u_n)u_n)\,\mathrm{d}x \\ &\quad + \int_{B_R(0)}(G_2'(x,u_n)u_n - pG_2(x,u_n))\,\mathrm{d}x \\ &= \int_{\Omega'_\Gamma}|u_n|^p\,\mathrm{d}x + \int_{B_R(0)\setminus\Omega'_\Gamma\cap[|u_n|>a_0]}(|u_n|^p + pF_2(u_n) - F_2'(u_n)u_n)\,\mathrm{d}x\end{aligned}$$

$$+\int_{B_R(0)\setminus\Omega'_\Gamma\cap[|u_n|>a_0]} \left(G_2'(x,u_n)u_n - pG_2(x,u_n)\right) \mathrm{d}x.$$

Using the fact

$$|t|^p + [pF_2(t) - F_2'(t)t + G_2'(x,t)t - pG_2(x,t)] \geq 0, \quad t \in C,\ x \in \mathbb{R}^N,$$

one gets

$$p\mathcal{I}_{\lambda,R}(u_n) - \mathcal{I}'_{\lambda,R}(u_n)u_n \geq \int_{\Omega'_\Gamma} |u_n|^p \,\mathrm{d}x.$$

So (3.3) implies that

$$pc_{\lambda,R} + 1 + o_n(1)\,\|u_n\|_{\lambda,R} \geq \int_{\Omega'_\Gamma} |u_n|^p \,\mathrm{d}x. \tag{3.4}$$

Let us employ the following logarithmic Sobolev inequality found in Alves-Ambrosio [1],

$$\int_\Lambda \frac{|u|^p}{\|u\|^p_{L^p(\Omega'_\Gamma)}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\Omega'_\Gamma)}}\right) \mathrm{d}x \leq K \log\left(\frac{\|u\|^p_{L^{p_s^*}(\Omega'_\Gamma)}}{\|u\|^p_{L^p(\Omega'_\Gamma)}}\right),$$

for all $u \in L^p(\Omega'_\Gamma) \cap L^{p_s^*}(\Omega'_\Gamma)$. Now, using $\|u_n\|_{L^{p_s^*}(\Omega'_\Gamma)} \leq \left(S_*^{-1}\right)^{\frac{1}{p}} \|u_n\|_{\lambda,R}$, we find

$$\begin{aligned}
\int_{\Omega'_\Gamma} (u_n)^p \log(u_n)^p \,\mathrm{d}x &\leq \left(\|u_n\|^p_{L^p(\Omega'_\Gamma)} - K\|u_n\|^p_{L^p(\Omega'_\Gamma)}\right) \log\left(\|u_n\|^p_{L^p(\Omega'_\Gamma)}\right)\\
&\quad + K\|u_n\|^p_{L^p(\Omega'_\Gamma)} \log\left(\|u_n\|^p_{L^{p^p}(\Omega'_\Gamma)}\right)\\
&\leq C\|u_n\|^p_{L^p(\Omega'_\Gamma)} \left|\log\left(\|u_n\|^p_{L^p(\Omega'_\Gamma)}\right)\right|\\
&\quad + C\|u_n\|_{\lambda,R} \left|\log\left(C\|u_n\|_{\lambda,R}\right)\right| + C + \|u_n\|_{\lambda,R},
\end{aligned}$$

for all $n \in \mathbb{N}$ and for some $C > 0$. Observe that, for all $r \in (0,1)$, there exists $A > 0$ such that

$$|t \log t| \leq A(1+t)^{r+1} \quad \text{for all } t \geq 0. \tag{3.5}$$

Then, employing (3.5)

$$\begin{aligned}
\|u_n\|^p_{L^p(\Omega'_\Gamma)} \left|\log\left(\|u_n\|^p_{L^p(\Omega'_\Gamma)}\right)\right| &= \left|\|u_n\|^p_{L^p(\Omega'_\Gamma)} \log\left(\|u_n\|^p_{L^p(\Omega'_\Gamma)}\right)\right|\\
&\leq \frac{A}{p}\left(1 + \|u_n\|^p_{L^p(\Omega'_\Gamma)}\right)^{r+1} \quad \text{for all } n \in \mathbb{N},
\end{aligned}$$

which combined with (3.4) leads to

$$\|u_n\|^p_{L^p(\Omega'_\Gamma)} \left|\log\left(\|u_n\|^p_{L^p(\Omega'_\Gamma)}\right)\right| \leq C\left(1 + \|u_n\|_{\lambda,R}\right)^{r+1} \quad \text{for all } n \in \mathbb{N}.$$

A similar argument shows that

$$\|u_n\|^p_{L^p(\Omega'_\Gamma)} \left|\log\left(\|u_n\|^p_{L^p(\Omega'_\Gamma)}\right)\right| \leq C\left(1 + \|u_n\|_{\lambda,R}\right)^{r+1} \quad \text{for all } n \in \mathbb{N},$$

and

$$\|u_n\|_{\lambda,R} \left|\log\left(C\|u_n\|_{\lambda,R}\right)\right| \leq C\left(1 + \|u_n\|_{\lambda,R}\right)^{r+1} \quad \text{for all } n \in \mathbb{N},$$

for some generic constant $C > 0$. The above analysis ensures that

$$\int_{\Omega'_\Gamma} (u_n)^p \log(u_n)^p \,\mathrm{d}x \leq C\left(1 + \|u_n\|_{\lambda,R}\right)^{r+1} \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

On the other hand,

$$c_{\lambda,R} + o_n(1) = \mathcal{I}_{\lambda,R}(u_n) \geq \frac{1}{p}\|u_n\|_{\lambda,R}^p - \frac{1}{p}\int_{\Omega'_\Gamma} (u_n)^p \log(u_n)^p \,\mathrm{d}x - \int_{B_R(0)\setminus\Omega'_\Gamma} G_2(x,u_n)\,\mathrm{d}x,$$

and recalling that

$$G_2(x,t) \leq \frac{\ell}{p}t^p \quad \text{for all } (x,t) \in B_R(0)\setminus\Omega'_\Gamma \times \mathbb{R},$$

we deduce that

$$c_{\lambda,R} + o_n(1) = \mathcal{I}_{\lambda,R}(u_n) \geq C\|u_n\|_{\lambda,R}^p - \frac{1}{p}\int_{\Omega'_\Gamma} (u_n)^p \log(u_n)^p \,\mathrm{d}x.$$

This fact together with (3.6) yields

$$\begin{aligned}\|u_n\|_{\lambda,R}^p &\leq \frac{1}{p}\int_{\Omega'_\Gamma} (u_n)^p \log(u_n)^p \,\mathrm{d}x + c_{\lambda,R} + o_n(1)\\ &\leq C\left(1+\|u_n\|_{\lambda,R}\right)^{r+1} + C + C\|u_n\|_{\lambda,R} + o_n(1),\end{aligned}$$

showing the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $E_{\lambda,R}$. □

Our next lemma shows that $\mathcal{I}_{\lambda,R}$ verifies the (PS) condition.

Lemma 3.4. *The functional $\mathcal{I}_{\lambda,R}$ verifies the* (PS) *condition on $E_{\lambda,R}$ at any level $c_{\lambda,R} \in \mathbb{R}$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS)-sequence for $\mathcal{I}_{\lambda,R}$ at the level $c_{\lambda,R}$, i.e.,

$$\mathcal{I}_{\lambda,R}(u_n) \to c_{\lambda,R} \quad \text{and} \quad \mathcal{I}'_{\lambda,R}(u_n) \to 0.$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $E_{\lambda,R}$, see Lemma 3.3, up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u & \text{in } E_{\lambda,R},\\ u_n \to u & \text{in } L^r(B_R(0)), \text{ for all } r \in [1,p_s^*)\\ u_n(x) \to u(x) & \text{a.e. in } B_R(0).\end{cases}$$

For all $\tau \in \mathbb{R}$ and fixed $q \in (p,p_s^*)$, there exists $C>0$ such that

$$|G_2'(x,\tau)| \leq \theta|\tau| + C|\tau|^{q-1},$$

and

$$|F_1'(\tau)| \leq C(1+|\tau|^q).$$

Hence, by the Sobolev embeddings, one has

$$\begin{aligned}\int_{B_R(0)} G_2'(x,u_n)u_n\,\mathrm{d}x &\to \int_{B_R(0)} G_2'(x,u)u\,\mathrm{d}x,\\ \int_{B_R(0)} F_1'(x,u_n)u_n\,\mathrm{d}x &\to \int_{B_R(0)} F_1'(x,u)u\,\mathrm{d}x,\\ \int_{B_R(0)} G_2'(x,u_n)\omega\,\mathrm{d}x &\to \int_{B_R(0)} G_2'(x,u)\omega\,\mathrm{d}x,\\ \int_{B_R(0)} F_1'(x,u_n)\omega\,\mathrm{d}x &\to \int_{B_R(0)} F_1'(x,u)\omega\,\mathrm{d}x,\end{aligned}$$

for all $\omega \in E_{\lambda,R}$.

Since $\mathcal{I}'_{\lambda,R}(u_n)u_n = \mathcal{I}'_{\lambda,R}(u_n)u = o_n(1)$, we get

$$\begin{aligned}\|u_n-u\|_{\lambda,R}^p = &\int_{B_R(0)} \left(G_2'(x,u_n) - G_2'(x,u)\right)(u_n-u)\,\mathrm{d}x\\ &- \int_{B_R(0)} \left(F_1'(x,u_n) - F_1'(x,u)\right)(u_n-u)\,\mathrm{d}x + o_n(1) = o_n(1),\end{aligned}$$

which shows the desired result. □

3.1. **The** $(\mathrm{PS})_{\infty,R}$ **condition.** In the sequel, for each $R>0$, we study the behavior of a $(\mathrm{PS})_{\infty,R}$-sequence for $\mathcal{I}_{\lambda,R}$, i.e., a sequence $\{u_n\}_{n\in\mathbb{N}}\subset W_0^{s,p}(B_R(0))$ satisfying

$$\begin{aligned}&u_n\in E_{\lambda_n,R}\text{ and }\lambda_n\to\infty,\\&\mathcal{I}_{\lambda_n,R}(u_n)\to c,\ \left\|\mathcal{I}'_{\lambda_n,R}(u_n)\right\|\to 0.\end{aligned}$$

Lemma 3.5. *Let $\{u_n\}_{n\in\mathbb{N}}\subset W_0^{s,p}(B_R(0))$ be a $(\mathrm{PS})_{\infty,R}$ sequence. Then, for some subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, there exists $u\in W_0^{s,p}(B_R(0))$ such that*

$$u_n\rightharpoonup u\quad\text{in } W_0^{s,p}(B_R(0)).$$

Moreover, the following hold:

(i) *u_n converges to u in the strong sense, i.e.,*

$$\|u_n-u\|_{\lambda_n,R}\to 0.$$

Hence,

$$u_n\to u\quad\text{in } W_0^{s,p}(B_R(0)).$$

(ii) *$u\equiv 0$ in $B_R(0)\setminus\Omega_\Gamma$ and u is a solution of*

$$\begin{cases}(-\Delta)_p^s u=|u|^{p-2}u\log|u|^p & \text{in }\Omega_\Gamma,\\ u=0 & \text{on }\partial\Omega_\Gamma.\end{cases}\tag{3.7}$$

(iii) *u_n also satisfies*

$$\begin{aligned}&\lambda_n\int_{B_R(0)}V(x)|u_n|^p\,\mathrm{d}x\to 0,\\&\|u_n\|^p_{\lambda_n,B_R(0)\setminus\Omega_\Gamma}\to 0,\\&\|u_n\|^p_{\lambda_n,\Omega'_j}\to\int_{\Omega_j}\int_{\Omega_j}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_j}|u|^p\,\mathrm{d}x\quad\text{for all } j\in\Gamma.\end{aligned}$$

Proof. By using Lemma 3.3, there exists $K>0$ such that

$$\|u_n\|^p_{\lambda_n,R}\le K\quad\text{for all } n\in\mathbb{N}.$$

Thus $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{s,p}(B_R(0))$ and we can assume that for some $u\in W_0^{s,p}(B_R(0))$,

$$\begin{aligned}&u_n\rightharpoonup u\quad\text{weakly in } W_0^{s,p}(B_R(0)),\\&u_n(x)\to u(x)\quad\text{a.e. in } B_R(0).\end{aligned}$$

Fixing $C_m=\left\{x\in B_R(0)\colon V(x)\ge\frac{1}{m}\right\}$, one has

$$\int_{C_m}|u_n|^p\,\mathrm{d}x\le\frac{m}{\lambda_n}\int_{B_R(0)}\lambda_nV(x)|u_n|^p\,\mathrm{d}x,$$

that is,

$$\int_{C_m}|u_n|^p\,\mathrm{d}x\le\frac{m}{\lambda_n}\|u_n\|^p_{\lambda_n,R},$$

which yields from Fatou's lemma that

$$\int_{C_m}|u|^p\,\mathrm{d}x=0\quad\text{for all } m\in\mathbb{N}.$$

Then $u(x)=0$ on $\bigcup_{m=1}^{+\infty}C_m=B_R(0)\setminus\overline{\Omega}$, and so, $u\big|_{\Omega_j}\in W_0^{s,p}(\Omega_j)$ for $j\in\{1,\dots,k\}$. From this, we will prove (i)–(iii).

(i) Since $u=0$ in $B_R(0)\backslash\overline{\Omega}$ and $\mathcal{I}'_{\lambda_n,R}(u_n)u_n=\mathcal{I}'_{\lambda_n,R}(u_n)u=o_n(1)$, similar to the proof of Lemma 3.4, it holds

$$\|u_n-u\|_{\lambda_n,R}\to 0,$$

which implies that $u_n\to u$ in $W_0^{s,p}(B_R(0))$.

(ii) Since $u\in W_0^{s,p}(B_R(0))$ and $u=0$ in $B_R(0)\setminus\overline{\Omega}$, we deduce $u\in W_0^{s,p}(\Omega)$, or equivalently $u\big|_{\Omega_j}\in W_0^{s,p}(\Omega_j)$ for $j=1,\dots,k$. Moreover, $u_n\to u$ in $W_0^{s,p}(B_R(0))$ combined with $\mathcal{I}'_{\lambda_n,R}(u_n)\varphi\to 0$ as $n\to+\infty$ for each $\varphi\in C_0^\infty(\Omega_\Gamma)$ implies that

$$\begin{aligned}&\int_{\Omega_\Gamma}\int_{\Omega_\Gamma}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\\&\quad+\int_{\Omega_\Gamma}|u|^{p-2}u\varphi\,\mathrm{d}x+\int_{\Omega_\Gamma}F_1'(u)\varphi\,\mathrm{d}x-\int_{\Omega_\Gamma}F_2'(u)\,\varphi\,\mathrm{d}x=0,\end{aligned}$$

from which it follows that $u\big|_{\Omega_\Gamma}$ is a solution for (3.7). On the other hand, for each $j\in\{1,2,\dots,k\}\setminus\Gamma$, we have that

$$\int_{\Omega_j}\int_{\Omega_j}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_j}|u|^p\,\mathrm{d}x+\int_{\Omega_j}F_1'(u)u\,\mathrm{d}x-\int_{\Omega_j}\tilde{F}_2'(u)\,u\,\mathrm{d}x=0.$$

By the fact that $F_1'(\varsigma)\varsigma\geq 0$ and $\tilde{F}_2'(\varsigma)\varsigma\leq\ell|\varsigma|^p$ for all $\varsigma\in\mathbb{R}$, we derive that

$$\int_{\Omega_j}\int_{\Omega_j}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_j}|u|^p\,\mathrm{d}x\leq\int_{\Omega_j}\tilde{F}_2'(u)\,u\,\mathrm{d}x\leq\theta\int_{\Omega_j}|u|^p\,\mathrm{d}x.$$

Since $\ell<1$, $u=0$ in Ω_j for $j\in\{1,2,\dots,k\}\setminus\Gamma$, which shows (ii).

(iii) Note that, from (i),

$$\int_{B_R(0)}\lambda_n V(x)\,|u_n|^p\,\mathrm{d}x=\int_{B_R(0)}\lambda_n V(x)\,|u_n-u|^p\,\mathrm{d}x\leq C\,\|u_n-u\|^p_{\lambda_n,R},$$

which shows that

$$\int_{B_R(0)}\lambda_n V(x)\,|u_n|^p\,\mathrm{d}x\to 0\quad\text{as } n\to+\infty.$$

Moreover, from (i) and (ii), it is easy to check that

$$\|u_n\|^p_{\lambda_n,B_R(0)\backslash\Omega_\Gamma}\to 0,$$

and

$$\|u_n\|^p_{\lambda_n,\Omega_j'}\to\int_{\Omega_j}\int_{\Omega_j}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega_j}|u|^p\,\mathrm{d}x\qquad\text{for all } j\in\Gamma.$$

This completes the proof. □

With a few modifications to the arguments in the proof of Lemma 3.5 and using Lemma 3.3, we also have the following result.

Lemma 3.6. *Let $\{u_n\}_{n\in\mathbb{N}}\subset E_{\lambda_n,R_n}$ be a $(\mathrm{PS})_{\infty,R_n}$ sequence with $R_n\to+\infty$, i.e.,*

$$u_n\in E_{\lambda_n,R_n}\ \text{and}\ \lambda_n\to\infty,\quad\mathcal{I}_{\lambda_n,R_n}(u_n)\to c,\quad\left\|\mathcal{I}'_{\lambda_n,R_n}(u_n)\right\|\to 0.$$

Then, for some subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, there exists $u\in W^{s,p}(\mathbb{R}^N)$ such that

$$u_n\rightharpoonup u\quad\text{in } W^{s,p}(\mathbb{R}^N).$$

Moreover, the following hold:

(i) $\|u_n-u\|_{\lambda_n,R_n}\to 0$, *and so,*

$$u_n\to u\quad\text{in } W^{s,p}(\mathbb{R}^N).$$

(ii) $u \equiv 0$ *in* $\mathbb{R}^N \setminus \Omega_\Gamma$ *and* u *is a solution of*

$$\begin{cases} (-\Delta)_p^s u = |u|^{p-2}u \log|u|^p & \text{in } \Omega_\Gamma, \\ u = 0 & \text{on } \partial\Omega_\Gamma. \end{cases}$$

(iii) u_n *also satisfies*

$$\begin{aligned}
&\lambda_n \int_{B_{R_n}(0)} V(x)\,|u_n|^p \,\mathrm{d}x \to 0,\\
&\|u_n\|^p_{\lambda_n, B_{R_n}(0)\setminus\Omega_\Gamma} \to 0,\\
&\|u_n\|^p_{\lambda_n,\Omega_j'} \to \int_{\Omega_j}\int_{\Omega_j} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_j} |u|^p\,\mathrm{d}x \quad \text{for all } j \in \Gamma.
\end{aligned}$$

Proof. First of all, the boundedness of $\{\mathcal{I}_{\lambda_n,R_n}(u_n)\}_{n\in\mathbb{N}}$ shows that there exists $K > 0$ such that

$$\|u_n\|^p_{\lambda_n,R_n} \le K \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,p}(\mathbb{R}^N)$ and we can assume that for some $u \in W^{s,p}(\mathbb{R}^N)$,

$$\begin{aligned}
u_n &\rightharpoonup u \quad \text{in } W^{s,p}(\mathbb{R}^N),\\
u_n(x) &\to u(x) \quad \text{a.e. in } \mathbb{R}^N,
\end{aligned}$$

and $u(x) = 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$.

(i) For any $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} \left(\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y + (\lambda_n V(x)+1)\,|u_n|^p \right) \mathrm{d}x < \zeta.$$

Let $0 < R < R_n$ and $\psi = \psi_R \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $\psi \equiv 0$ if $x \in B_{\frac{R}{2}}(0)$, $\psi \equiv 1$ if $x \notin B_R(0)$ with $0 \le \psi(x) \le 1$, and $\|\nabla\psi(x)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{R}$, where C is a constant independent of R. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded, the sequence $\{\psi u_n\}_{n\in\mathbb{N}}$ is also bounded. This shows that $\mathcal{I}'_{\lambda_n,R_n}(u_n)(\psi u_n) = o_n(1)$, namely,

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\psi(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} (\lambda_n V(x)+1)\,|u_n|^p\psi(x)\,\mathrm{d}x\\
&= \int_{\Omega_\Gamma'} F_2'(u_n)u_n\psi(x)\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus\Omega_\Gamma'} \tilde{F}_2'(u_n)u_n\psi(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} F_1'(u_n)u_n\psi(x)\,\mathrm{d}x\\
&\quad - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}u_n(y)\,\mathrm{d}x\,\mathrm{d}y + o_n(1).
\end{aligned}$$

Take $R > 0$ such that $\Omega_\Gamma' \subset B_{\frac{R}{2}}(0)$. Then, by (2.2) and the definitions of $\tilde{F}_2'$, we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\psi(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} (\lambda_n V(x)+1)\,|u_n|^p\psi(x)\,\mathrm{d}x\\
&\le \ell \int_{\mathbb{R}^N} |u_n|^p\psi(x)\,\mathrm{d}x - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}u_n(y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + o_n(1).
\end{aligned}$$

By Hölder's inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, we arrive at

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}u_n(y)\,\mathrm{d}x\,\mathrm{d}y \right|\\
&\le \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{p-1}{p}} \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\,|u_n(y)|^p\,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq C\left(\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\left|u_n(y)\right|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \leq \frac{C}{R^s},$$

where we have used that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\left|u_n(y)\right|^p \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^N}\left|u_n(y)\right|^p\left(\int_{|x-y|>R} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x+\int_{|x-y| \leq R} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\right) \mathrm{d}y \\
&\leq C \int_{\mathbb{R}^N}\left|u_n(y)\right|^p \,\mathrm{d}y\left(\int_R^{\infty} \frac{1}{r^{sp+1}} \,\mathrm{d}r+R^{-p} \int_0^R \frac{1}{r^{sp-p+1}} \,\mathrm{d}r\right) \\
&\leq \frac{C}{R^{sp}}.
\end{aligned}$$

Now, fixing $\zeta>0$ and passing to the limit in the last inequality, it follows that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \backslash B_R}\left(\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}y+\left(\lambda_n V(x)+1\right)\left|u_n\right|^p\right) \mathrm{d}x \leq \frac{C}{R^s}<\zeta, \tag{3.8}$$

whenever $R>0$ is sufficiently large.

Since G_2' has a subcritical growth, the above estimate (3.8) ensures that

$$\begin{aligned}
\int_{\mathbb{R}^N} G_2'\left(x, u_n\right) w \,\mathrm{d}x &\rightarrow \int_{\mathbb{R}^N} G_2'(x, u)\, w \,\mathrm{d}x \quad \text{for all } w \in C_0^{\infty}(\mathbb{R}^N), \\
\int_{\mathbb{R}^N} G_2'\left(x, u_n\right) u_n \,\mathrm{d}x &\rightarrow \int_{\mathbb{R}^N} G_2'(x, u)\, u \,\mathrm{d}x, \\
\int_{\mathbb{R}^N} G_2\left(x, u_n\right) \mathrm{d}x &\rightarrow \int_{\mathbb{R}^N} G_2(x, u) \,\mathrm{d}x.
\end{aligned}$$

Now, recalling that $\lim_{n\to\infty} \mathcal{I}_{\lambda_n,R_n}'(u_n)w=0$ for all $w \in C_0^{\infty}\left(\mathbb{R}^N\right)$ and $\|u_n\|_{\lambda_n,R_n}^p \leq K$ for all $n \in \mathbb{N}$, we deduce that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{R}^N}|u|^{p-2} u w \,\mathrm{d}x \\
&+\int_{\mathbb{R}^N} F_1'(u) w \,\mathrm{d}x-\int_{\mathbb{R}^N} G_2'(x, u) w \,\mathrm{d}x=0,
\end{aligned}$$

and so,

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{R}^N}|u|^p \,\mathrm{d}x+\int_{\mathbb{R}^N} F_1'(u) u \,\mathrm{d}x-\int_{\mathbb{R}^N} G_2'(x, u) u \,\mathrm{d}x=0.$$

This together with the equality $\lim_{n\to\infty} \mathcal{I}_{\lambda_n,R_n}'(u_n)u_n=0$, i.e.,

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{R}^N}\left(\lambda_n V(x)+1\right)\left|u_n\right|^p \mathrm{d}x+\int_{\mathbb{R}^N} F_1'(u_n) u_n \,\mathrm{d}x \\
&=\int_{\mathbb{R}^N} G_2'(x, u_n) u_n \,\mathrm{d}x+o_n(1),
\end{aligned}$$

leads to

$$\begin{aligned}
&\lim_{n \rightarrow+\infty}\left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{R}^N}\left(\lambda_n V(x)+1\right)\left|u_n\right|^p \mathrm{d}x+\int_{\mathbb{R}^N} F_1'(u_n) u_n \,\mathrm{d}x\right) \\
&=\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{R}^N}|u|^p \,\mathrm{d}x+\int_{\mathbb{R}^N} F_1'(u) u \,\mathrm{d}x,
\end{aligned}$$

from which it follows that for some subsequence,

$$u_n \rightarrow u \quad \text{in } W^{s,p}(\mathbb{R}^N), \quad \lambda_n \int_{\mathbb{R}^N} V(x)\left|u_n\right|^p \,\mathrm{d}x \rightarrow 0,$$

and

$$F_1'(u_n)u_n \to F_1'(u)u \quad \text{in } L^1(\mathbb{R}^N).$$

Since F_1 is convex, even and $F(0)=0$, we know that $F_1'(\tau)\tau \geq F_1(\tau) \geq 0$ for all $\tau \in \mathbb{R}$. Thus, the last limit together with Lebesgue's dominated convergence theorem yields

$$F_1(u_n) \to F_1(u) \quad \text{in } L^1(\mathbb{R}^N).$$

Since

$$\|u_n - u\|_{\lambda_n,R_n}^p = \iint_{\mathbb{R}^{2N}} \frac{|(u_n(x)-u(x))-(u_n(y)-u(y))|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} (\lambda_n V(x)+1)\,|u_n-u|^p\,\mathrm{d}x,$$

it follows that

$$\|u_n - u\|_{\lambda_n,R_n}^p \to 0,$$

which implies (i). The proofs of (ii) and (iii) are similar to that of Lemma 3.5 and so we omit it. ☐

3.2. **The L^∞-boundedness of solutions to (3.1).** Next, we investigate the boundedness outside Ω_Γ' for the solutions of (3.1). The following lemma is crucial to show that the solutions of the auxiliary problem (3.1) are the solutions of the original problem (1.1). Furthermore, we define

$$|u|_{q,R} = \left(\int_{B_R(0)} u^q\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Lemma 3.7. *Let $\{u_{\lambda,R}\}$ be a family of nodal solutions of (3.1) such that $\{\mathcal{I}_{\lambda,R}(u_{\lambda,R})\}$ is bounded in $\mathbb{R}$ for any $\lambda>0$ and $R>0$ large enough. Then, there exists $K>0$ that does not depend on $\lambda>0$ and $R^*>0$ such that*

$$|u_{\lambda,R}|_{\infty,R} \leq K \quad \text{for all } \lambda>0 \text{ and } R \geq R^*.$$

Proof. For each $L>0$, let $u_L^+ := \min\{u_{\lambda,R}^+, L\}$ and define the function

$$\mathcal{E}(u_{\lambda,R}) := \mathcal{E}_{L,\sigma}(u_{\lambda,R}) = u_{\lambda,R}(u_L^+)^{p(\sigma-1)},$$

with $\sigma>1$ to be determined later. Note that $\mathcal{E}$ is increasing, thus we have

$$(a-b)(\mathcal{E}(a)-\mathcal{E}(b)) \geq 0 \quad \text{for any } a,b \in \mathbb{R}.$$

Consider the functions

$$\mathcal{Q}(t) := \frac{|t|^p}{p} \quad \text{and} \quad \mathcal{L}(t) := \int_0^t (\mathcal{E}'(\tau))^{\frac{1}{p}}\,\mathrm{d}\tau,$$

and note that

$$\mathcal{L}(u_{\lambda,R}) \geq \frac{C}{\sigma} u_{\lambda,R}(u_L^+)^{\sigma-1}.$$

Hence, from Lemma 2.1, we obtain

$$[\mathcal{L}(u_{\lambda,R})]^p \geq S_*\,|\mathcal{L}(u_{\lambda,R})|_{p_s^*,R}^p \geq S_* \frac{1}{\sigma^p} |u_{\lambda,R}(u_L^+)^{\sigma-1}|_{p_s^*,R}^p. \tag{3.9}$$

In addition, for any $a,b \in \mathbb{R}$, it holds

$$\mathcal{Q}'(a-b)(\mathcal{E}(a)-\mathcal{E}(b)) \geq |\mathcal{L}(a)-\mathcal{L}(b)|^p.$$

In fact, suppose that $a>b$, it follows from Jensen's inequality that

$$\begin{aligned}\mathcal{Q}'(a-b)(\mathcal{E}(a)-\mathcal{E}(b)) &= (a-b)(\mathcal{E}(a)-\mathcal{E}(b)) = (a-b)\int_b^a \mathcal{E}'(\tau)\,\mathrm{d}\tau\\ &= (a-b)\int_b^a (\mathcal{L}'(\tau))^p\,\mathrm{d}\tau \geq \left(\int_b^a \mathcal{L}'(\tau)\,\mathrm{d}\tau\right)^p = (\mathcal{L}(a)-\mathcal{L}(b))^p.\end{aligned}$$

A similar argument holds if $a \leq b$. Thus, we infer that

$$|\mathcal{L}(u_{\lambda,R})(x) - \mathcal{L}(u_{\lambda,R})(y)|^p$$
$$\leq |u_{\lambda,R}(x) - u_{\lambda,R}(y)|^{p-2}\,(u_{\lambda,R}(x) - u_{\lambda,R}(y))\left(u_{\lambda,R}(x)(u_L^+)^{p(\sigma-1)}(x) - u_{\lambda,R}(y)(u_L^+)^{p(\sigma-1)}(y)\right).$$

Using $\mathcal{E}(u_{\lambda,R})$ as test function in (3.1), in view of the above inequality, we get that

$$\begin{aligned}
&[\mathcal{L}(u_{\lambda,R})]^p + \int_{B_R(0)} (\lambda V(x)+1)\, u_{\lambda,R}^p (u_L^+)^{p(\sigma-1)}\,\mathrm{d}x + \int_{B_R(0)} F_1'(u_{\lambda,R}) u_{\lambda,R}(u_L^+)^{p(\sigma-1)}\,\mathrm{d}x\\
&\leq \iint_{B_R(0)\times B_R(0)} \frac{|u_{\lambda,R}(x) - u_{\lambda,R}(y)|^{p-2}\,(u_{\lambda,R}(x) - u_{\lambda,R}(y))}{|x-y|^{N+sp}}\\
&\quad\times\left(u_{\lambda,R}(x)(u_L^+)^{p(\sigma-1)}(x) - u_{\lambda,R}(y)(u_L^+)^{p(\sigma-1)}(y)\right)\mathrm{d}x\,\mathrm{d}y\\
&\quad+ \int_{B_R(0)} (\lambda V(x)+1)\, u_{\lambda,R}^p (u_L^+)^{p(\sigma-1)}\,\mathrm{d}x + \int_{B_R(0)} F_1'(u_{\lambda,R}) u_{\lambda,R}(u_L^+)^{p(\sigma-1)}\,\mathrm{d}x\\
&\leq \int_{B_R(0)} G_2'(x,u) u_{\lambda,R}(u_L^+)^{p(\sigma-1)}\,\mathrm{d}x.
\end{aligned}$$

By the definition of G_2', for fixed $q \in (p, p_s^*)$, there exists $C > 0$ such that

$$0 \leq G_2'(x,\tau) \leq \theta\tau + C\tau^{q-1} \quad \text{for } (x,\tau) \in B_R(0)\times[0,\infty).$$

The above estimates and (3.9) provide

$$|u_{\lambda,R}^+(u_L^+)^{\sigma-1}|_{p_s^*,R}^p \leq \sigma^p S_*^{-1}\left[\mathcal{L}(u_{\lambda,R}^+)\right]^p \leq C\sigma^p \int_{B_R(0)} (u_{\lambda,R}^+)^q (u_L^+)^{p(\sigma-1)}\,\mathrm{d}x. \tag{3.10}$$

Since

$$(u_{\lambda,R}^+)^q (u_L^+)^{p(\sigma-1)} = (u_{\lambda,R}^+)^{q-p}(u_{\lambda,R}^+(u_L^+)^{\sigma-1})^p,$$

we can use (3.10) and Hölder's inequality to deduce that

$$|u_{\lambda,R}^+(u_L^+)^{\sigma-1}|_{p_s^*,R}^p \leq C\sigma^p |u_{\lambda,R}^+|_{p_s^*,R}^{q-p} |u_{\lambda,R}^+(u_L^+)^{\sigma-1}|_{\alpha_s^*,R}^p,$$

where

$$\alpha_s^* = \frac{pp_s^*}{p_s^* - (q-p)} \in (p, p_s^*).$$

Since $\{u_{\lambda,R}\}$ is bounded, we conclude that

$$|u_{\lambda,R}^+(u_L^+)^{\sigma-1}|_{p_s^*,R}^p \leq C\sigma^p |u_{\lambda,R}^+(u_L^+)^{\sigma-1}|_{\alpha_s^*,R}^p.$$

Note that, if $u_{\lambda,R} \in L^{\sigma\alpha_s^*}(B_R(0))$, using the fact that $u_L^+ \leq u_{\lambda,R}^+$, then

$$|u_{\lambda,R}^+(u_L^+)^{\sigma-1}|_{p_s^*,R}^p \leq C\sigma^p |u_{\lambda,R}^+|_{\sigma\alpha_s^*,R}^{p\sigma} < \infty,$$

which together with Faton's lemma implies

$$|u_{\lambda,R}^+|_{\sigma p_s^*,R}^{p\sigma} \leq C\sigma^p |u_{\lambda,R}^+|_{\sigma\alpha_s^*,R}^{p\sigma},$$

as $L \to \infty$. Now, taking $\sigma = p_s^*/\alpha_s^* > 0$, we have

$$|u_{\lambda,R}^+|_{\sigma p_s^*,R}^{p\sigma} \leq C\sigma^p |u_{\lambda,R}^+|_{p_s^*,R}^{p\sigma},$$

and replacing σ by σ^j, $j \in \mathbb{N}$, in the above inequality, we obtain that

$$|u_{\lambda,R}^+|_{\sigma^j p_s^*,R}^{p\sigma^j} \leq C(\sigma^j)^p |u_{\lambda,R}^+|_{p_s^*,R}^{p\sigma^j}.$$

Then, by an argument of induction, we may verify that

$$|u_{\lambda,R}^+|_{p_s^*\sigma^j,R} \leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}+\cdots+\frac{j}{\sigma^j}}(pC)^{\frac{1}{p}(\frac{1}{\sigma}+\frac{1}{\sigma^p}+\cdots+\frac{1}{\sigma^j})}|u_{\lambda,R}^+|_{p_s^*,R}, \tag{3.11}$$

for every $j \in \mathbb{N}$. Note that

$$\sum_{j=1}^{\infty} \frac{1}{\sigma^j} = \frac{1}{\sigma - 1} \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{i}{\sigma^j} = \frac{\sigma}{(\sigma-1)^2}.$$

Since $\sigma > 1$, passing to the limit as $j \to \infty$ in (3.11), we may infer that $u \in L^\infty(B_R(0))$ and

$$|u^+_{\lambda,R}|_{\infty,R} \leq \sigma^{\frac{\sigma}{(\sigma-1)^2}} (pC)^{\frac{1}{\sigma-1}} |u^+_{\lambda,R}|_{p^*_s,R}.$$

Using $|u^+_{\lambda,R}|_{p^*_s,R} \leq M$, fixing any sequences $\lambda_n \to +\infty$ and $R_n \to +\infty$, it is easy to see there exists a constant $K_1 > 0$ such that

$$|u^+_{\lambda_n,R_n}|_\infty \leq K_1 \quad \text{for all } n \in \mathbb{N}.$$

A similar argument can be used to prove that

$$|u^-_{\lambda_n,R_n}|_\infty \leq K_2 \quad \text{for all } n \in \mathbb{N}$$

for a suitable constant K_2. The proof is complete. □

Lemma 3.8. *Let $\{u_{\lambda,R}\}$ be a family of nodal solutions of (3.1) such that $\{\mathcal{I}_{\lambda,R}(u_{\lambda,R})\}$ is bounded in $\mathbb{R}$ for any $\lambda > 0$ and $R > 0$ large enough. Then, there exist $\lambda' > 0$ and $R' > 0$ such that*

$$|u_{\lambda,R}|_{\infty,B_R(0)\setminus\Omega'_\Gamma} \leq a_0 \quad \textit{for all } \lambda \geq \lambda' \textit{ and } R \geq R'.$$

In particular, $u_{\lambda,R}$ solves the original problem (3.2) for $\lambda \geq \lambda'$ and $R \geq R'$.

Proof. Choose $R_0 > 0$ large such that $\overline{\Omega'_\Gamma} \subset B_{R_0}(0)$. Since $\partial\Omega'_\Gamma$ is a compact set, we fix a neighborhood of $\mathcal{B}$ of $\partial\Omega'_\Gamma$ such that

$$\mathcal{B} \subset B_{R_0}(0) \setminus \Omega_\Gamma.$$

The Moser iteration technique implies that there exists $C > 0$, which is independent of λ, such that

$$\left|u^+_{\lambda,R}\right|_{L^\infty\left(\partial\Omega'_\Gamma\right)} \leq C \left|u^+_{\lambda,R}\right|_{L^{p^*_s}(\mathcal{B})} \quad \text{for all } R \geq R_0.$$

Fixing two sequences $\lambda_n \to +\infty$ and $R_n \to +\infty$, by Lemma 3.6 we have that for some subsequence $u_{\lambda_n,R_n} \to 0$ in $W^{s,p}(B_{R_n}(0) \setminus \Omega_\Gamma)$, then $u_{\lambda_n,R_n} \to 0$ in $W^{s,p}(B_{R_0}(0) \setminus \Omega_\Gamma)$, and so,

$$\left|u^+_{\lambda_n,R_n}\right|_{L^{p^*_s}(\mathcal{B})} \to 0 \quad \text{as } n \to \infty.$$

Hence, there is $n_0 \in \mathbb{N}$ such that

$$\left|u^+_{\lambda_n,R_n}\right|_{L^\infty\left(\partial\Omega'_\Gamma\right)} \leq a_0 \quad \text{for all } n \geq n_0.$$

Now, for $n \geq n_0$, we set $\widetilde{u}^+_{\lambda_n,R_n} : B_{R_n}(0) \setminus \Omega'_\Gamma \to \mathbb{R}$ given by

$$\tilde{u}^+_{\lambda_n,R_n}(x) = \left(u^+_{\lambda_n,R_n} - a_0\right)^+(x).$$

Then, $\tilde{u}^+_{\lambda_n,R_n} \in W^{s,p}_0(B_{R_n}(0) \setminus \Omega'_\Gamma)$. Our goal is to show that $\tilde{u}^+_{\lambda_n,R_n}(x) = 0$ in $B_{R_n}(0) \setminus \Omega'_\Gamma$, because this will ensure that

$$\left|u^+_{\lambda_n,R_n}\right|_{\infty,B_{R_n}(0)\setminus\Omega'_\Gamma} \leq a_0.$$

Indeed, extending $\tilde{u}^+_{\lambda_n,R_n}(x) = 0$ in Ω'_Γ and taking $\tilde{u}^+_{\lambda,R}$ as a test function, we have

$$\int_{B_{R_n}(0)\setminus\Omega'_\Gamma} \frac{|u_{\lambda_n,R_n}(x) - u_{\lambda_n,R_n}(y)|^{p-1}(u_{\lambda_n,R_n}(x) - u_{\lambda_n,R_n}(y))(\tilde{u}^+_{\lambda_n,R_n}(x) - \tilde{u}^+_{\lambda_n,R_n}(y))}{|x-y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{B_{R_n}(0)\setminus\Omega'_\Gamma} (\lambda_n V(x) + 1) |u_{\lambda_n,R_n}|^{p-2} u_{\lambda_n,R_n} \tilde{u}^+_{\lambda_n,R_n} \,\mathrm{d}x$$

$$\leq \int_{B_{R_n}(0)\setminus\Omega'_\Gamma} \tilde{F}_2'(u_{\lambda_n,R_n})\tilde{u}^+_{\lambda_n,R_n}\,\mathrm{d}x.$$

Since

$$\int_{B_{R_n}(0)\setminus\Omega'_\Gamma} \frac{|u_{\lambda_n,R_n}(x)-u_{\lambda_n,R_n}(y)|^{p-1}(u_{\lambda_n,R_n}(x)-u_{\lambda_n,R_n}(y))(\tilde{u}^+_{\lambda_n,R_n}(x)-\tilde{u}^+_{\lambda_n,R_n}(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{B_{R_n}(0)\setminus\Omega'_\Gamma} \frac{|\tilde{u}^+_{\lambda_n,R_n}(x)-\tilde{u}^+_{\lambda_n,R_n}(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y,$$

we have

$$\int_{B_{R_n}(0)\setminus\Omega'_\Gamma} (\lambda_n V(x)+1)\,|u_{\lambda_n,R_n}|^{p-2}u_{\lambda_n,R_n}\tilde{u}^+_{\lambda_n,R_n}\,\mathrm{d}x$$
$$= \int_{\left(B_{R_n}(0)\setminus\Omega'_\Gamma\right)_+} (\lambda_n V(x)+1)\,|\tilde{u}^+_{\lambda_n,R_n}+a_0|^{p-2}\left(\tilde{u}^+_{\lambda_n,R_n}+a_0\right)\tilde{u}^+_{\lambda_n,R_n}\,\mathrm{d}x,$$

and

$$\int_{B_{R_n}(0)\setminus\Omega'_\Gamma} \tilde{F}_2'\left(u_{\lambda_n,R_n}\right)\tilde{u}^+_{\lambda_n,R_n}\,\mathrm{d}x$$
$$= \int_{\left(B_{R_n}(0)\setminus\Omega'_\Gamma\right)_+} \frac{\tilde{F}_2'\left(u_{\lambda_n,R_n}\right)}{|u_{\lambda_n,R_n}|^{p-2}u_{\lambda_n,R_n}}|\tilde{u}^+_{\lambda_n,R_n}+a_0|^{p-2}\left(\tilde{u}^+_{\lambda_n,R_n}+a_0\right)\tilde{u}^+_{\lambda_n,R_n}\,\mathrm{d}x,$$

where

$$\left(B_{R_n}(0)\setminus\Omega'_\Gamma\right)_+ = \left\{x\in B_{R_n}(0)\setminus\Omega'_\Gamma\colon u_{\lambda_n,R_n}(x)>a_0\right\}.$$

From the above equalities, we have

$$\int_{B_{R_n}(0)\setminus\Omega'_\Gamma} \frac{|\tilde{u}^+_{\lambda_n,R_n}(x)-\tilde{u}^+_{\lambda_n,R_n}(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y$$
$$+\int_{\left(B_{R_n}(0)\setminus\Omega'_\Gamma\right)_+}\left((\lambda_n V(x)+1)-\frac{\tilde{F}_2'\left(u_{\lambda_n,R_n}\right)}{|u_{\lambda_n,R_n}|^{p-2}u_{\lambda_n,R_n}}\right)$$
$$\times\left(|\tilde{u}^+_{\lambda_n,R_n}+a_0|^{p-2}\left(\tilde{u}^+_{\lambda_n,R_n}+a_0\right)\right)\tilde{u}^+_{\lambda_n,R_n}\,\mathrm{d}x=0.$$

By the definition of $\tilde{F}_2'$, we obtain

$$(\lambda_n V(x)+1)-\frac{\tilde{F}_2'\left(u_{\lambda_n,R_n}\right)}{|u_{\lambda_n,R_n}|^{p-2}u_{\lambda_n,R_n}} \geq 1-\ell>0 \quad \text{in } \left(B_{R_n}(0)\setminus\Omega'_\Gamma\right)_+.$$

Thus, $\tilde{u}^+_{\lambda_n,R_n}=0$ in $\left(B_{R_n}(0)\setminus\Omega'_\Gamma\right)_+$ and $\tilde{u}^+_{\lambda_n,R_n}=0$ in $B_{R_n}(0)\setminus\Omega'_\Gamma$. From the above argument we conclude that there are $\lambda'>0$ and $R'>0$ such that

$$\left|u^+_{\lambda,R}\right|_{\infty,B_R(0)\setminus\Omega'_\Gamma}\leq a_0 \quad \text{for all } \lambda\geq\lambda' \text{ and } R\geq R'.$$

A similar argument can be used to prove that

$$\left|u^-_{\lambda,R}\right|_{\infty,B_R(0)\setminus\Omega'_\Gamma}\leq a_0 \quad \text{for all } \lambda\geq\lambda' \text{ and } R\geq R',$$

if necessary, λ' and R' can be increased. Thus,

$$|u_{\lambda,R}|_{\infty,B_R(0)\setminus\Omega'_\Gamma}\leq a_0 \quad \text{for all } \lambda\geq\lambda' \text{ and } R\geq R'.$$

This finished the proof. □

4. A special minimax level

In the section, for any $\lambda > 0$ and $j \in \Gamma$, let us denote by $\mathcal{I}_j \colon W_0^{s,p}(\Omega_j) \to \mathbb{R}$ and $\mathcal{I}_{\lambda,j} \colon W^{s,p}(\Omega'_j) \to \mathbb{R}$ the functionals given by

$$\mathcal{I}_j(u) = \frac{1}{p}[u]^p_{\Omega_j} + \int_{\Omega_j} |u|^p \,\mathrm{d}x - \frac{1}{p}\int_{\Omega_j} |u|^p \log |u|^p \,\mathrm{d}x,$$
$$\mathcal{I}_{\lambda,j} = \frac{1}{p}[u]^p_{\Omega'_j} + \int_{\Omega'_j} (\lambda V(x)+1)\,|u|^p \,\mathrm{d}x - \frac{1}{p}\int_{\Omega'_j} |u|^p \log |u|^p \,\mathrm{d}x,$$

where

$$[u]^p_Y = \iint_{Y\times Y} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y,$$

which are the energy functionals associated with the following logarithmic systems:

$$\begin{cases} (-\Delta)^s_p\, u = |u|^{p-2}u \log |u|^p & \text{in } \Omega_j,\\ u = 0 & \text{on } \partial\Omega_j, \end{cases} \tag{4.1}$$

and

$$\begin{cases} (-\Delta)^s_p\, u + \lambda V(x)|u|^{p-2}u = |u|^{p-2}u \log |u|^p & \text{in } \Omega'_j,\\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega'_j, \end{cases} \tag{4.2}$$

respectively. It is obvious that $\mathcal{I}_j$ and $\mathcal{I}_{\lambda,j}$ satisfy the Mountain Pass geometry, since Ω_j and Ω'_j are bounded, and $\mathcal{I}_j$ and $\mathcal{I}_{\lambda,j}$ satisfy the (PS) condition. Using the same arguments as in Section 3, there exist two nontrivial functions $\omega_j \in W^{s,p}(\Omega_j)$ and $\omega_{\lambda,j} \in W^{s,p}(\Omega'_j)$ satisfying

$$\mathcal{I}_j(\omega_j) = c_j, \quad \mathcal{I}_{\lambda,j}(\omega_{\lambda,j}) = c_{\lambda,j} \quad \text{and} \quad \mathcal{I}'_j(\omega_j) = \mathcal{I}'_{\lambda,j}(\omega_{\lambda,j}) = 0,$$

where

$$c_j = \min_{u\in\mathcal{N}_j} \mathcal{I}_j(u), \quad c_{\lambda,j} = \min_{u\in\mathcal{N}_{\lambda,j}} \mathcal{I}_{\lambda,j}(u),$$

and

$$\mathcal{N}_j = \left\{u \in W_0^{s,p}(\Omega_j) \colon u^\pm \neq 0 \text{ and } \mathcal{I}'_j(u^\pm)u^\pm = 0\right\},$$
$$\mathcal{N}_{\lambda,j} = \left\{u \in W^{s,p}(\Omega'_j) \colon u^\pm \neq 0 \text{ and } \mathcal{I}'_{\lambda,j}(u^\pm)u^\pm = 0\right\}.$$

In what follows, without loss of any generality, we consider $\Gamma = \{1,2,\ldots,l\}$ with $l \leq k$, $c_\Gamma = \sum_{j=1}^l c_j$ and $T > 0$ is a constant large enough, which does not depend on $R > 0$ large enough, such that

$$0 < \mathcal{I}_j\left(\frac{1}{T}\omega_j^\pm\right), \quad \mathcal{I}_j\left(T\omega_j^\pm\right) < \frac{\mathcal{I}_j\left(\omega_j^\pm\right)}{2} \quad \text{for all } j \in \Gamma. \tag{4.3}$$

We define

$$\gamma_0\left(\varsigma_1,\ldots,\varsigma_l,\tau_1,\ldots,\tau_l\right)(x) = \sum_{j=1}^l \varsigma_j T\omega_j^+(x) + \sum_{j=1}^l \tau_j T\omega_j^-(x)$$

for all $(\varsigma_1,\ldots,\varsigma_l,\tau_1,\ldots,\tau_l) \in \left[1/T^2,1\right]^{2l}$,

$$\Gamma_{\lambda,R} = \left\{\gamma \in C\left(\left[1/T^2,1\right]^{2l}, E_{\lambda,R}\right) \colon \left.\gamma^\pm\right|_{\Omega'_j} \neq 0 \text{ for all } j\in\Gamma, \gamma = \gamma_0 \text{ on } \partial\left(\left[1/T^2,1\right]^{2l}\right)\right\},$$

and

$$b_{\lambda,R,\Gamma} = \inf_{\gamma\in\Gamma_{\lambda,R}} \max_{(\vec{\varsigma},\vec{\tau})\in[1/T^2,1]^{2l}} \mathcal{I}_{\lambda,R}(\gamma(\vec{\varsigma},\vec{\tau})),$$

where $(\vec{\varsigma},\vec{\tau}) = (\varsigma_1,\ldots,\varsigma_l,\tau_1,\ldots,\tau_l)$. Note that $\gamma_0 \in \Gamma_{\lambda,R}$, so $\Gamma_{\lambda,R} \neq \emptyset$ and $b_{\lambda,R,\Gamma}$ is well defined.

Lemma 4.1. *For each $\gamma \in \Gamma_{\lambda,R}$, there exists $(\vec{\varsigma_*}, \vec{\tau_*}) \in \left[1/T^2, 1\right]^{2l}$ such that*
$$\mathcal{I}'_{\lambda,j}\left(\gamma^{\pm}\left(\vec{\varsigma_*}, \vec{\tau_*}\right)\right)\left(\gamma^{\pm}\left(\vec{\varsigma_*}, \vec{\tau_*}\right)\right) = 0 \quad \textit{for all } j \in \{1, \ldots, l\}.$$

Proof. Given $\gamma \in \Gamma_{\lambda,R}$, we consider the map $\widetilde{H}\colon \left[1/T^2, 1\right]^{2l} \to \mathbb{R}^{2l}$ defined as
$$\widetilde{H}(\vec{\varsigma}, \vec{\tau}) = \left(\mathcal{I}'_{\lambda,1}\left(\gamma^+\right)\cdot\left(\gamma^+\right), \ldots, \mathcal{I}'_{\lambda,l}\left(\gamma^+\right)\cdot\left(\gamma^+\right), \mathcal{I}'_{\lambda,1}\left(\gamma^-\right)\cdot\left(\gamma^-\right), \ldots, \mathcal{I}'_{\lambda,l}\left(\gamma^-\right)\cdot\left(\gamma^-\right)\right),$$
where
$$\mathcal{I}'_{\lambda,j}\left(\gamma^{\pm}\right)\cdot\left(\gamma^{\pm}\right) = \mathcal{I}'_{\lambda,j}\left(\gamma^{\pm}(\vec{\varsigma}, \vec{\tau})\right)\cdot\left(\gamma^{\pm}(\vec{\varsigma}, \vec{\tau})\right) \quad \text{for all } j \in \Gamma.$$
For $(\vec{\varsigma}, \vec{\tau}) \in \partial\left(\left[1/T^2, 1\right]^{2l}\right)$, since
$$\widetilde{H}(\vec{\varsigma}, \vec{\tau}) = H_0(\vec{\varsigma}, \vec{\tau}),$$
where
$$H_0(\vec{\varsigma}, \vec{\tau}) = \left(\mathcal{I}'_{\lambda,1}\left(\gamma_0^+\right)\cdot\left(\gamma_0^+\right), \ldots, \mathcal{I}'_{\lambda,l}\left(\gamma_0^+\right)\cdot\left(\gamma_0^+\right), \mathcal{I}'_{\lambda,1}\left(\gamma_0^-\right)\cdot\left(\gamma_0^-\right), \ldots, \mathcal{I}'_{\lambda,l}\left(\gamma_n^-\right)\cdot\left(\gamma_n^-\right)\right)$$
and by the properties of F'_2, $\deg\left(H_0, \left(1/T^2, 1\right)^{2l}, 0\right) = 1$. Therefore, using topological degree properties, we derive that $\deg\left(\widetilde{H}, \left(1/T^2, 1\right)^{2l}, 0\right) = 1$. This shows that there is $(\vec{\varsigma_*}, \vec{\tau_*}) \in \left[1/T^2, 1\right]^{2l}$ such that $\widetilde{H}\left((\vec{\varsigma_*}, \vec{\tau_*})\right) = (0, \ldots, 0)$, which proves the lemma. □

Lemma 4.2. *The following assertions hold:*

(a) *For any $\lambda > 0$ and $R > 0$ large enough, $\sum_{j=1}^{l} c_{\lambda,j} \leq b_{\lambda,R,\Gamma} \leq c_\Gamma$;*
(b) *$b_{\lambda,R,\Gamma} \to c_\Gamma$, when $\lambda \to +\infty$ uniformly for $R > 0$ large.*

Proof. (a) Since $\gamma_0 \in \Gamma_{\lambda,R}$, we have
$$\begin{aligned} b_{\lambda,R,\Gamma} &\leq \max_{(\vec{\varsigma}, \vec{\tau}) \in [1/T^2, 1]^{2l}} \mathcal{I}_{\lambda,R}\left(\gamma_0(\vec{\varsigma}, \vec{\tau})\right) \\ &= \max_{(\varsigma_1, \ldots, \varsigma_l) \in \left[\frac{1}{T^2}, 1\right]^l} \sum_{j=1}^{l} \mathcal{I}_j\left(\varsigma_j T w_j^+\right) + \max_{(\tau_1, \ldots, \tau_l) \in \left[\frac{1}{T^2}, 1\right]^l} \sum_{j=1}^{l} I_j\left(\tau_j T w_j^-\right). \end{aligned}$$
From the definition of w_j, we have
$$\max_{\varsigma \in \left[\frac{1}{R^2}, 1\right]} \mathcal{I}_j\left(\varsigma_j R w_j^{\pm}\right) = \mathcal{I}_j\left(w_j^{\pm}\right) \quad \text{for all } j \in \Gamma, \tag{4.4}$$
and thus
$$b_{\lambda,R,\Gamma} \leq \sum_{j=1}^{l} c_j = c_\Gamma.$$
Taking $(\vec{\varsigma_*}, \vec{\tau_*}) \in \left[1/T^2, 1\right]^{2l}$ as given in Lemma 4.1, this shows that
$$\mathcal{I}_{\lambda,j}\left(\gamma\left(\vec{\varsigma_*}, \vec{\tau_*}\right)\right) \geq c_{\lambda,j} \quad \text{for all } j \in \Gamma.$$
On the other hand, it is easy to verify that $\mathcal{I}_{\lambda, B_R(0)\setminus\Omega'_\Gamma}(u) \geq 0$ for all $u \in W^{s,p}\left(B_R(0) \setminus \Omega'_\Gamma\right)$. Thus, we obtain that
$$\mathcal{I}_{\lambda,R}\left(\gamma\left(\vec{\varsigma_*}, \vec{\tau_*}\right)\right) \geq \sum_{j=1}^{l} \mathcal{I}_{\lambda,j}\left(\gamma\left(\vec{\varsigma_*}, \vec{\tau_*}\right)\right).$$
Then
$$\max_{(\vec{\varsigma}, \vec{\tau}) \in [1/T^2, 1]^{2l}} \mathcal{I}_{\lambda,R}(\gamma(\vec{\varsigma}, \vec{\tau})) \geq \mathcal{I}_{\lambda,R}\left(\gamma\left(\vec{\varsigma_*}, \vec{\tau_*}\right)\right) \geq \sum_{j=1}^{l} c_{\lambda,j}.$$

From the definition of $b_{\lambda,R,\Gamma}$, we can obtain

$$b_{\lambda,R,\Gamma} \geq \sum_{j=1}^{l} c_{\lambda,j},$$

which completes the proof of (a).

(b) Let λ_n be an arbitrary sequence with $\lambda_n \to +\infty$ and assume $\omega_{\lambda_n,j} \in W^{s,p}\left(\Omega'_j\right)$ to be least energy nodal solutions of problem (4.2), with $\lambda = \lambda_n$, that is

$$\mathcal{I}_{\lambda_n,j}\left(\omega_{\lambda_n,j}\right) = c_{\lambda_n,j} \quad \text{and} \quad \mathcal{I}'_{\lambda_n,j}\left(\omega_{\lambda_n,j}\right) = 0 \quad \text{for all } j \in \Gamma.$$

Using the same arguments as in the proof of Lemma 3.5, for each $j \in \Gamma$ and for a subsequence $\{\omega_{\lambda_{n_k},j}\}$, there exists $\omega_{0,j}$ such that

$$\omega_{\lambda_{n_k},j} \to \omega_{0,j} \quad \text{in } W^{s,p}\left(\Omega'_j\right) \text{ as } n_k \to \infty.$$

Moreover, $\omega_{0,j} \in W_0^{s,p}\left(\Omega_j\right)$ is a nodal solution of problem (4.1). Thus,

$$\lim_{k\to\infty} \mathcal{I}_{\lambda_{n_k},j}\left(\omega_{\lambda_{n_k},j}\right) = \mathcal{I}_j\left(\omega_{0,j}\right) \geq c_j.$$

Since $c_{\lambda,j} \leq c_j$,we conclude that $c_{\lambda,j} \to c_j$ as $\lambda \to \infty$, from where it follows that

$$\sum_{j=1}^{l} c_{\lambda,j} \to c_\Gamma \quad \text{as } \lambda \to \infty.$$

The last limit together with (a) implies that (b) holds. □

5. Uniform estimates

In the following, let us denote

$$F_\lambda(\Omega'_\Gamma) := \left\{u \in W^{s,p}(\Omega'_\Gamma)\colon \int_{\Omega'_\Gamma} V(x)|u|^p \,\mathrm{d}x < \infty\right\},$$

endowed with the norm

$$\|u\|^p_{\lambda,\Omega'_\Gamma} := \int_{\Omega'_\Gamma}\int_{\Omega'_\Gamma} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega'_\Gamma} \left(\lambda V(x)+1\right)|u|^p\,\mathrm{d}x.$$

Moreover, $F^+_{\lambda,j}$ and $F^-_{\lambda,j}$ denote the cone of nonnegative and nonpositive functions belonging to $F_\lambda(\Omega'_j)$, respectively, that is

$$F^+_\lambda = \left\{u \in F_\lambda(\Omega'_\Gamma)\colon u(x) \geq 0 \text{ a.e. in } \Omega'_j\right\},$$
$$F^-_\lambda = \left\{u \in F_\lambda(\Omega'_\Gamma)\colon u(x) \leq 0 \text{ a.e. in } \Omega'_j\right\}.$$

From the definition of γ_0, there are positive constants ν and $\lambda^* > 0$ such that

$$\operatorname{dist}_{\lambda,j}\left(\gamma_0(\vec{\varsigma},\vec{\tau}), F^{\pm}_{\lambda,j}\right) > \nu, \quad \text{for all } (\vec{\varsigma},\vec{\tau}) \in \left[1/T^2,1\right]^{2l}, \ j \in \Gamma \text{ and } \lambda \geq \lambda^*,$$

where $\operatorname{dist}_{\lambda,j}(K,F)$ denotes the distance between sets of $F_\lambda\left(\Omega'_j\right)$. Taking the number ν obtained in the last inequality, we define

$$\Upsilon = \left\{u \in E_{\lambda,R}\colon \operatorname{dist}_{\lambda,j}\left(u|_{\Omega'_j}, F^{\pm}_{\lambda,j}\right) \geq \nu \text{ for all } j \in \Gamma\right\}.$$

Moreover, for any constants d, $\mu > 0$ and $0 < \kappa < \nu/2$, we consider the sets

$$\mathcal{I}^{c_\Gamma}_{\lambda,R} = \left\{u \in E_{\lambda,R}\colon \mathcal{I}_{\lambda,R}(u) \leq c_\Gamma\right\},$$
$$\mathcal{A}^\lambda_{\mu,R} = \left\{u \in \Upsilon_{2\kappa}\colon \mathcal{I}_{\lambda,B_R(0)\setminus\Omega'_\Gamma}(u) \geq 0, \|u\|^p_{\lambda,B_R(0)\setminus\Omega_\Gamma} \leq \mu, |\mathcal{I}_{\lambda,j}(u) - b_{\lambda,R,\Gamma}| \leq \mu \text{ for all } j \in \Gamma\right\},$$

where Υ_r for $r > 0$ denotes the set

$$\Upsilon_r = \left\{u \in E_{\lambda,R}\colon \operatorname{dist}(u,\Upsilon) \leq r\right\}.$$

Notice that for each $\mu > 0$, there exists $\Lambda^* = \Lambda^*(\mu) > 0$ such that $w = \sum_{j=1}^{l} w_j \in \mathcal{A}^{\lambda}_{\mu,R}$ for all $\lambda \geq \Lambda^*$. Because $\omega \in \Upsilon$, $\mathcal{I}_{\lambda,R}(\omega) = c_\Gamma$ and $b_{\lambda,R,\Gamma} \to c_\Gamma$, when $\lambda \to +\infty$ uniformly for R large. Thus, $\mathcal{A}^{\lambda}_{\mu,R} \neq \emptyset$ for λ sufficiently large.

In what follows, for $M > 0$, let us consider

$$\mathcal{B}_{M+1} = \{u \in E_{\lambda,R} \colon \|u\|_{\lambda,R} \leq M+1\},$$

where M is a constant large enough independent of λ and R satisfying

$$\|\gamma(\vec{\varsigma},\vec{\tau})\|_{\lambda,R} \leq \frac{M}{2} \quad \text{for all } (\vec{\varsigma},\vec{\tau}) \in \left[1/T^2, 1\right]^{2l},$$

and

$$\left\| \sum_{j=1}^{k} w_j \right\|_{\lambda,R} \leq \frac{M}{2}.$$

Now let us set μ^* as

$$\mu^* = \min\left\{ \frac{\mathcal{I}_j(\omega^{\pm}) + M + \kappa}{4}, j \in \Gamma \right\}. \tag{5.1}$$

Next, we will establish uniform estimates of $\|\mathcal{I}'_{\lambda,R}(u)\|$ in the set $\left(\mathcal{A}^{\lambda}_{2\mu,R} \setminus \mathcal{A}^{\lambda}_{\mu,R}\right) \cap \mathcal{B}_{M+1} \cap \mathcal{I}^{c_\Gamma}_{\lambda,R}$.

Lemma 5.1. *For each $\mu \in (0,\mu^*)$, there are $\lambda^* > 0$, $R^* > 0$ large enough and $\sigma_0 > 0$ independent of λ and $R > 0$ large enough such that*

$$\left\|\mathcal{I}'_{\lambda,R}(u)\right\| \geq \sigma_0 \quad \textit{for } \lambda \geq \lambda^*, R \geq R^* \textit{ and } u \in \left(\mathcal{A}^{\lambda}_{2\mu,R} \setminus \mathcal{A}^{\lambda}_{\mu,R}\right) \cap \mathcal{B}_{M+1} \cap \mathcal{I}^{c_\Gamma}_{\lambda,R}.$$

Proof. Arguing by contradiction, assume that there are $\lambda_n, R_n \to \infty$ and $u_n \in \left(\mathcal{A}^{\lambda_n}_{2\mu,R_n} \setminus \mathcal{A}^{\lambda_n}_{\mu,R_n}\right) \cap \mathcal{B}_{M+1} \cap \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n}$ such that

$$\left\|\mathcal{I}'_{\lambda_n,R_n}(u_n)\right\| \to 0.$$

Since $u_n \in \mathcal{A}^{\lambda_n}_{2\mu,R_n}$, we have that $\left\{\|u_n\|_{\lambda_n,R_n}\right\}_{n\in\mathbb{N}}$ and $\{\mathcal{I}_{\lambda_n,R_n}(u_n)\}_{n\in\mathbb{N}}$ are both bounded. Then, up to a subsequence if necessary, assume that $\{\mathcal{I}_{\lambda_n,R_n}(u_n)\}_{n\in\mathbb{N}}$ is a convergent sequence. Hence, by Lemma 3.6, there exists $u \in W^{s,p}(\Omega_\Gamma)$ such that u is a solution for (4.1) and

$$u_n \to u \quad \text{in } W^{s,p}(\mathbb{R}^N), \quad \|u_n\|^p_{\lambda_n, B_{R_n}(0)\setminus\Omega_\Gamma} \to 0 \quad \text{and} \quad \mathcal{I}_{\lambda_n,R_n}(u_n) \to \mathcal{I}_\Gamma(u) \in (-\infty, c_\Gamma].$$

Note that $\{u_n\}_{n\in\mathbb{N}} \subset \Upsilon_{2\kappa}$, we derive that $\|u_n^{\pm}\|_{\lambda_n,\Omega'_j} \nrightarrow 0$ for all $j \in \Gamma$, from where it follows that $\|u^{\pm}\|_{\Omega_j} \neq 0$ for all $j \in \Gamma$. Thus u is a nodal solution of (4.1) for all $j \in \Gamma$ and

$$\sum_{j=1}^{l} c_j \leq \sum_{j=1}^{l} \mathcal{I}_j(u|_{\Omega_j}) \leq c_\Gamma,$$

which shows that $\mathcal{I}_j(u|_{\Omega_j}) = c_j$ for all $j \in \Gamma$. Hence $\mathcal{I}_{\lambda_n,R_n}(u_n) \to \mathcal{I}_\Gamma(u)$ as $n \to +\infty$. On the other hand, since $b_{\lambda,R,\Gamma} \to c_\Gamma$, when $\lambda \to +\infty$ uniformly for R large, we derive that $\mathcal{A}^{\lambda_n}_{\mu,R_n} \cap \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n}$ for large n, which is a contradiction. □

Lemma 5.2. *Assume $\mu \in (0,\mu^*)$, $\lambda^* > 0$ and $R^* > 0$ sufficiently large as given in Lemma 5.1. Then, the functional $\mathcal{I}_{\lambda,R}$ has a critical point $u_{\lambda,R}$ satisfying $u_\lambda \in \mathcal{A}^{\lambda}_{\mu,R} \cap \mathcal{I}^{c_\Gamma}_{\lambda,R} \cap \mathcal{B}_{M+1}$ for each $\lambda \geq \lambda^*$ and $R \geq R^*$.*

Proof. Assume by contradiction that there are $\mu \in (0,\mu^*)$ and a sequence $\lambda_n \to \infty$ such that the functional $\mathcal{I}_{\lambda_n,R_n}(u)$ has no critical points in $\mathcal{A}^{\lambda_n}_{\mu,R_n} \cap \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n} \cap \mathcal{B}_{M+1}$. Since $\mathcal{I}_{\lambda_n,R_n}$ satisfies the (PS) condition, there exists a constant $d_{\lambda_n,R_n} > 0$ such that

$$\left\|\mathcal{I}'_{\lambda_n,R_n}(u)\right\| \geq d_{\lambda_n,R_n} \quad \text{for all } u \in \mathcal{A}^{\lambda_n}_{\mu,R_n} \cap \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n} \cap \mathcal{B}_{M+1}.$$

By Lemma 5.1, we have that

$$\left\|\mathcal{I}'_{\lambda_n,R_n}(u)\right\| \geq \sigma_0 \quad \text{for all } u \in \left(\mathcal{A}^{\lambda_n}_{2\mu,R_n} \setminus \mathcal{A}^{\lambda_n}_{\mu,R_n}\right) \cap \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n} \cap \mathcal{B}_{M+1},$$

where $\sigma_0 > 0$ is independent of λ_n and R_n for n large enough. Now, we define a continuous functional $\Phi_n\colon E_{\lambda_n,R_n} \to \mathbb{R}$ such that

$$\begin{cases} \Phi_n(u) = 1 & \text{for } u \in \mathcal{A}^{\lambda_n}_{3\mu/2,R_n} \cap \Upsilon_\kappa \cap \mathcal{B}_M, \\ \Phi_n(u) = 0 & \text{for } u \notin \mathcal{A}^{\lambda_n}_{2\mu,R_n} \cap \Upsilon_{2\kappa} \cap \mathcal{B}_{M+1}, \\ 0 \leq \Phi_n(u) \leq 1, & \text{for } u \in E_{\lambda_n,R_n}, \end{cases}$$

and $\mathcal{H}_n\colon \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n} \to E_{\lambda_n}\left(B_{R_n}(0)\right)$ is a function given by

$$\mathcal{H}_n(u) := \begin{cases} -\Phi_n(u)\frac{Y_n(u)}{\|Y_n(u)\|}, & u \in \mathcal{A}^{\lambda_n}_{2\mu,R_n} \cap \mathcal{B}_{M+1}, \\ 0, & u \notin \mathcal{A}^{\lambda_n}_{2\mu,R_n} \cap \mathcal{B}_{M+1}, \end{cases}$$

where Y_n is a pseudo-gradient vector field for $\mathcal{I}_{\lambda_n,R_n}$ on $\mathcal{K}_{\lambda_n} = \left\{u \in E_{\lambda_n,R_n} : \mathcal{I}'_{\lambda_n,R_n}(u) \neq 0\right\}$. It is obvious that $\mathcal{H}_n$ is well defined, since $\mathcal{I}'_{\lambda_n,R_n}(u) \neq 0$ for $u \in \mathcal{A}^{\lambda_n}_{2\mu,R_n} \cap \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n}$. Hereafter, we denote by m_0^n the real number given by

$$m_0^n = \left\{\mathcal{I}_{\lambda_n,R_n}(u)\colon u \in \gamma_0\left([1/T^2,1]^{2l} \setminus \mathcal{A}^{\lambda_n}_{\mu,R_n} \cap \mathcal{B}_M\right)\right\}$$

which verifies $\limsup_{n\to\infty} m_0^n < c_\Gamma$. Moreover, let us define $K_n > 0$ satisfying

$$|\mathcal{I}_{\lambda_n,j}(u) - \mathcal{I}_{\lambda_n,j}(v) \leq \|u-v\|_{\lambda_n,\Omega'_j} \quad \text{for all } u,v \in \mathcal{B}_{M+1} \text{ and } j \in \Gamma.$$

Note that

$$\|\mathcal{H}_n(u)\| \leq 1 \quad \text{for all } n \in \mathbb{N} \text{ and } u \in \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n},$$

so

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{I}_{\lambda_n,R_n}(\eta_n(\tau,u)) \leq -\Phi_n(\eta_n(\tau,u))\left\|\mathcal{I}'_{\lambda_n,R_n}(\eta_n(\tau,u))\right\| \leq 0,$$

$$\left\|\frac{\mathrm{d}\eta_n}{\mathrm{d}\tau}\right\|_\lambda = \|\mathcal{H}_n(\eta_n)\|_\lambda \leq 1,$$

and

$$\eta_n(\tau,u) = u \quad \text{for all } \tau \geq 0 \quad \text{and} \quad u \notin \mathcal{A}^{\lambda_n}_{2\mu,R_n} \cap \mathcal{B}_{M+1},$$

where the deformation flow $\eta_n\colon [0,\infty) \times \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n} \to \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n}$ is defined by

$$\frac{\mathrm{d}\eta_n}{\mathrm{d}\tau} = \mathcal{H}_n(\eta_n) \quad \text{and} \quad \eta_n(0,u) = u \in \mathcal{I}^{c_\Gamma}_{\lambda_n,R_n}.$$

Claim: There exists $T_n = T\left(\lambda_n, R_n\right) > 0$ and $\varepsilon^* > 0$ independent of n such that

$$\limsup_{n\to\infty}\left[\max_{(\vec{\varsigma},\vec{\tau})\in[1/T^2,1]^{2l}} \mathcal{I}_{\lambda_n,R_n}\left(\eta_n\left(T_n, \gamma_0(\vec{\varsigma},\vec{\tau})\right)\right)\right] < c_\Gamma - \varepsilon^*. \tag{5.2}$$

Indeed, assume $u = \gamma_0(\vec{\varsigma},\vec{\tau})$, $\widetilde{d}_{\lambda_n,R_n} = \min\left\{d_{\lambda_n,R_n}, \sigma_0\right\}$, $T_n = \sigma_0\mu/2\widetilde{d}_{\lambda_n,R_n}$ and $\widetilde{\eta}_n(\tau) = \eta_n(\tau,u)$. If $u \notin \mathcal{A}^{\lambda_n}_{\mu,R_n} \cap \mathcal{B}_M \cap \Upsilon_\kappa$, by the definition of m_0^n, we have

$$\mathcal{I}_{\lambda_n,R_n}\left(\eta_n(\tau,u)\right) \leq \mathcal{I}_{\lambda_n,R_n}(u) \leq m_0^n \quad \text{for all } \tau \geq 0.$$

On the other hand, if $u \in \mathcal{A}^{\lambda_n}_{\mu,R_n} \cap \mathcal{B}_M \cap \Upsilon_\kappa$, we need to consider two cases:
Case 1: $\widetilde{\eta}_n(\tau) \in \mathcal{A}^{\lambda_n}_{3\mu/2,R_n} \cap \mathcal{B}_M \cap \Upsilon_\kappa$ for all $\tau \in [0,T_n]$.
This case shows that there is $\varepsilon^* > 0$ independent of n such that

$$\mathcal{I}_{\lambda_n,R_n}\left(\widetilde{\eta}_n\left(T_n\right)\right) \leq c_\Gamma - \varepsilon^*.$$

Case 2: $\widetilde{\eta}_n(\tau_0) \notin \mathcal{A}^{\lambda_n}_{3\mu/2,R_n} \cap \mathcal{B}_M \cap \Upsilon_\kappa$ for some $\tau_0 \in [0, T_n]$.
Related to this case, we have the following situations:

(i) There exists $\tau_2 \in [0, T_n]$ such that $\widetilde{\eta}_n(\tau_2) \notin \Upsilon_\kappa$, and thus for $\tau_1 = 0$, it holds
$$\|\widetilde{\eta}_n(\tau_2) - \widetilde{\eta}_n(\tau_1)\|_{\lambda_n,R_n} \geq \delta > \mu,$$
since $\widetilde{\eta}_n(\tau_1) = u \in \Upsilon$.

(ii) There exists $\tau_2 \in [0, T_n]$ such that $\widetilde{\eta}_n(\tau_2) \notin \mathcal{B}_M$, so that for $\tau_1 = 0$, we have
$$\|\widetilde{\eta}_n(\tau_2) - \widetilde{\eta}_n(\tau_1)\|_{\lambda_n,R_n} \geq \frac{M}{2} > \mu,$$
since $\widetilde{\eta}_n(\tau_1) = u \in \mathcal{B}_{M/2}$.

(iii) $\widetilde{\eta}_n(\tau) \notin \Upsilon_\kappa \cap \mathcal{B}_M$, and there exist $0 \leq \tau_1 < \tau_2 \leq T_n$ such that $\widetilde{\eta}_n(\tau) \in \mathcal{A}^{\lambda_n}_{3\mu/2,R_n} \setminus \mathcal{A}^{\lambda_n}_{\mu,R_n}$ for all $\tau \in [\tau_1, \tau_2]$ with
$$|\mathcal{I}_{\lambda_n,R_n}(\widetilde{\eta}_n(\tau_1)) - b_{\lambda,R,\Gamma}| = \mu \quad \text{and} \quad |\mathcal{I}_{\lambda_n,R_n}(\widetilde{\eta}_n(\tau_2)) - b_{\lambda,R,\Gamma}| = \frac{3\mu}{2}.$$

According to the definition of K_n, we have
$$\begin{aligned}
\|\widetilde{\eta}_n(\tau_2) - \widetilde{\eta}_n(\tau_1)\|_{\lambda,R} &\geq \frac{1}{K_n}\left|\mathcal{I}_{\lambda_n,R_n}(\widetilde{\eta}_n(\tau_2)) - \mathcal{I}_{\lambda_n,R_n}(\widetilde{\eta}_n(\tau_1))\right| \\
&\geq \frac{1}{K_n}\left(|\mathcal{I}_{\lambda_n,R_n}(\widetilde{\eta}_n(\tau_2)) - b_{\lambda,R,\Gamma}| - |\mathcal{I}_{\lambda_n,R_n}(\widetilde{\eta}_n(\tau_1)) - b_{\lambda,R,\Gamma}|\right) \\
&\geq \frac{1}{2K_n}\mu.
\end{aligned}$$

The estimates in (i)–(iii) show that $\tau_2 - \tau_1 \geq \frac{1}{2K_n}\mu$. From the mean value theorem, it follows that
$$\begin{aligned}
\mathcal{I}_{\lambda_n,R_n}(\widetilde{\eta}_n(T_n)) &= \mathcal{I}_{\lambda_n,R_n}(u) + \int_0^{T_n} \frac{\mathrm{d}}{\mathrm{d}\zeta}\mathcal{I}_{\lambda_n,R_n}(\widetilde{\eta}_n(\zeta))\,\mathrm{d}\zeta \\
&\leq \mathcal{I}_{\lambda_n,R_n}(u) - \int_0^{T_n} \Phi(\widetilde{\eta}_n(\zeta))\left\|\mathcal{I}'_{\lambda_n,R_n}(\widetilde{\eta}_n(\zeta))\right\|\,\mathrm{d}\zeta \\
&\leq c_\Gamma - \int_{\tau_1}^{\tau_2} \sigma_0\,\mathrm{d}\zeta \\
&= c_\Gamma - \sigma_0(\tau_2 - \tau_1) \\
&\leq c_\Gamma - \frac{\sigma_0\mu}{2K_n},
\end{aligned}$$
which proves (5.2) and shows the Claim.

Now, we prove that $(\vec{\varsigma}, \vec{\tau}) \to \mu_n(T_n, \gamma_0(\vec{\varsigma}, \vec{\tau}))$ belongs to Γ_{λ_n,R_n} for n large. First, it is easy to prove that $\eta_n(\gamma_0(\vec{\varsigma}, \vec{\tau}))$ is a continuous function in $\left[1/T^2, 1\right]^{2l}$. Hence, we have to show that
$$\eta_n(T_n, \gamma_0(\vec{\varsigma}, \vec{\tau})) = \gamma_0(\vec{\varsigma}, \vec{\tau}) \quad \text{for all } (\vec{\varsigma}, \vec{\tau}) \in \partial\left(\left[1/T^2, 1\right]^{2l}\right),$$
and
$$(\eta_n(T_n, \gamma_0(\vec{\varsigma}, \vec{\tau})))^{\pm} \in W^{s,p}\left(\Omega'_j\right) \setminus \{0\},$$
for all $j \in \Gamma$ and for all $(\vec{\varsigma}, \vec{\tau}) \in \left[1/T^2, 1\right]^{2l}$.

From $\mu \in (0, \mu^*)$, (4.3), (4.4) and (5.1) we obtain
$$|\mathcal{I}_{\lambda_n,R_n}(\gamma_0(\vec{\varsigma}, \vec{\tau})) - c_\Gamma| \geq 2\mu^* \quad \text{for all } (\vec{\varsigma}, \vec{\tau}) \in \partial\left(\left[1/T^2, 1\right]^{2l}\right) \text{ and } n \in \mathbb{N}.$$

Hence, using again the fact that $b_{\lambda,R,\Gamma} \to c_\Gamma$, when $\lambda \to +\infty$ uniformly for R large, there is $n_0 > 0$ such that
$$|\mathcal{I}_{\lambda_n,R_n}(\gamma_0(\vec{\varsigma}, \vec{\tau})) - s_{\lambda_n,R_n,\Gamma}| > 2\mu \quad \text{for all } (\vec{\varsigma}, \vec{\tau}) \in \partial\left(\left[1/T^2, 1\right]^{2l}\right) \text{ and } n \geq n_0,$$

which shows that $\gamma_0(\vec{\varsigma},\vec{\tau}) \notin \mathcal{A}^{\lambda_n}_{2\mu,R_n}$ for all $(\vec{\varsigma},\vec{\tau}) \in \partial\left(\left[1/T^2,1\right]^{2l}\right)$ and $n \geq n_0$. From this,

$$\eta_n\left(T_n,\gamma_0(\vec{\varsigma},\vec{\tau})\right) = \gamma_0(\vec{\varsigma},\vec{\tau}) \quad \text{for all } (\vec{\varsigma},\vec{\tau}) \in \partial\left(\left[1/T^2,1\right]^{2l}\right) \text{ and } n \geq n_0.$$

On the other hand, since $\eta_n\left(T_n,\gamma_0(\vec{\varsigma},\vec{\tau})\right) \in \Upsilon_{2\kappa}$ for all n, we have

$$\operatorname{dist}_{\lambda_n,j}\left(\eta_n\left(T_n,\gamma_0(\vec{\varsigma},\vec{\tau})\right), F^{\pm}_{\lambda_n,j}\right) \geq \nu - 2\kappa > 0.$$

Then, $\left(\eta_n\left(T_n,\gamma_0(\vec{\varsigma},\vec{\tau})\right)\right)^{\pm}\Big|_{\Omega_j} \neq 0$ for all $j \in \Gamma$, and we can get $\eta_n\left(T_n,\gamma_0(\vec{\varsigma},\vec{\tau})\right) \in \Gamma_{\lambda_n,R_n}$ for n large enough. Combining the definition of $b_{\lambda,R,\Gamma}$ with the Claim and the fact that $\eta_n\left(T_n,\gamma_0(\vec{\varsigma},\vec{\tau})\right) \in \Gamma_{\lambda_n,R_n}$ for n large enough, we have the following inequality

$$\limsup_{n\to\infty} b_{\lambda_n,R_n,\Gamma} < c_\Gamma - \varepsilon^*,$$

which is a contradiction. Thus, the lemma holds. □

From the last lemma, we have the following corollary.

Corollary 5.3. *For each $\mu \in (0,\mu^*)$, there exist $\lambda^* > 0$ and $R^* > 0$ large enough as given in the previous lemma. Then, problem (3.2) has a nodal solution $u_{\lambda,R} \in \mathcal{A}^{\lambda}_{\mu,R}$ for all $\lambda \geq \lambda^*$ and $R \geq R^*$.*

Proof. From Lemma 5.2, there exists a nodal solution $u_{\lambda,R} \in \mathcal{A}^{\lambda}_{\mu,R} \cap \mathcal{I}^{c_\Gamma}_{\lambda,R} \cap \mathcal{B}_{M+1}$ to problem (3.1). Then, by Remark 3.1 and Lemma 3.8, the solution $u_{\lambda,R}$ is also a nodal solution of problem (3.2). □

6. The proof of Theorem 1.1

By Corollary 5.3, for any $\mu \in (0,\mu^*)$, there exist $\lambda^* > 0$ and $R^* > 0$, such that we can find a nodal solution $u_{\lambda,R} \in \mathcal{A}^{\lambda}_{\mu,R} \cap \mathcal{I}^{c_\Gamma}_{\lambda,R} \cap \mathcal{B}_{M+1}$ of problem (3.2) for all $\lambda \geq \lambda^*$ and $R \geq R^*$.

Fixing $\lambda \geq \lambda^*$ and taking a sequence $R_n \to +\infty$, there exists a solution $u_{\lambda,n} = u_{\lambda,R_n}$ for the problem (3.2) with

$$u_{\lambda,n} \in \mathcal{A}^{\lambda}_{\mu,R_n} \cap \mathcal{I}^{c_\Gamma}_{\lambda,R_n} \cap \mathcal{B}_{M+1} \quad \text{for all } n \in \mathbb{N}.$$

Since $\{u_{\lambda,n}\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$, we can assume that for some $u_\lambda \in W^{s,p}(\mathbb{R}^N)$,

$$\begin{cases} \mathcal{I}_{\lambda,R_n}\left(u_{\lambda,n}\right) \to c \leq c_\Gamma, \\ u_{\lambda,n} \rightharpoonup u_\lambda \quad \text{in } W^{s,p}(\mathbb{R}^N), \\ u_{\lambda,n} \to u_\lambda \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for any } q \in [1,p^*_s), \\ u_{\lambda,n}(x) \to u_\lambda(x) \quad \text{a.e. } x \in \mathbb{R}^N. \end{cases}$$

Recalling Lemma 3.8, we obtain

$$|u_{\lambda,n}(x)| \leq a_0 \quad \text{for all } x \in \mathbb{R}^N \setminus \Omega_\Gamma,$$

then,

$$|u_\lambda(x)| \leq a_0 \quad \text{for all } x \in \mathbb{R}^N \setminus \Omega_\Gamma.$$

The next two lemmas play a fundamental role in the proof of Theorem 1.1. Their proofs follow from similar arguments as in the proof of Lemma 3.6, so we omit them.

Lemma 6.1. *For any fixed $\zeta > 0$, there is an $R > 0$ satisfying*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} \left(\int_{\mathbb{R}^N} \frac{|u_{\lambda,n}(x) - u_{\lambda,n}(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y + (\lambda V(x)+1)\,|u_{\lambda,n}|^p\right)\mathrm{d}x < \zeta.$$

Lemma 6.2. *$u_{\lambda,n} \to u_\lambda$ in $W^{s,p}(\mathbb{R}^N)$. In addition,*

$$F_1(u_{\lambda,n}) \to F_1(u_\lambda) \quad \textit{and} \quad F_1'(u_{\lambda,n})u_{\lambda,n} \to F_1'(u_\lambda)u_\lambda \quad \textit{in } L^1(\mathbb{R}^N).$$

As a consequence, we consider the energy functional $\mathcal{I}_\lambda$, which is defined in (2.1). It is easy to see that u_λ is a critical point of $\mathcal{I}_\lambda$ satisfying

$$u_\lambda \in \mathcal{A}_\mu^\lambda = \left\{ u \in (\Upsilon_\infty)_{2\kappa} : \mathcal{I}_{\lambda,\mathbb{R}^N\setminus\Omega'_\Gamma}(u) \geq 0, \|u\|^p_{\mathbb{R}^N\setminus\Omega_\Gamma} \leq \mu, |\mathcal{I}_{\lambda,R}(u) - b_{\lambda,R,\Gamma}| \leq \mu, \text{ for all } j \in \Gamma \right\},$$

where

$$\Upsilon_\infty = \left\{ u \in E_\lambda :\ \mathrm{dist}_{\lambda,j}\left(u, F^\pm_{\lambda,j}\right) \geq \nu, \forall j \in \Gamma \right\},$$

$$(\Upsilon_\infty)_r = \left\{ u \in E_\lambda :\ \inf_{v\in\Upsilon_\infty} \|u-v\|_{\lambda,\Omega'_j} \leq r, \forall j \in \Gamma \right\}.$$

Here, by a critical point we understand that u_λ satisfies the inequality

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))((v(x)-u_\lambda(x))-(v(y)-u_\lambda(y)))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{\mathbb{R}^N}(\lambda V(x)+1)|u_\lambda|^{p-2}u_\lambda(v-u_\lambda)\,\mathrm{d}x+\int_{\mathbb{R}^N}F_1(v)\,\mathrm{d}x-\int_{\mathbb{R}^N}F_1(u_\lambda)\,\mathrm{d}x\\ &\geq \int_{\mathbb{R}^N}F_2'(u_\lambda)(v-u_\lambda)\,\mathrm{d}x\end{aligned}$$

for all $v \in E_\lambda$. Hence, u_λ satisfies the equality

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{R}^N}\lambda V(x)|u_\lambda|^{p-2}u_\lambda v\,\mathrm{d}x\\ &=\int_{\mathbb{R}^N}|u_\lambda|^{p-2}u_\lambda v\log|u_\lambda|^p\,\mathrm{d}x,\end{aligned}$$

for all $v \in C_0^\infty(\mathbb{R}^N)$

Now, we are ready to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Letting $\lambda_n \to +\infty$ and $\mu_n \in (0,\mu^*)$ with $\mu_n \to 0$, we can find a solution $u_n \in \mathcal{A}^{\lambda_n}_{\mu_n}$ of problem (1.1) with $\lambda = \lambda_n$. Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,p}(\mathbb{R}^N)$ such that

(a) $\left\|\mathcal{I}'_{\lambda_n}(u_{\lambda_n})\right\| = 0$ for all $n \in \mathbb{N}$;
(b) $\|u_{\lambda_n}\|_{\lambda_n,\mathbb{R}^N\setminus\Omega_\Gamma} \to 0$;
(c) $\mathcal{I}_{\lambda_n}(u_n) \to c \leq c_\Gamma$,

where

$$\|\mathcal{I}'_\lambda(u)\| = \sup\left\{\langle \mathcal{I}'_\lambda(u), z\rangle : z \in W^{s,p}_c(\mathbb{R}^N) \text{ and } \|z\|_\lambda \leq 1\right\}.$$

Arguing as in Lemma 3.6, there is a $u \in W^{s,p}(\mathbb{R}^N)$ satisfying $u_{\lambda_n} \to u$ in $W^{s,p}(\mathbb{R}^N)$, and $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega_\Gamma$ and u is a nontrivial solution of

$$\begin{cases}(-\Delta)^s_p u = |u|^{p-2}u\log|u|^p & \text{in } \Omega_\Gamma,\\ u = 0 & \text{on } \partial\Omega_\Gamma,\end{cases} \tag{6.1}$$

and so,

$$\mathcal{I}_\Gamma(u) \geq c_\Gamma.$$

Moreover, since $\{u_{\lambda_n}\}$ verifies

$$\mathrm{dist}_{\lambda,j}\left(u_{\lambda_n}, F^\pm_{\lambda,j}\right) \geq \nu - 2\kappa > 0 \quad \text{for all } j \in \Gamma,$$

we derive that $\left\|u^\pm_{\lambda_n}\right\|_{\lambda_n,\Omega'_j} \to 0$ for all $j \in \Gamma$. Hence, from the definition of G'_2, it follows that there exists $\nu_* > 0$ such that

$$\int_{\Omega'_j} \left|u^\pm_{\lambda_n}\right|^{q+1}\mathrm{d}x \geq \nu_* \quad \text{for all } n \in \mathbb{N} \text{ and for all } j \in \Gamma.$$

Therefore

$$\int_{\Omega_j'} \left|u^{\pm}\right|^{q+1} \mathrm{d}x \geq \nu_* \quad \text{for all } j \in \Gamma.$$

Thus, u changes its sign on Ω_j for all $j \in \Gamma$, and

$$\mathcal{I}_j(u) \geq c_j \quad \text{for all } j \in \Gamma.$$

Note that

$$\mathcal{I}_{\lambda_n}\left(u_{\lambda_n}\right) \to \mathcal{I}_\Gamma(u),$$

which shows that

$$\mathcal{I}_\Gamma(u) = c \quad \text{and} \quad c \geq c_\Gamma.$$

Due to $c \leq c_\Gamma$, it follows that $\mathcal{I}_\Gamma(u) = c_\Gamma$, which implies that $u|_{\Omega_j}$ is a least energy nodal solution of problem (6.1). This concludes the proof of the theorem. □

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