EXISTENCE AND ASYMPTOTIC PROPERTIES FOR QUASILINEAR ELLIPTIC EQUATIONS WITH GRADIENT DEPENDENCE

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ABSTRACT. The existence of solutions of opposite constant sign is proved for a Dirichlet problem driven by the weighted (p,q)-Laplacian with $q < p$ and exhibiting a $(q-1)$-order term as well as a convection term. The approach is based on the method of sub-supersolution. Extremal solutions in relevant ordered intervals are obtained as well.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We consider the following quasilinear Dirichlet problem

$$\begin{aligned} -\Delta_p u - \mu(x)\Delta_q u &= a|u|^{q-2}u - g(x,u,\nabla u) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{$P_{\mu,a}$}$$

with $1 < q < p < +\infty$, $a > 0$, and a weight function $\mu : \Omega \to \mathbb{R}$ with $\mu \in L^\infty(\Omega)$ and $\operatorname{ess\,inf}_\Omega \mu > 0$. Here, for $r = p, q$, Δ_r stands for the r-Laplace differential operator. The case $\mu \equiv 1$ is fundamental giving rise to the problem driven by the (p,q)-Laplacian. In the statement of ($P_{\mu,a}$) we also have a Carathéodory function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, i.e., $g(\cdot, s, \xi)$ is measurable for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $g(x,\cdot,\cdot)$ is continuous for a.a. $x \in \Omega$, describing dependence on u and its gradient ∇u which is called convection term. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem ($P_{\mu,a}$) if it fulfills

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx + \int_\Omega \mu(x)|\nabla u|^{q-2}\nabla u \cdot \nabla\varphi dx\\ &\quad = a\int_\Omega |u|^{q-2}u\varphi dx - \int_\Omega g(x,u,\nabla u)\varphi dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \end{aligned} \tag{1.1}$$

Problem ($P_{\mu,a}$) belongs to the class of quasilinear elliptic equations

$$\begin{aligned} \operatorname{div} A(x,u,\nabla u) &= f(x,u,\nabla u) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

with Carathéodory mappings $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. Generally, (1.2) does not have variational structure, so non-variational methods must be used, see Averna-Motreanu-Tornatore [1], Carl-Le-Motreanu [2], Faraci-Motreanu-Puglisi [3], Faria-Miyagaki-Motreanu [4], Faria-Miyagaki-Motreanu-Tanaka [5],

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Motreanu [9], Motreanu-Tornatore [10] and Tanaka [11]. A leading part is represented by the sub-supersolution approach, which in addition allows the location of solutions within ordered intervals determined by sub-supersolutions. This enclosure principle is useful for instance to find positive solutions. A frequent assumption is that $f(x,s,\xi)$ in (1.2) is bounded from below with respect to $s>0$ near zero by a term of order s^r with $r<q-1$, see Faraci-Motreanu-Puglisi [3], Faria-Miyagaki-Motreanu [4], Faria-Miyagaki-Motreanu-Tanaka [5], Motreanu [9], Motreanu-Tornatore [10] and Tanaka [11]. Such a condition is not applicable to ($P_{\mu,a}$) due to the term $a|u|^{q-2}u$ matching the weighted q-Laplacian $\mu(x)\Delta_q$.

The objective of the present paper is to establish the existence of a positive solution and of a negative solution to problem ($P_{\mu,a}$) through an adequate set-up for the method of sub-supersolution. As mentioned before, these results cannot be deduced from what it is known for the more general problem (1.2). Our main contribution consists in dealing with the possibly concave term $a|u|^{q-2}u$ against the convection $g(x,u,\nabla u)$. There is a balance between the roles of the reals p and q. For instance, we argue in the space $W_0^{1,p}(\Omega)$ but assume that the parameter a is above the first eigenvalue of $-\Delta_q$ with weight μ. We are able to provide precise bounds for the obtained solutions. Moreover, we show the existence of extremal (i.e., the greatest and smallest) solutions in relevant ordered intervals.

2. Preliminaries

For a bounded domain $\Omega\subset\mathbb{R}^N$ and a real $1<r<+\infty$, we denote by $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ the usual Sobolev spaces. Recall that the negative r-Laplacian $-\Delta_r$ is the mapping $-\Delta_r: W_0^{1,r}(\Omega)\to (W_0^{1,r}(\Omega))^*=W^{-1,r'}(\Omega)$ given by

$$\Delta_r u=\operatorname{div}\left(|\nabla u|^{r-2}\nabla u\right).$$

Regarding the weighted eigenvalue problem

$$\begin{aligned} -\mu(x)\Delta_r u&=\lambda|u|^{r-2}u \qquad &&\text{in } \Omega,\\ u&=0 &&\text{on } \partial\Omega, \end{aligned}\tag{2.1}$$

with $\lambda\in\mathbb{R}$ and a weight function $\mu:\Omega\to\mathbb{R}$ as in ($P_{\mu,a}$), we say that λ is an eigenvalue and $u\in W^{1,r}(\Omega)$ an associated eigenfunction if $u\neq 0$ and

$$\int_\Omega \mu(x)|\nabla u|^{r-2}\nabla u\cdot\nabla\varphi dx=\lambda\int_\Omega |u|^{r-2}u\varphi dx$$

for all $\varphi\in W_0^{1,r}(\Omega)$. Based on the Ljusternik-Schnirelman principle, see, e.g., Lê [6], we can construct a sequence $\{\lambda_{n,r,\mu}\}_{n\geq 1}$ of eigenvalues for problem (2.1). The first eigenvalue $\lambda_{1,r,\mu}$ admits the variational representation

$$\lambda_{1,r,\mu}=\inf_{u\in W_0^{1,r}(\Omega),u\neq 0}\left\{\frac{\int_\Omega \mu(x)|\nabla u|^r dx}{\int_\Omega |u|^r dx}\right\}>0.\tag{2.2}$$

In the study of problem ($P_{\mu,a}$), we make use of (2.2) in the case $r=q$.

An element $v\in W_0^{1,p}(\Omega)$ with $v\big|_{\partial\Omega}\geq 0$ ($v\big|_{\partial\Omega}\leq 0$) is a supersolution (subsolution) of problem ($P_{\mu,a}$) if it satisfies

$$\begin{aligned} &\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi dx+\int_\Omega \mu(x)|\nabla v|^{q-2}\nabla v\cdot\nabla\varphi dx\\ &\geq (\leq)\ a\int_\Omega |v|^{q-2}v\varphi dx-\int_\Omega g(x,v,\nabla v)\varphi dx \end{aligned}\tag{2.3}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$. Corresponding to an ordered pair $\underline{u} \leq \overline{u}$ a.e. in Ω consisting of a subsolution $\underline{u}$ and a supersolution $\overline{u}$ for problem ($P_{\mu,a}$), we introduce the ordered interval

$$[\underline{u}, \overline{u}] = \left\{ u \in W_0^{1,p}(\Omega) : \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ for a.a. } x \in \Omega \right\}. \tag{2.4}$$

The positive and negative parts of any $r \in \mathbb{R}$ are denoted by $r^{\pm}$, that is, $r^{\pm} = \max\{\pm r, 0\}$. In the sequel, for any $r > 1$ the notation r' stands for the Hölder conjugate of r, i.e., $r' = r/(r-1)$. In particular, this applies to the Sobolev critical exponent p^* with its conjugate $(p^*)'$. Recall that $p^* = \frac{pN}{N-p}$ if $N > p$ and $p^* = +\infty$ if $N \leq p$. For a later use, it is worth pointing out that $p - 1 < p/(p^*)'$. The strong convergence and the weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively.

3. Two solutions of opposite constant sign

The following conditions on the nonlinearity $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ in ($P_{\mu,a}$) are required:

H(g) $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying

(i) there exist constants $b > 0$ and $\delta > 0$ such that

$$g(x, s, \xi)s \leq b|s|^p \tag{3.1}$$

for a.a. $x \in \Omega$, for all $|s| \leq \delta$, for all $\xi \in \mathbb{R}^N$, and

$$\left(\frac{a}{b}\right)^{\frac{1}{p-q}} \leq \delta; \tag{3.2}$$

(ii) there exist constants $M > 0$, $\gamma \in [0, \frac{p}{(p^*)'})$ and $c_1, c_2 \geq \delta$, with $\delta > 0$ in (i), for which one has

$$ac_1^{q-1} \leq g(x, c_1, 0) \qquad \text{for a.a. } x \in \Omega, \tag{3.3}$$

$$-ac_2^{q-1} \geq g(x, -c_2, 0) \qquad \text{for a.a. } x \in \Omega, \tag{3.4}$$

$$|g(x, s, \xi)| \leq M\left(1 + |\xi|^{\gamma}\right) \qquad \text{for a.a. } x \in \Omega, \tag{3.5}$$

for all $|s| \leq max\{c_1, c_2\}$ and for all $\xi \in \mathbb{R}^N$.

Theorem 3.1. *Assume that hypotheses H(g) hold. If $a > \lambda_{1,q,\mu}$, then problem ($P_{\mu,a}$) has at least two solutions $u, v \in C^{1,\beta}(\overline{\Omega})$ of opposite constant sign satisfying*

$$0 < u \leq c_1 \quad \text{and} \quad -c_2 \leq v < 0 \quad \text{in } \Omega,$$

with some $\beta \in (0, 1)$, where c_1 and c_2 are given in (3.3) and (3.4).

Proof. We start with the existence of a positive solution through the method of sub-supersolution. To this end we formulate the auxiliary problem

$$\begin{aligned} -\Delta_p u - \mu(x)\Delta_q u + b|u|^{p-2}u &= a\left(u^+\right)^{q-1} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.6}$$

for $b > 0$ as in assumption H(g)(i). Notice that (3.6) has variational structure and its corresponding energy functional $J_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ is expressed as

$$J_+(w) = \frac{1}{p}\int_\Omega (|\nabla w|^p + b|w|^p)\,dx + \frac{1}{q}\int_\Omega \mu(x)|\nabla w|^q dx - \frac{a}{q}\int_\Omega \left(w^+\right)^q dx.$$

Since $p > q$ and $b > 0$, the functional J_+ is coercive and weakly sequentially lower semicontinuous. Hence a global minimizer $u_+ \in W_0^{1,p}(\Omega)$ of J_+ exists. It follows that u_+ is a weak solution of problem (3.6), that is,

$$\begin{aligned} &\int_\Omega |\nabla u_+|^{p-2}\nabla u_+ \cdot \nabla\varphi dx + \int_\Omega \mu(x)|\nabla u_+|^{q-2}\nabla u_+ \cdot \nabla\varphi dx \\ &\quad + b\int_\Omega |u_+|^{p-2}u_+\varphi dx = a\int_\Omega \left((u_+)^+\right)^{q-1}\varphi dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.7}$$

The hypothesis $a > \lambda_{1,q,\mu}$, in conjunction with (2.2) for $r = q$, enables us to fix $w \in W_0^{1,p}(\Omega)$ with $w > 0$ a.e. in Ω such that

$$\lambda_{1,q,\mu} < \frac{\int_\Omega \mu(x)|\nabla w|^q dx}{\int_\Omega w^q dx} < a. \tag{3.8}$$

From (3.8) and $q < p$, for $t > 0$ sufficiently small, we get

$$J_+(tw) = \frac{t^p}{p}\int_\Omega (|\nabla w|^p + bw^p)\, dx + \frac{1}{q}t^q\int_\Omega \mu(x)|\nabla w|^q dx - \frac{a}{q}t^q\int_\Omega w^q dx < 0.$$

We infer that $J_+(u_+) < 0$, thus the solution u_+ of (3.6) is nontrivial.

Testing (3.7) with $\varphi = -(u_+)^-$ we see that $u_+ \geq 0$. Then, in view of (3.6), u_+ is a weak solution of

$$\begin{aligned} -\Delta_p u - \mu(x)\Delta_q u + bu^{p-1} &= au^{q-1} && \text{in } \Omega,\\ u &\geq 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.9}$$

Through Moser's iteration, see, e.g., Marino-Winkert [8], applied to (3.9) we note that $u_+ \in L^\infty(\Omega)$. At this point, the regularity up to the boundary, see Lieberman [7, p. 320], ensures that $u_+ \in C_0^1(\overline{\Omega})\setminus\{0\}$. Then, the strong maximum in Motreanu [9, Theorem 2.19] enables us to conclude that $u_+ > 0$ in Ω.

Let us act with $\varphi = u_+^{\alpha+1}$ as test functions in (3.9) for each $\alpha > 0$. By Hölder's inequality, this leads to

$$b\int_\Omega u_+^{p+\alpha}dx \leq a\int_\Omega u_+^{q+\alpha}dx \leq a\left(\int_\Omega u_+^{p+\alpha}dx\right)^{\frac{q+\alpha}{p+\alpha}}|\Omega|^{\frac{p-q}{p+\alpha}},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. This results in

$$b\|u_+\|_{L^{p+\alpha}(\Omega)}^{p-q} \leq a|\Omega|^{\frac{p-q}{p+\alpha}}.$$

Letting $\alpha \to +\infty$ implies

$$b\|u_+\|_{L^\infty(\Omega)}^{p-q} \leq a. \tag{3.10}$$

Using (3.1), (3.2), (3.10) and the fact that u_+ is a solution of (3.9), we find

$$-\Delta_p u_+ - \mu(x)\Delta_q u_+ = au_+^{q-1} - bu_+^{p-1} \leq au_+^{q-1} - g(x, u_+, \nabla u_+).$$

According to (2.3), this means that $\underline{u} = u_+$ is a subsolution of problem ($P_{\mu,a}$).

Thanks to assumption (3.3) it turns out that $\overline{u} \equiv c_1$ is a supersolution of ($P_{\mu,a}$). By means of (3.10) and (3.2), as well as the assumption $c_1 \geq \delta$, we note that

$$\underline{u}(x) \leq \|\underline{u}\|_{L^\infty(\Omega)} \leq \left(\frac{a}{b}\right)^{\frac{1}{p-q}} \leq \delta \leq c_1 = \overline{u}(x) \quad \text{for all } x \in \Omega.$$

We have thus a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of problem ($P_{\mu,a}$) satisfying $\underline{u} \le \overline{u}$. Therefore, taking into account (3.5), the general method of sub-supersolution for quasilinear elliptic equations as presented in Motreanu-Tornatore [10, Theorem 3.1] (see also Carl-Le-Motreanu [2, Theorem 3.17]) can be carried out to problem ($P_{\mu,a}$) with the ordered pair $\underline{u} \le \overline{u}$. It gives the existence of a weak solution u with the enclosure property $0 < u_+ \le u \le c_1$. Again by the nonlinear regularity up to the boundary, we have that $u \in C^{1,\beta}(\overline{\Omega})$ with some $\beta \in (0,1)$.

Let us prove the existence of a negative solution to problem ($P_{\mu,a}$). Consider the auxiliary problem

$$\begin{aligned} -\Delta_p u - \mu(x)\Delta_q u + b|u|^{p-2}u &= -a(u^-)^{q-1} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.11}$$

The energy functional $J_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated to (3.11) is defined by

$$J_-(v) = \frac{1}{p}\int_\Omega \left(|\nabla v|^p + b|v|^p\right)dx + \frac{1}{q}\int_\Omega \mu(x)|\nabla v|^q dx - \frac{a}{q}\int_\Omega \left(v^-\right)^q dx.$$

As before we can show that there exists a global minimizer v_- of the functional J_-, which is a nontrivial weak solution of (3.11) belonging to $C_0^1(\overline{\Omega})$. Upon acting on (3.11) with $(v_-)^+$, it readily follows that v_- turns out to be a negative weak solution of (3.11). Along the lines of the first part of the proof, arguing this time with the test function $\varphi = |v_-|^\alpha v_-$ in (3.11) for each $\alpha > 0$, we arrive at

$$b\|v_-\|_{L^\infty(\Omega)}^{p-q} \le a. \tag{3.12}$$

Through (3.11), (3.1), (3.2), and (3.12), we find that

$$-\Delta_p v_- - \mu(x)\Delta_q v_- = a|v_-|^{q-2}v_- - b|v_-|^{p-2}v_- \ge a|v_-|^{q-2}v_- - g(x, v_-, \nabla v_-).$$

This amounts to saying that v_- is a negative supersolution of problem ($P_{\mu,a}$).

From (2.3) and (3.4), it is clear that the negative constant $-c_2$ is a subsolution of problem ($P_{\mu,a}$). On the basis of (3.2), (3.12) and because $c_2 \ge \delta$, we see that

$$-c_2 \le -\delta \le -\left(\frac{a}{b}\right)^{\frac{1}{p-q}} \le -\|v_-\|_{L^\infty(\Omega)} \le v_-(x) \quad \text{for all } x \in \Omega.$$

On account of (3.5), we are thus able to implement the method of sub-supersolution in the form of Motreanu-Tornatore [10, Theorem 3.1] (see also Carl-Le-Motreanu [2, Theorem 3.17]) to the quasilinear elliptic problem ($P_{\mu,a}$) with the ordered pair $-c_2 \le v_-$, which leads to the existence of a weak solution to ($P_{\mu,a}$) with $-c_2 \le v \le v_- < 0$ in Ω. The fact that $v \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ is the consequence of the nonlinear regularity theory up to the boundary applied to problem ($P_{\mu,a}$) with the weak solution v. The proof is complete. □

Finally, we focus on extremal solutions to problem ($P_{\mu,a}$).

Corollary 3.2. *Under hypotheses $H(g)$ and $a > \lambda_{1,q,\mu}$, problem ($P_{\mu,a}$) possesses extremal solutions (i.e., the smallest and greatest solution) in each of the ordered sub-supersolution interval $[\underline{u}, \overline{u}]$ obtained by Theorem 3.1.*

Proof. We only prove the existence of the smallest solution in the ordered interval $[u_+, c_1]$. The proof for the existence of the greatest solution in $[u_+, c_1]$, as well as for the extremal solutions in the ordered interval $[-c_2, v_-]$, can be done analogously.

Denote by $\mathcal{S}$ the set of solutions to problem ($P_{\mu,a}$) belonging to $[u_+, c_1]$. Theorem 3.1 ensures that $\mathcal{S}$ is nonempty. It is well-known that there exists a sequence

$\{u_n\}_{n\geq 1}$ in $\mathcal{S}$ such that with respect to the pointwise order in $W_0^{1,p}(\Omega)$ and the pointwise convergence it holds

$$\inf \mathcal{S} = \lim_{n\to\infty} u_n. \tag{3.13}$$

Since $u_n \in \mathcal{S}$, by (1.1), it satisfies (1.1), that is

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi dx + \int_\Omega \mu(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla\varphi dx \\ = a\int_\Omega |u_n|^{q-2}u_n\varphi dx - \int_\Omega g(x,u_n,\nabla u_n)\varphi dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega).\end{aligned} \tag{3.14}$$

If we insert $\varphi = u_n$ in (3.14) and use that the sequence $\{u_n\}_{n\geq 1}$ is uniformly bounded, namely $u_+ \leq u_n \leq c_1$ (see (2.4)), by (3.5) we infer that

$$\int_\Omega |\nabla u_n|^p dx + \int_\Omega \mu(x)|\nabla u_n|^q dx \leq a\int_\Omega |u_n|^q dx + C\int_\Omega (1+|\nabla u_n|^\gamma)dx,$$

with a constant $C > 0$. Due to $\gamma < p$, it turns out that the sequence $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$, thus, up to a subsequence, $u_n \rightharpoonup u$ for some $u \in W_0^{1,p}(\Omega)$. Through (3.14) with $\varphi = u_n - u$, in conjunction with (3.5) and Hölder's inequality, we derive that

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u)dx \leq 0.$$

Then the S_+-property of $-\Delta_p$ on $W_0^{1,p}(\Omega)$, see Carl-Le-Motreanu [2, Theorem 2.109], implies the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$. We can pass to the limit as $n \to \infty$ in (3.14), whence $u \in \mathcal{S}$. In view of (3.13), the desired conclusion ensues. $\square$

We end the paper with a simple example of term $g(x,s,\xi)$ in problem ($P_{\mu,a}$) verifying assumptions H(g). For simplicity we drop the dependence on $x \in \Omega$ in $g(x,s,\xi)$.

Example 3.3. *Let $1 < q < p < +\infty$, a weight function $\mu : \Omega \to \mathbb{R}$ with $\mu \in L^\infty(\Omega)$ and $\operatorname{ess\,inf}_\Omega \mu > 0$, and $a > \lambda_{1,q,\mu}$, for which we state problem ($P_{\mu,a}$). For fixed constants $b_1, b_2 \geq a$, $0 < r_1, r_2 < p-1$, $\gamma_1,\gamma_2 \in [0, \frac{p}{(p^*)'})$ and $d_1, d_2 > 0$, we introduce the continuous function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by*

$$g(x,s,\xi) = \begin{cases} -as^{r_1} + b_1 s^{p-1} - d_1|\xi|^{\gamma_1}s & \text{if } s \geq 0 \\ as^{r_2} - b_2|s|^{p-1} - d_2|\xi|^{\gamma_2}s & \text{if } s < 0. \end{cases}$$

Condition H(g) is satisfied taking for instance $\delta = 1$, $b = \max\{b_1,b_2\}$, $\gamma = \max\{\gamma_1,\gamma_2\}$ and a sufficiently large $M > 0$.

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