CONSTANT SIGN SOLUTIONS FOR DOUBLE PHASE PROBLEMS WITH VARIABLE EXPONENTS

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Abstract. In this paper we study quasilinear elliptic equations driven by the variable exponent double phase operator and a right-hand side that contains a parametric term and a superlinear perturbation with a growth that need not necessarily be polynomial. Under a certain behavior near the origin of the perturbation we prove the existence of at least two constant sign solutions by using truncation arguments and comparison methods.

1. Introduction and results

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For $r \in C(\overline{\Omega})$ we define

$$r^- = \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r^+ = \max_{x\in\overline{\Omega}} r(x).$$

In this paper, we study the following variable exponent double phase Dirichlet problem with parameter dependence in the reaction term

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) &= \lambda |u|^{p^- -2}u - f(x,u) \qquad && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ to be specified and the exponents as well as the weight function are supposed to satisfy the following conditions:

(H1) $p, q \in C(\overline{\Omega})$ such that $1 < p(x) < N$ and $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$, where p^* is given by

$$p^*(x) = \frac{Np(x)}{N - p(x)} \quad \text{for } x \in \overline{\Omega}.$$

(H2) There exists $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$ the function $p_x \colon \Omega_x \to \mathbb{R}$ defined by $p_x(z) = p(x + z\xi_0)$ is monotone, where $\Omega_x := \{z \in \mathbb{R} : x + z\xi_0 \in \Omega\}$.

From hypothesis (H2) we know that

$$\hat{\lambda} := \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x}{\displaystyle\int_\Omega |u|^{p(x)}\,\mathrm{d}x} > 0,$$

due to Fan-Zhang-Zhao [7, Theorem 3.3].

Let

$$\mathcal{B} := \left\{ u \in C_0^1(\overline{\Omega})_+ \setminus \{0\} \,:\, \int_\Omega |u|^{p^-}\,\mathrm{d}x \geq \int_\Omega |u|^{p(x)}\,\mathrm{d}x \right\},$$

where $C_0^1(\overline{\Omega})$ and $C_0^1(\overline{\Omega})_+$ are defined by

$$C_0^1(\overline{\Omega}) := \left\{ u \in C^1(\overline{\Omega}) \,:\, u\big|_{\partial\Omega} = 0 \right\} \quad \text{and} \quad C_0^1(\overline{\Omega})_+ := \left\{ u \in C_0^1(\overline{\Omega}) \,:\, u(x) \geq 0 \text{ for all } x \in \overline{\Omega} \right\}.$$

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Then we put

$$\overline{\lambda} := \inf_{u\in\mathcal{B}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x}{\displaystyle\int_\Omega |u|^{p(x)}\,\mathrm{d}x}$$

According to the above definitions, we easily see that $0 < \hat{\lambda} \le \overline{\lambda}$.

In addition, the hypotheses on f are the following ones:

(H3) $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that
- (i) f is bounded on bounded sets;
- (ii)
$$\lim_{s\to\pm\infty}\frac{f(x,s)}{|s|^{q^+-2}s} = +\infty \quad \text{uniformly for a. a. } x\in\Omega;$$
- (iii)
$$\lim_{s\to 0}\frac{f(x,s)}{|s|^{p^--2}s} = 0 \quad \text{uniformly for a. a. } x\in\Omega.$$

We say that a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a weak solution of problem (1.1) if

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x = \int_\Omega \left(\lambda|u|^{p^--2}u - f(x,u)\right)v\,\mathrm{d}x$$

is satisfied for all $v \in W_0^{1,\mathcal{H}}(\Omega)$ and all the integrals are finite.

Our main result reads as follows.

Theorem 1.1. *Let hypotheses (H1), (H2) and (H3) be satisfied and let $\lambda > \overline{\lambda}$. Then problem (1.1) admits at least two nontrivial bounded weak solutions $u_1, u_2 \in W_0^{1,\mathcal{H}}(\Omega)$ such that that $u_1 \ge 0$ and $u_2 \le 0$ in Ω.*

The proof of Theorem 1.1 relies on an appropriate combination of variational tools like truncation and comparison methods and on the properties of the Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$. The operator in problem (1.1) is the so-called variable exponent double phase operator given by

$$\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right), \quad u\in W_0^{1,\mathcal{H}}(\Omega), \tag{1.2}$$

which has been recently studied by Crespo-Blanco-Gasiński-Harjulehto-Winkert [5]. Clearly, if $\inf \mu \ge \mu_0 > 0$ or $\mu \equiv 0$, (1.2) reduces to the $(q(\cdot), p(\cdot))$-Laplacian or the $p(\cdot)$-Laplacian, respectively, and if in addition the exponents are constants, we get the (q,p)-Laplacian or the usual p-Laplacian, respectively. Operators of type (1.2) have their origin in the study of functionals of type

$$\omega \mapsto \int_\Omega \left(|\nabla\omega|^p + \mu(x)|\nabla\omega|^q\right)\mathrm{d}x, \quad 1<p<q<N,$$

due to Zhikov [25] in order to describe models for strongly anisotropic materials. Double phase differential operators and corresponding energy functionals appear in several physical applications. For example, in the elasticity theory, the modulating coefficient $\mu(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents q and p, see Zhikov [27]. But also in other mathematical applications such kind of functional plays an important role, for example, in the study of duality theory and of the Lavrentiev gap phenomenon, see Papageorgiou-Rădulescu-Repovš [16], Ragusa-Tachikawa [21] and Zhikov [26, 27].

So far, there are only few results involving the variable exponent double phase operator given in (1.2). We refer to the recent results of Aberqi-Bennouna-Benslimane-Ragusa [1] for existence results in complete manifolds, Albalawi-Alharthi-Vetro [2] for convection problems with $(p(\cdot), q(\cdot))$-Laplace type problems, Bahrouni-Rădulescu-Winkert [3] for double phase problems of Baouendi-Grushin type operator, Crespo-Blanco-Gasiński-Harjulehto-Winkert [5] for double phase convection problems, Kim-Kim-Oh-Zeng [13] for concave-convex-type double-phase problems, Leonardi-Papageorgiou [14] for concave-convex problems and Zeng-Rădulescu-Winkert [24] for multivalued problems, see also the references therein. In the context of double phase problems with constant exponents or variable exponent double phase problems with balanced growth we refer to the papers of Colasuonno-Squassina [4] for eigenvalue problems of double phase type, Farkas-Winkert [8] for Finsler double phase problems, Gasiński-Papageorgiou [9] locally Lipschitz right-hand sides, Gasiński-Winkert [11] for convection problems, Liu-Dai [15] for a Nehari manifold approach, Papageorgiou-Vetro [17] for superlinear problems, Papageorgiou-Vetro-Vetro [18] for parametric Robin problems, Perera-Squassina [19] for Morse theoretical approach, Vetro-Winkert [22] for parametric convective problems, Zeng-Bai-Gasiński-Winkert [23] for implicit obstacle problems with multivalued operators and the references therein.

We point out that the nonlinearity f in problem (1.1) does not need to satisfy any polynomial growth. The shape of f is just fixed by the behavior at $\pm\infty$ and near the origin. With this work we extend the results of Gasiński-Winkert [10] to the variable exponent double phase operator. Indeed, if p and q are constants, the condition $\lambda > \overline{\lambda}$ in Theorem 1.1 becomes $\lambda > \lambda_{1,p}$ where $\lambda_{1,p} > 0$ is the first eigenvalue of the p-Laplacian with Dirichlet boundary condition. In our setting, in order to have $\hat{\lambda} > 0$, we need the additional condition in (H2) on the exponent $p(\cdot)$.

2. Preliminaries

In this section we recall some basic facts about variable exponent Sobolev spaces and Musielak-Orlicz Sobolev spaces. We refer to the monographs of Diening-Harjulehto-Hästö-Růžička [6], Harjulehto-Hästö [12] and Rădulescu-Repovš [20], see also the recent paper of Crespo-Blanco-Gasiński-Harjulehto-Winkert [5].

Let $M(\Omega)$ be the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. For $r \in C(\overline{\Omega})$ with $r(x) > 1$ for all $x \in \overline{\Omega}$, we denote by $L^{r(\cdot)}(\Omega)$ the usual variable exponent Lebesgue space defined by

$$L^{r(\cdot)}(\Omega) = \left\{ u \in M(\Omega) \,:\, \varrho_r(u) := \int_\Omega |u(x)|^{r(x)} \,\mathrm{d}x < +\infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{r(\cdot)} := \inf \left\{ \tau > 0 \,:\, \varrho_r\left(\frac{u}{\tau}\right) \le 1 \right\}$$

Also, $W^{1,r(\cdot)}(\Omega)$ and $W_0^{1,r(\cdot)}(\Omega)$ stand for the corresponding Sobolev spaces.

Suppose hypothesis (H1) and let $\mathcal{H}\colon \Omega \times [0,+\infty) \to [0,+\infty)$ be the nonlinear function defined by

$$\mathcal{H}(x,t) = t^{p(x)} + \mu(x) t^{q(x)}.$$

Moreover, the modular function $\rho_{\mathcal{H}}$ is given by

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)} \right) \mathrm{d}x.$$

Then, the Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega) = \{ u \in M(\Omega) \,:\, \rho_{\mathcal{H}}(u) < +\infty \}$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf \left\{ \tau > 0 \,:\, \rho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \le 1 \right\}.$$

The corresponding Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \| \, |\nabla u| \, \|_{\mathcal{H}}$. Furthermore, the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$ is denoted by $W_0^{1,\mathcal{H}\cdot}(\Omega)$. It is clear that the norm $\|\cdot\|_{\mathcal{H}}$ defined on $L^{\mathcal{H}}(\Omega)$ is uniformly convex and hence the spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive Banach spaces, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [5, Proposition 2.12].

We have the following embedding results for the spaces $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [5, Propositions 2.16 and 2.18].

Proposition 2.1. *Let hypotheses* (H1) *be satisfied. Then the following hold:*

(i) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ *are continuous for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p(x)$ *for all* $x \in \overline{\Omega}$*;*
(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *are compact for* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$*;*
(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is compact;*
(iv) *There exists a constant* $C > 0$ *independent of* u *such that*

$$\|u\|_{\mathcal{H}} \leq C\|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega).$$

According to the previous proposition, we can equip the space $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm $\|\cdot\|_{1,\mathcal{H},0} = \|\nabla \cdot\|_{\mathcal{H}}$.

Finally for any $s \in \mathbb{R}$ we denote $s_\pm = \max\{\pm s, 0\}$, that means $s = s_+ - s_-$ and $|s| = s_+ + s_-$. For any function $v\colon \Omega \to \mathbb{R}$ we denote $v_\pm(\cdot) = [v(\cdot)]_\pm$.

3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By assumption (H3)(ii), we know that for each $a > 0$, there exists a constant $M > 1$ such that

$$f(x,s)s \geq a|s|^{q^+} \quad \text{for a. a. } x \in \Omega \text{ and for all } |s| \geq M. \tag{3.1}$$

Since $p(x) < q^+$ for all $x \in \overline{\Omega}$ and $M > 1$, choosing $a = \lambda$ and taking a constant function $\overline{u} \in [M,\infty)$, from (3.1) we obtain that

$$0 \geq \lambda \overline{u}^{p^- - 1} - f(x,\overline{u}) \quad \text{for a. a. } x \in \Omega. \tag{3.2}$$

Let $k_+\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the truncation function given by

$$k_+(x,s) := \begin{cases} 0 & \text{if } s < 0,\\ \lambda s^{p^- - 1} - f(x,s) & \text{if } 0 \leq s \leq \overline{u},\\ \lambda \overline{u}^{p^- - 1} - f(x,\overline{u}) & \text{if } \overline{u} < s, \end{cases} \tag{3.3}$$

and let $K_+(x,s) = \int_0^s k_+(x,t)\,\mathrm{d}t$. We denote by $\phi_+\colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ the C^1-functional defined by

$$\phi_+(u) = \int_\Omega \left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right] \mathrm{d}x - \int_\Omega K_+(x,u)\,\mathrm{d}x.$$

It is clear that the functional ϕ_+ is coercive because of the truncation in (3.3) and the estimate

$$\rho_{\mathcal{H}}(\nabla u) \geq \|\nabla u\|_{\mathcal{H}}^{p^-} \qquad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega) \text{ with } \|u\|_{1,\mathcal{H},0} > 1,$$

see Crespo-Blanco-Gasiński-Harjulehto-Winkert [5, Proposition 2.13]. In addition, it is sequentially weakly lower semicontinuous due to the compactness of the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ for any $r \in C(\overline{\Omega})$ with $1 \leq r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, see Proposition 2.1(ii). Hence, there exists $u_1 \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\phi_+(u_1) = \inf \left[\phi_+(u) \,:\, u \in W_0^{1,\mathcal{H}}(\Omega)\right].$$

Now, we show that u_1 is nontrivial. Thanks to assumption (H3)(iii), we can find for each $\varepsilon > 0$ a number $\delta \in (0,\overline{u})$ depending on ε such that

$$F(x,s) \leq \frac{\varepsilon}{p^-}|s|^{p^-} \quad \text{for a. a. } x \in \Omega \text{ and for all } |s| \leq \delta \tag{3.4}$$

with $F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t$.

Let $u_* \in \mathcal{B}$ be a function such that

$$\int_\Omega |\nabla u_*|^{p(x)}\,\mathrm{d}x < (\overline{\lambda} + \varepsilon)\int_\Omega |u_*|^{p(x)}\,\mathrm{d}x \leq (\overline{\lambda} + \varepsilon)\int_\Omega |u_*|^{p^-}\,\mathrm{d}x. \tag{3.5}$$

Now we take $t \in (0,1)$ small enough such that $tu_*(x) \in [0,\delta]$ for all $x \in \overline{\Omega}$. Hence, by applying (3.4) and (3.5), we get

$$\begin{aligned}
\phi_+(tu_*) &= \int_\Omega \left[\frac{1}{p(x)}|\nabla(tu_*)|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla(tu_*)|^{q(x)}\right] \mathrm{d}x - \int_\Omega K_+(x, tu_*)\,\mathrm{d}x \\
&\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla u_*|^{p(x)}\,\mathrm{d}x + \frac{t^{q^-}}{q^-}\int_\Omega \mu(x)|\nabla u_*|^{q(x)}\,\mathrm{d}x - \frac{\lambda t^{p^-}}{p^-}\int_\Omega |u_*|^{p^-}\,\mathrm{d}x + \frac{\varepsilon t^{p^-}}{p^-}\int_\Omega |u_*|^{p^-}\,\mathrm{d}x \\
&\leq \frac{t^{p^-}}{p^-}\left(\overline{\lambda} - \lambda + 2\varepsilon\right)\int_\Omega |u_*|^{p^-}\,\mathrm{d}x + \frac{t^{q^-}}{q^-}\int_\Omega \mu(x)|\nabla u_*|^{q(x)}\,\mathrm{d}x.
\end{aligned}$$

If we choose $\varepsilon \in (0, \frac{1}{2}(\lambda - \overline{\lambda}))$, then $\overline{\lambda} - \lambda + 2\varepsilon < 0$ and since $p^- < q^-$, for $t > 0$ sufficiently small, we have

$$t^{p^-}\left[\frac{\overline{\lambda} - \lambda + 2\varepsilon}{p^-}\right]\int_\Omega |u_*|^{p^-}\,\mathrm{d}x + \frac{t^{q^-}}{q^-}\int_\Omega \mu(x)|\nabla u_*|^{q(x)}\,\mathrm{d}x < 0.$$

This shows that for $t \in (0,1)$ sufficiently small $\phi_+(tu_*) < 0 = \phi_+(0)$. Hence, $u_1 \neq 0$.

Now, taking into account that u_1 is a global minimizer of ϕ_+, we have $\phi_+'(u_1) = 0$, that is,

$$\int_\Omega \left(|\nabla u_1|^{p(x)-2}\nabla u_1 + \mu(x)|\nabla u_1|^{q(x)-2}\nabla u_1\right)\cdot\nabla v\,\mathrm{d}x = \int_\Omega k_+(x,u_1)v\,\mathrm{d}x \quad \text{for all } v \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.6}$$

Note that $\pm u_\pm \in W_0^{1,\mathcal{H}}(\Omega)$ for any $u \in W_0^{1,\mathcal{H}}(\Omega)$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [5, Proposition 2.17]. Choosing the test function $v = -(u_1)_-$ in (3.6) we obtain that $(u_1)_- = 0$ and hence $u_1 \geq 0$. On the other hand, taking $v = (u_1 - \overline{u})_+$ in (3.6) and using (3.2) as well as (3.3), we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_1|^{p(x)-2}\nabla u_1 + \mu(x)|\nabla u_1|^{q(x)-2}\nabla u_1\right)\cdot\nabla(u_1 - \overline{u})_+\,\mathrm{d}x \\
&= \int_\Omega k_+(x,u_1)(u_1 - \overline{u})_+\,\mathrm{d}x \\
&= \int_\Omega \left(\lambda\overline{u}^{p^- - 1} - f(x,\overline{u})\right)(u_1 - \overline{u})_+\,\mathrm{d}x \leq 0.
\end{aligned}$$

This implies that

$$\int_{\{x\in\Omega\,:\,u_1(x)>\overline{u}(x)\}} |\nabla u_1|^{p(x)}\,\mathrm{d}x + \int_{\{x\in\Omega\,:\,u_1(x)>\overline{u}(x)\}} \mu(x)|\nabla u_1|^{q(x)}\,\mathrm{d}x \leq 0$$

and hence we deduce that $u_1 \leq \overline{u}$. Finally, since $0 \leq u_1 \leq \overline{u}$ and taking into account the definition of the truncation function k_+ in (3.3) and hypothesis (H3)(i), we conclude that $u_+ \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of problem (1.1).

In a similar way we show the existence of a negative solution of problem (1.1). Indeed, we can take $a = \lambda$ and choose a constant function $\underline{u} \in (-\infty, -M]$ so that $0 \leq \lambda|\underline{u}|^{p^- - 2}\underline{u} - f(x,\underline{u})$ for a. a. $x \in \Omega$. Then, we consider the truncation function $k_-\colon \Omega\times\mathbb{R}\to\mathbb{R}$ given by

$$k_-(x,s) := \begin{cases} \lambda|\underline{u}|^{p^- - 2}\underline{u} - f(x,\underline{u}) & \text{if } s < \underline{u}, \\ \lambda|s|^{p^- - 2}s - f(x,s) & \text{if } \underline{u} \leq s \leq 0, \\ 0 & \text{if } 0 < s, \end{cases}$$

and the C^1-functional $\phi_-\colon W_0^{1,\mathcal{H}}(\Omega)\to\mathbb{R}$ defined by

$$\phi_-(u) = \int_\Omega \left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right]\mathrm{d}x - \int_\Omega K_-(x,u)\,\mathrm{d}x,$$

where $K_-(x,s) = \int_0^s k_-(x,t)\,\mathrm{d}t$. Thus, we can show that the global minimizer of $\phi_-(u)$ is nontrivial and it is a bounded negative weak solution for our problem. $\square$

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