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Sign changing solutions for critical double phase problems with variable exponent

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Abstract. In this paper, we deal with a double phase problem with variable exponent and a right-hand side consisting of a Carathéodory perturbation defined only locally and of a critical term. We stress that the presence of the critical term inhibits the possibility to apply results of the critical point theory to the corresponding energy functional. Instead, we use suitable cut-off functions and truncation techniques in order to work with a coercive functional. Then, using variational tools and an appropriate auxiliary coercive problem, we can produce a sequence of sign changing solutions to our main problem converging to 0 in L^∞ and in the Musielak–Orlicz Sobolev space.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with Lipschitz boundary $\partial\Omega$. In this paper, we study the following critical double phase Dirichlet problem:

$$\begin{aligned} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\big) &= f(x,u) + |u|^{p^*-2}u \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where the exponents and the weight function satisfy the following condition:

(H1) $q \in C(\overline{\Omega})$ is such that $1 < p < N$, $p < q(x) < p^* := \frac{Np}{N-p}$ for all $x \in \overline{\Omega}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$.

For $r \in C(\overline{\Omega})$, we put

$$r^- = \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r^+ = \max_{x\in\overline{\Omega}} r(x).$$

Then we assume the following hypotheses on $f(\cdot,\cdot)$:

(H2) $f\colon \Omega \times [-\eta_0, \eta_0] \to \mathbb{R}$ is a Carathéodory function for $\eta_0 > 0$ with $f(x,0) = 0$, $f(x,\cdot)$ is odd for a.a. $x \in \Omega$ and

(i) there exists $a_0 \in L^\infty(\Omega)$ such that

$$|f(x,s)| \leq a_0(x) \quad \text{for a.a. } x \in \Omega \text{ and for all } |s| \leq \eta_0;$$

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(ii) there exists $\tau \in (1, \min\{p, \frac{p^2}{N-p} + 1\})$ such that

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{\tau-2}s} = 0 \quad \text{uniformly for a.a. } x \in \Omega;$$

(iii)

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$

Remark 1.1. Note that f is defined only locally. Therefore, according to

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,$$

we can suppose, without any loss of generality, that

$$\frac{f(x,s)}{|s|^{p-2}s} > 0 \quad \text{for a.a. } x \in \Omega \text{ and all } |s| \leq \eta_0,$$

which implies

$$f(x,s) > 0 \quad \text{for all } 0 < s \leq \eta_0 \quad \text{and} \quad f(x,s) < 0 \quad \text{for all } -\eta_0 \leq s < 0.$$

We call a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ a weak solution of problem (1.1) if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla h \,\mathrm{d}x = \int_\Omega \left(f(x,u) + |u|^{p^*-2}u\right)h \,\mathrm{d}x$$

is satisfied for all $h \in W_0^{1,\mathcal{H}}(\Omega)$.

Our main result reads as follows.

Theorem 1.2. *Let hypotheses (H1) and (H2) be satisfied. Then problem (1.1) has a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ of sign-changing solutions such that $\|w_n\| \to 0$ and $\|w_n\|_\infty \to 0$ as $n \to \infty$.*

In the right-hand side of (1.1), we have the combined effects of a Carathéodory perturbation $f(x,\cdot)$ which is defined only locally and of a critical term $u \to |u|^{p^*-2}u$, where $p^* := \frac{Np}{N-p}$ is the critical exponent corresponding to p. We note that the presence of the critical term inhibits the possibility to apply results of the critical point theory to the corresponding energy functional. Consequently, here we introduce suitable cut-off functions and truncation techniques to deal with a coercive functional so that we can act by using variational tools. Thus, we work on an auxiliary coercive problem and we show the existence of extremal constant sign solutions for such a problem (see Section 3). Then we apply these extremal solutions and a generalized version of the symmetric mountain pass theorem due to Kajikiya [18, Theorem 1] in order to produce a sequence of sign changing solutions for problem (1.1). In this way, we extend the results of Liu–Papageorgiou [24] to the double phase operator with one variable exponent, and we were able to skip condition H_1 (iii) in [24].

Recall that functionals of type

$$\omega \mapsto \int_\Omega \left(|\nabla \omega|^p + \mu(x)|\nabla \omega|^q\right) \mathrm{d}x, \quad 1 < p < q < N,$$

were first considered by Zhikov [36] in order to describe strongly anisotropic materials in the context of homogenization and elasticity; we refer also to applications in the study of duality theory and of the Lavrentiev gap phenomenon; see Zhikov [37, 38]. A first mathematical framework for such type of functionals has been done by Baroni–Colombo–Mingione [4]; see also the related works by the same authors in [5, 6] and of De Filippis–Mingione [10] about nonautonomous integrals.

Even though double phase differential operators and corresponding energy functionals appear in several physical applications, there are only few results involving the variable exponent double phase operator. We refer to the recent results of Aberqi–Bennouna–Benslimane–Ragusa [1] for existence results in complete manifolds, Albalawi–Alharthi–Vetro [2] for convection problems with $(p(\cdot), q(\cdot))$-Laplace type problems, Bahrouni–Rădulescu–Winkert [3] for double phase problems of Baouendi–Grushin type operator, Crespo–Blanco–Gasiński–Harjulehto–Winkert [8] for double phase convection problems, Kim–Kim–Oh–Zeng [19] for concave-convex-type double phase problems, Leonardi–Papageorgiou [21] for concave-convex problems, Vetro–Winkert [33] for parametric problems involving superlinear nonlinearities and Zeng–Rădulescu-Winkert [35] for multivalued problems; see also the references therein. In order to enlarge the literature on the topic, we refer to the papers of Colasuonno–Squassina [7] for eigenvalue problems of double phase type, Farkas–Winkert [12] for Finsler double phase problems, Gasiński–Papageorgiou [13] for locally Lipschitz right-hand sides, Gasiński–Winkert [14, 15] for convection problems and constant sign-solutions, Liu–Dai [23] for a Nehari manifold approach, Papageorgiou–Vetro [27] for superlinear problems, Papageorgiou–Vetro–Vetro [28] for parametric Robin problems, Perera–Squassina [30] for a Morse theoretical approach, Vetro–Winkert [32] for parametric convective problems, Zeng–Bai–Gasiński–Winkert [34] for implicit obstacle problems with multivalued operators.

2. Preliminaries

In this section, we recall the main properties of the Musielak–Orlicz Sobolev spaces and tools which we will need later. To this end, let $M(\Omega)$ be the set of all measurable functions $u\colon \Omega \to \mathbb{R}$. For a given $r \in C(\overline{\Omega})$ with $r(x) > 1$ for all $x \in \overline{\Omega}$, we denote by $L^{r(\cdot)}(\Omega)$ the usual variable exponent Lebesgue space defined by

$$L^{r(\cdot)}(\Omega) = \left\{ u \in M(\Omega) : \varrho_r(u) := \int_\Omega |u|^{r(x)} \,\mathrm{d}x < +\infty \right\}$$

and equip it with the Luxemburg norm

$$\|u\|_{r(\cdot)} = \inf\left\{ \beta > 0 : \varrho_r\left(\frac{u}{\beta}\right) \leq 1 \right\}.$$

Similarly, we can define the corresponding Sobolev spaces $W^{1,r(\cdot)}(\Omega)$ and $W_0^{1,r(\cdot)}(\Omega)$ endowed with the norms $\|\cdot\|_{1,r(\cdot)}$ and $\|\nabla\cdot\|_{r(\cdot)}$, respectively; see Diening–Harjulehto–Hästö–Růžička [11] or Harjulehto–Hästö [16].

Now, under assumption (H1), we introduce the nonlinear function $\mathcal{H}\colon \Omega \times [0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{H}(x,t) = t^p + \mu(x)t^{q(x)} \quad \text{for all } x \in \Omega \text{ and for all } t \geq 0.$$

Then we can introduce the Musielak–Orlicz space $L^{\mathcal{H}}(\Omega)$ by

$$L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) : \rho_{\mathcal{H}}(u) < +\infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf\left\{\beta > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\beta}\right) \leq 1\right\},$$

where the modular $\rho_{\mathcal{H}}(\cdot)$ is given by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^{q(x)}\right)\mathrm{d}x.$$

Using the Musielak–Orlicz space, we define the corresponding Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ by

$$W^{1,\mathcal{H}}(\Omega) = \{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\}$$

and endow it with the norm

$$\|u\|_{1,\mathcal{H}} := \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} := \||\nabla u|\|_{\mathcal{H}}$. Furthermore, we denote by $W_0^{1,\mathcal{H}}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. We point out that the norm $\|\cdot\|_{\mathcal{H}}$ defined on $L^{\mathcal{H}}(\Omega)$ is uniformly convex and hence the spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive Banach spaces; see Crespo–Blanco–Gasiński–Harjulehto–Winkert [8, Proposition 2.12]. In addition, based on [8, Proposition 2.18], we can equip the space $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\|u\| := \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega).$$

The next proposition gives some important embedding results for the space $W_0^{1,\mathcal{H}}(\Omega)$; see Crespo–Blanco–Gasiński–Harjulehto–Winkert [8, Proposition 2.16].

Proposition 2.1. *Let hypothesis (H1) be satisfied. Then the following hold:*

(i) $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ *is continuous for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p$ *for all* $x \in \overline{\Omega}$*;*

(ii) $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is compact for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p^*$ *for all* $x \in \overline{\Omega}$.

Now, we point out the relation between the modular $\rho_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$; see Crespo–Blanco–Gasiński–Harjulehto–Winkert [8, Proposition 2.13].

Proposition 2.2. *Let hypothesis (H1) be satisfied. Then the following hold:*

(i) $\|u\|_{\mathcal{H}} < 1$ *(resp.* $\|u\|_{\mathcal{H}} > 1$, $\|u\|_{\mathcal{H}} = 1$*) if and only if* $\rho_{\mathcal{H}}(u) < 1$ *(resp.* $\rho_{\mathcal{H}}(u) > 1$, $\rho_{\mathcal{H}}(u) = 1$*);*

(ii) *if* $\|u\|_{\mathcal{H}} < 1$ *then* $\|u\|_{\mathcal{H}}^{q^+} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{p}$*;*

(iii) *if* $\|u\|_{\mathcal{H}} > 1$ *then* $\|u\|_{\mathcal{H}}^{p} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{q^+}$*;*

(iv) $\|u\|_{\mathcal{H}} \to 0$ *if and only if* $\rho_{\mathcal{H}}(u) \to 0$*;*

(v) $\|u\|_{\mathcal{H}} \to +\infty$ *if and only if* $\rho_{\mathcal{H}}(u) \to +\infty$*.*

Let $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A(u), v\rangle_{\mathcal{H}} = \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla v \,\mathrm{d}x$$

for all $u, v \in W_0^{1,\mathcal{H}}(\Omega)$ with $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ being the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. The properties of the operator $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ are summarized in the next proposition; see Crespo–Blanco–Gasiński–Harjulehto–Winkert [8, Theorem 3.3].

Proposition 2.3. *Let hypothesis (H1) be satisfied. Then the operator A is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone (hence maximal monotone), of type* (S_+)*, coercive and a homeomorphism.*

As usual, we denote by $C_0^1(\overline{\Omega})$ the ordered Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\},$$

with positive cone

$$C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \; \forall x \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \left\{u \in C_0^1(\overline{\Omega}) : u(x) > 0 \; \forall x \in \Omega \text{ and } \frac{\partial u}{\partial n}(x) < 0 \; \forall x \in \partial\Omega\right\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$.

We complete this section with some known results on the spectrum of the r-Laplacian with $1 < r < \infty$ and homogeneous Dirichlet boundary condition given by

$$\begin{aligned} -\Delta_r u &= \lambda |u|^{r-2}u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

We call a number $\lambda \in \mathbb{R}$ an eigenvalue of (2.1) if problem (2.1) has a nontrivial solution $u \in W_0^{1,r}(\Omega)$. Such a solution is called an eigenfunction corresponding to the eigenvalue λ.

From Lê [20], we know that there exists a smallest eigenvalue $\lambda_{1,r}$ of (2.1) which is positive, isolated, simple and it can be variationally characterized through

$$\lambda_{1,r} = \inf\left\{ \frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega),\ u \neq 0 \right\}. \tag{2.2}$$

In what follows, we denote by $u_{1,r}$ the L^r-normalized (i.e., $\|u_{1,r}\|_r = 1$) positive eigenfunction corresponding to $\lambda_{1,r}$. The nonlinear regularity theory and the nonlinear maximum principle imply that $u_{1,r} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$; see Lieberman [22] and Pucci–Serrin [31].

For any $s \in \mathbb{R}$, we put $s_\pm = \max\{\pm s, 0\}$, that means, $s = s_+ - s_-$ and $|s| = s_+ + s_-$. Also, for any function $v\colon \Omega \to \mathbb{R}$, we put $v_\pm(\cdot) = [v(\cdot)]_\pm$.

Given a Banach space X and its dual space X^*, we say that a functional $\varphi \in C^1(X)$ satisfies the Palais–Smale condition (PS-condition for short) if every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(x_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(x_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence. Moreover, we denote by K_φ the set of all critical points of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

We also recall that a set $\mathcal{S} \subseteq X$ is said to be downward directed if, for given $u_1, u_2 \in \mathcal{S}$, we can find $u \in \mathcal{S}$ such that $u \le u_1$ and $u \le u_2$. Analogously, $\mathcal{S} \subseteq X$ is said to be upward directed if, for given $v_1, v_2 \in \mathcal{S}$, we can find $v \in \mathcal{S}$ such that $v_1 \le v$ and $v_2 \le v$.

3. An auxiliary problem

In this section, we consider an auxiliary problem in order to prove Theorem 1.2 in the next section. For this purpose, let $\theta \in C^1(\mathbb{R})$ be an even cut-off function satisfying the following conditions:

$$\operatorname{supp}\theta \subseteq [-\eta_0, \eta_0], \quad \theta|_{[\frac{-\eta_0}{2}, \frac{\eta_0}{2}]} \equiv 1 \quad \text{and} \quad 0 < \theta \le 1 \quad \text{on } (-\eta_0, \eta_0). \tag{3.1}$$

Taking θ into account, we define the Carathéodory function $k\colon \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$k(x,s) = \theta(s)[f(x,s) + |s|^{p^*-2}s] + (1-\theta(s))|s|^{\tau-2}s \tag{3.2}$$

for all $(x,s) \in \Omega \times \mathbb{R}$, where τ is given in (H2) (ii). Note that, from (3.1) and (H2) (ii), we get that

$$|k(x,s)| \le c(1 + |s|^{\tau-1}) \tag{3.3}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$ with some $c > 0$.

Next, we study the following auxiliary double phase Dirichlet problem

$$\begin{aligned} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\big) &= k(x,u) \quad \text{in } \Omega,\\ u &= 0 \qquad\quad\ \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

Our aim is to show the existence of extremal constant sign solutions for problem (3.4). We are going to need these extremal solutions in order to produce sign changing solutions for problem (1.1).

Let $\mathcal{S}_+$ and $\mathcal{S}_-$ be the sets of positive and negative solutions of problem (3.4), respectively.

Proposition 3.1. *Let hypotheses (H1) and (H2) be satisfied. Then $\mathcal{S}_+$ and $\mathcal{S}_-$ are nonempty subsets in $W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.*

Proof. We start by proving that $\mathcal{S}_+ \neq \emptyset$ and denote by $\Phi_+ \colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ the C^1-functional defined by

$$\Phi_+(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right] \mathrm{d}x - \int_\Omega K(x, u_+) \, \mathrm{d}x$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$, where $K(x,s) = \int_0^s k(x,t)\,\mathrm{d}t$. First, we have

$$\begin{aligned}
\Phi_+(u) &\geq \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q^+}\int_\Omega \mu(x)|\nabla u|^{q(x)}\,\mathrm{d}x - \int_\Omega K(x,u_+)\,\mathrm{d}x \\
&\geq \frac{1}{q^+}\rho_{\mathcal{H}}(|\nabla u|) - \int_\Omega K(x,u_+)\,\mathrm{d}x.
\end{aligned}$$

Combining this and (3.3) along with $\tau < p$ (see (H2) (ii)) and Proposition 2.2 (iii), it is clear that Φ_+ is coercive. In addition, thanks to the compactness of the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ for any $r \in C(\overline{\Omega})$ with $1 \leq r(x) < p^*$ for all $x \in \overline{\Omega}$ (see Proposition 2.1 (ii)), we conclude that the functional Φ_+ is sequentially weakly lower semicontinuous. Then there exists $u_0 \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\Phi_+(u_0) = \inf[\Phi_+(u) : u \in W_0^{1,\mathcal{H}}(\Omega)].$$

Let us prove that u_0 is nontrivial. From hypothesis (H2) (iii), we can find for each $\eta > 0$ a number $\delta \in (0, \min\{\frac{\eta_0}{2}, 1\})$ such that

$$F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t \geq \frac{\eta}{p}|s|^p \quad \text{for all } |s| \leq \delta. \tag{3.5}$$

Further, we can take $t \in (0,1)$ small enough so that $tu_{1,p}(x) \in (0,\delta]$ for all $x \in \overline{\Omega}$, where $u_{1,p} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ is the L^p-normalized positive eigenfunction corresponding to $\lambda_{1,p}$ (see Section 2). Thus, we have, using (2.2),

$$\begin{aligned}
\Phi_+(tu_{1,p}) &= \int_\Omega \left[\frac{1}{p}|\nabla(tu_{1,p})|^p + \frac{\mu(x)}{q(x)}|\nabla(tu_{1,p})|^{q(x)}\right]\mathrm{d}x - \int_\Omega K(x,tu_{1,p})\,\mathrm{d}x \\
&\leq \frac{t^p}{p}\int_\Omega |\nabla u_{1,p}|^p\,\mathrm{d}x + \frac{t^{q^-}}{q^-}\int_\Omega \mu(x)|\nabla u_{1,p}|^{q(x)}\,\mathrm{d}x - \int_\Omega K(x,tu_{1,p})\,\mathrm{d}x \\
&= \frac{t^p}{p}\lambda_{1,p} + \frac{t^{q^-}}{q^-}\int_\Omega \mu(x)|\nabla u_{1,p}|^{q(x)}\,\mathrm{d}x - \int_\Omega K(x,tu_{1,p})\,\mathrm{d}x.
\end{aligned} \tag{3.6}$$

Since $tu_{1,q} \in (0, \delta]$ and $\delta \in (0, \min\{\frac{\eta_0}{2}, 1\})$ from (3.1), we deduce that

$$k(x, tu_{1,p}) = f(x, tu_{1,p}) + (tu_{1,p})^{p^*-2} tu_{1,p} \geq f(x, tu_{1,p}). \tag{3.7}$$

Then, using (3.5) and (3.7) in (3.6), we obtain

$$\begin{aligned}\Phi_+(tu_{1,p}) &\leq \frac{t^p}{p}\lambda_{1,p} + \frac{t^{q^-}}{q^-}\int_\Omega \mu(x)|\nabla u_{1,p}|^{q(x)}\,\mathrm{d}x - \frac{t^p}{p}\eta \\ &= \frac{t^p}{p}(\lambda_{1,p} - \eta) + \frac{t^{q^-}}{q^-}\int_\Omega \mu(x)|\nabla u_{1,p}|^{q(x)}\,\mathrm{d}x.\end{aligned}$$

If we choose $\eta > \lambda_{1,p}$, then $\lambda_{1,p} - \eta < 0$, and thus, for $t > 0$ sufficiently small, we have

$$\frac{t^p}{p}(\lambda_{1,p} - \eta) + \frac{t^{q^-}}{q^-}\int_\Omega \mu(x)|\nabla u_{1,p}|^{q(x)}\,\mathrm{d}x < 0$$

since $p < q^-$. Hence, we have $\Phi_+(tu_{1,p}) < 0 = \Phi_+(0)$ for $t \in (0, 1)$ sufficiently small, which implies that $u_0 \neq 0$.

Recall that u_0 is a global minimizer of Φ_+. Hence, $\Phi'_+(u_0) = 0$, that is,

$$\begin{aligned}&\int_\Omega \big(|\nabla u_0|^{p-2}\nabla u_0 + \mu(x)|\nabla u_0|^{q(x)-2}\nabla u_0\big)\cdot\nabla h\,\mathrm{d}x \\ &\quad = \int_\Omega k\big(x, (u_0)_+\big)h\,\mathrm{d}x\end{aligned} \tag{3.8}$$

for all $h \in W_0^{1,\mathcal{H}}(\Omega)$. Note that $\pm u_\pm \in W_0^{1,\mathcal{H}}(\Omega)$ for any $u \in W_0^{1,\mathcal{H}}(\Omega)$; see Crespo–Blanco–Gasiński–Harjulehto–Winkert [8, Proposition 2.17]. So, if we choose $h = -(u_0)_-$ in (3.8), then we obtain that $(u_0)_- = 0$. This gives $u_0 \geq 0$. Taking into account that $u_0 \neq 0$, we conclude that u_0 is a nontrivial positive weak solution of problem (3.4). Hence, it follows that $\mathcal{S}_+ \neq \emptyset$. From Crespo–Blanco–Winkert [9, Theorem 3.1], we know that $u_0 \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.

In a similar way, we get the existence of a nontrivial negative weak solution of problem (3.4). In this case, we work with the C^1-functional $\Phi_-\colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined by

$$\Phi_-(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right]\mathrm{d}x - \int_\Omega K(x, -u_-)\,\mathrm{d}x$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$ and show that it has a global minimizer which turns out to be nontrivial and nonpositive. Hence, it must be a nontrivial negative weak solution of problem (3.4). ■

Now, we are going to prove the existence of extremal solution of (3.4), that is, the existence of a smallest positive solution $u_* \in \mathcal{S}_+$ and the existence of a largest negative solution $v_* \in \mathcal{S}_-$.

Proposition 3.2. *Let hypotheses (H1) and (H2) be satisfied. Then there exists $u_* \in \mathcal{S}_+$ such that $u_* \leq u$ for all $u \in \mathcal{S}_+$ and there exists $v_* \in \mathcal{S}_-$ such that $v_* \geq v$ for all $v \in \mathcal{S}_-$.*

Proof. We start by proving the existence of a smallest positive solution of (3.4). Similar to the proof of [25, Proposition 7] by Papageorgiou–Rădulescu–Repovš, we can show that $\mathcal{S}_+$ is downward directed. Then, from Hu–Papageorgiou [17, Lemma 3.10, p. 178], we know that we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf \mathcal{S}_+.$$

Also, since $u_n \in \mathcal{S}_+$, we have

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right) \cdot \nabla h \,\mathrm{d}x = \int_\Omega k(x,u_n)h \,\mathrm{d}x \tag{3.9}$$

for all $h \in W_0^{1,\mathcal{H}}(\Omega)$ and for all $n \in \mathbb{N}$. If we take $h = u_n$ in (3.9), using (3.3) and $0 \le u_n \le u_1$, we get that

$$\rho_{\mathcal{H}}(\nabla u_n) = \int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega \mu(x)|\nabla u_n|^{q(x)} \,\mathrm{d}x < c_1$$

for some $c_1 > 0$ and for all $n \in \mathbb{N}$. From this and Proposition 2.2, we deduce that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega)$$

is bounded. Moreover, due to hypothesis (H2), we have $\tau < \frac{p^2}{N-p} + 1$, which implies that $\frac{N}{p}(\tau-1) < p^*$. Now, we choose $s > \frac{N}{p}$ such that $s(\tau-1) < p^*$. Then, taking into account that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{H}}(\Omega)$ is bounded, we can assume that

$$u_n \rightharpoonup u_* \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad u_n \to u_* \quad \text{in } L^{s(\tau-1)}(\Omega).$$

From (3.1), (3.2) and hypothesis (H2) (i), it follows that

$$|k(x,s)| \le b_1|s|^{\tau-1} \tag{3.10}$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$ and for some $b_1 > 0$. Then, from (3.9) and (3.10) along with a Moser-iteration type argument as it was explained by Colasuonno–Squassina [7, Section 3.2], we obtain, as $s > \frac{N}{p}$, that

$$\|u_n\|_\infty \le b_2\|u_n\|_{s(\tau-1)}^{\frac{\tau-1}{p-1}}$$

for some $b_2 > 0$ and for all $n \in \mathbb{N}$.

Suppose now $u_* = 0$; then $\|u_n\|_\infty \to 0$ as $n \to +\infty$. This implies the existence of $n_0 \in \mathbb{N}$ such that

$$0 < u_n(x) \le \delta$$

for a.a. $x \in \Omega$ and for all $n \ge n_0$, where $\delta \in (0, \min\{\frac{\eta_0}{2}, 1\})$. Hence, in view of (3.1) and (3.2), it follows that

$$k(x,u_n(x)) = f(x,u_n(x)) + u_n(x)^{p^*-1} \tag{3.11}$$

for a.a. $x \in \Omega$ and for all $n \geq n_0$. Now, we put $y_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$; then $\|y_n\| = 1$ and $y_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \rightharpoonup y \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega)$$

with $y \geq 0$. From (3.9) and (3.11), we get

$$\begin{aligned}
&\int_\Omega \left(\|u_n\|^{p-1}|\nabla y_n|^{p-2}\nabla y_n + \mu(x)\|u_n\|^{q(x)-1}|\nabla y_n|^{q(x)-2}\nabla y_n\right)\cdot \nabla h \,\mathrm{d}x\\
&\quad = \int_\Omega \|u_n\|^{p-1}\left[\frac{f(x,u_n)}{u_n^{p-1}} + u_n^{p^*-p}\right] y_n^{p-1} h \,\mathrm{d}x
\end{aligned}$$

for all $h \in W_0^{1,\mathcal{H}}(\Omega)$ and for all $n \geq n_0$, which can be equivalently written as

$$\begin{aligned}
&\int_\Omega |\nabla y_n|^{p-2}\nabla y_n \cdot \nabla h \,\mathrm{d}x + \int_\Omega \|u_n\|^{q(x)-p}|\nabla y_n|^{q(x)-2}\nabla y_n \cdot \nabla h \,\mathrm{d}x\\
&\quad = \int_\Omega \left[\frac{f(x,u_n)}{u_n^{p-1}} + u_n^{p^*-p}\right] y_n^{p-1} h \,\mathrm{d}x
\end{aligned} \tag{3.12}$$

for all $h \in W_0^{1,\mathcal{H}}(\Omega)$ and for all $n \geq n_0$. We point out that the left-hand side of (3.12) is bounded for all $h \in W_0^{1,\mathcal{H}}(\Omega)$. From this, using hypothesis (H2) (ii), we infer

$$y = 0 \quad \text{and} \quad \frac{f(x,u_n)}{u_n^{p-1}} y_n^{p-1} \to 0 \quad \text{for a.a. } x \in \Omega.$$

In addition, if we take $h = y_n$ in (3.12) and pass to the limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \int_\Omega |\nabla y_n|^p = 0.$$

This implies, at least for a susequence, that $\nabla y_n(x) \to 0$ for a.a. $x \in \Omega$, and hence we deduce that $\mathcal{H}(\nabla y_n) \to 0$ for a.a. $x \in \Omega$. Taking into account that $\{\mathcal{H}(\nabla y_n)\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ is uniformly integrable by Vitali's convergence theorem, we get that

$$\rho_{\mathcal{H}}(\nabla y_n) \to 0 \quad \text{in } W_0^{1,\mathcal{H}}(\Omega). \tag{3.13}$$

Now, we recall that $\|y_n\| = 1$, and this implies that $\rho_{\mathcal{H}}(\nabla y_n) = 1$ for all $n \in \mathbb{N}$; see Proposition 2.2 (i). This gives a contradiction to (3.13). Therefore, $u_* \neq 0$, and so $u_* \in \mathcal{S}_+$ with u_* being the smallest positive solution of (1.1) in $\mathcal{S}_+$. Proceeding in a similar way, we can show that $v_* \in \mathcal{S}_-$ such that $v_* = \sup \mathcal{S}_-$. ∎

4. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2, that is, we prove the existence of a sequence of sign changing solutions for problem (1.1), which converges to 0 in $W_0^{1,\mathcal{H}}(\Omega)$ and in $L^\infty(\Omega)$. Our strategy is to use the extremal constant sign solutions u_* and v_*

obtained in Proposition 3.2 and focus on the order interval

$$[v_*, u_*] := \{u \in W_0^{1,\mathcal{H}}(\Omega) : v_*(x) \leq u(x) \leq u_*(x) \text{ for a.a. } x \in \Omega\}.$$

For this purpose, we use truncations of $k(x,\cdot)$ at $v_*(x)$ and $u_*(x)$, that is, we consider the function $k_*\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_*(x,s) := \begin{cases} k(x, v_*(x)) & \text{if } s < v_*(x), \\ k(x,s) & \text{if } v_*(x) \leq s \leq u_*(x), \\ k(x, u_*(x)) & \text{if } u_*(x) < s. \end{cases}$$

Then we introduce the C^1-functional $\Psi_*\colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ by

$$\Psi_*(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q(x)}|\nabla u|^{q(x)}\right] \mathrm{d}x - \int_\Omega K_*(x,u)\,\mathrm{d}x$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$, where $K_*(x,s) = \int_0^s k_*(x,t)\,\mathrm{d}t$.

First, we point out that $K_{\Psi_*} = \{u \in W_0^{1,\mathcal{H}}(\Omega) : (\Psi_*)'(u) = 0\}$ is contained in the order interval $[v_*, u_*]$. In fact, let $u \in K_{\Psi_*} \setminus \{u_*, v_*\}$; then we have

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla h\,\mathrm{d}x \\ &\quad = \int_\Omega k_*(x,u)h\,\mathrm{d}x \quad \text{for all } h \in W_0^{1,\mathcal{H}}(\Omega). \end{aligned} \tag{4.1}$$

Taking the function test $h = (u - u_*)_+$ in (4.1), we get

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla(u-u_*)_+\,\mathrm{d}x \\ &\quad = \int_\Omega k_*(x,u)(u-u_*)_+\,\mathrm{d}x \\ &\quad = \int_\Omega k(x,u_*)(u-u_*)_+\,\mathrm{d}x \\ &\quad = \int_\Omega \left(|\nabla u_*|^{p-2}\nabla u_* + \mu(x)|\nabla u_*|^{q(x)-2}\nabla u_*\right) \cdot \nabla(u-u_*)_+\,\mathrm{d}x \end{aligned}$$

since $u_* \in \mathcal{S}_+$. This implies that

$$\begin{aligned} &\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_*|^{p-2}\nabla u_*) \cdot \nabla(u-u_*)_+\,\mathrm{d}x \\ &\quad + \int_\Omega \mu(x)(|\nabla u|^{q(x)-2}\nabla u - |\nabla u_*|^{q(x)-2}\nabla u_*) \cdot \nabla(u-u_*)_+\,\mathrm{d}x = 0. \end{aligned}$$

Hence, we deduce that $u \leq u_*$. Similarly, if we choose the function test $h = (v_* - u)_+$ in (4.1), then we easily check that $v_* \leq u$.

Let $V \subseteq W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ be a finite-dimensional subspace. Then we have the following result.

Proposition 4.1. *Let hypotheses (H1) and (H2) be satisfied. Then we can find $r_V > 0$ such that*

$$\sup[\Psi_*(v) : v \in V,\ \|v\| = r_V] < 0.$$

Proof. Since, V is finite-dimensional, all the norms on V are equivalent; see, for example, Papageorgiou–Winkert [29, Proposition 3.1.17, p. 183]. This allows us to find $r_V > 0$ such that

$$v \in V \text{ and } \|v\| \leq r_V \text{ imply } |v(x)| \leq \delta \text{ for a.a. } x \in \Omega$$

with $\delta \in (0, \min\{\frac{\eta_0}{2}, 1\})$. In particular, we have $\delta < \frac{\eta_0}{2}$, which implies that $\theta(v(x)) = 1$ for a.a. $x \in \Omega$; see (3.1). Taking this into account, for $v \in V$ with $\|v\| \leq r_V$, we have

$$k_*(x, v(x)) = \begin{cases} f(x, v_*(x)) + |v_*(x)|^{p^*-2} v_*(x) & \text{if } v(x) < v_*(x), \\ f(x, v(x)) + |v(x)|^{p^*-2} v(x) & \text{if } v_*(x) \leq v(x) \leq u_*(x), \\ f(x, u_*(x)) + |u_*(x)|^{p^*-2} u_*(x) & \text{if } u_*(x) < v(x). \end{cases}$$

We denote by $f_* \colon \Omega \times \mathbb{R} \to \mathbb{R}$ the function given by

$$f_*(x, v(x)) = \begin{cases} f(x, v_*(x)) & \text{if } v(x) < v_*(x), \\ f(x, v(x)) & \text{if } v_*(x) \leq v(x) \leq u_*(x), \\ f(x, u_*(x)) & \text{if } u_*(x) < v(x) \end{cases}$$

and put $F_*(x, s) := \int_0^s f_*(x, t)\, \mathrm{d}t$. We point out that, for $v < v_*$, we have

$$\begin{aligned} F_*(x, v) &= \int_0^{v_*} f_*(x, s)\, \mathrm{d}s + \int_{v_*}^{v} f_*(x, s)\, \mathrm{d}s \\ &= \int_0^{v_*} f(x, s)\, \mathrm{d}s + \int_{v_*}^{v} f(x, v_*)\, \mathrm{d}s \\ &= F(x, v_*) + f(x, v_*)(v - v_*). \end{aligned}$$

We recall that $f(x, v_*)$ is negative (see Remark 1.1); hence, $f(x, v_*)(v - v_*) > 0$. Using this, we deduce

$$\begin{aligned} F(x, v) - F_*(x, v) &= F(x, v) - F(x, v_*) + f(x, v_*)(v_* - v) \\ &\leq F(x, v) - F(x, v_*), \end{aligned}$$

where $F(x, s) := \int_0^s f(x, t)\, \mathrm{d}t$. Similarly, for $u_* < v$, we have

$$F_*(x, v) = F(x, u_*) + f(x, u_*)(v - u_*),$$

which implies

$$\begin{aligned} F(x, v) - F_*(x, v) &= F(x, v) - F(x, u_*) + f(x, u_*)(u_* - v) \\ &\leq F(x, v) - F(x, u_*) \end{aligned}$$

since $f(x, u_*)(u_* - v) < 0$; see Remark 1.1.

On account of this, we can write

$$\begin{aligned}
\Psi_*(v) &= \int_\Omega \left[\frac{1}{p}|\nabla v|^p + \frac{\mu(x)}{q(x)}|\nabla v|^{q(x)}\right] \mathrm{d}x - \int_\Omega K_*(x,v)\,\mathrm{d}x \\
&\leq \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q^-}\int_\Omega \mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x \\
&\quad - \int_{\{v<v_*\}} \left[F_*(x,v) + \frac{1}{p^*}|v_*|^{p^*}\right] \mathrm{d}x \\
&\quad - \int_{\{v_*\leq v\leq u_*\}} \left[F(x,v) + \frac{1}{p^*}|v|^{p^*}\right] \mathrm{d}x \\
&\quad - \int_{\{u_*<v\}} \left[F_*(x,v) + \frac{1}{p^*}|u_*|^{p^*}\right] \mathrm{d}x \\
&\leq \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q^-}\int_\Omega \mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x - \int_{\{v<v_*\}} F_*(x,v)\,\mathrm{d}x \\
&\quad - \int_{\{v_*\leq v\leq u_*\}} F(x,v)\,\mathrm{d}x - \int_{\{u_*<v\}} F_*(x,v)\,\mathrm{d}x,
\end{aligned}$$

where we used the abbreviations

$$\begin{aligned}
\{v < v_*\} &:= \{x \in \Omega : v(x) < v_*(x)\}, \\
\{v_* \leq v \leq u_*\} &:= \{x \in \Omega : v_*(x) \leq v(x) \leq u_*(x)\}, \\
\{u_* < v\} &:= \{x \in \Omega : u_*(x) < v(x)\}
\end{aligned}$$

and the fact that the terms

$$\frac{1}{p^*}|v_*|^{p^*}, \quad \frac{1}{p^*}|v|^{p^*} \quad \text{and} \quad \frac{1}{p^*}|u_*|^{p^*}$$

are positive. Furthermore, we have

$$\begin{aligned}
\Psi_*(v) &\leq \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q^-}\int_\Omega \mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x - \int_\Omega F(x,v)\,\mathrm{d}x \\
&\quad + \int_{\{v<v_*\}} [F(x,v) - F_*(x,v)]\,\mathrm{d}x + \int_{\{u_*<v\}} [F(x,v) - F_*(x,v)]\,\mathrm{d}x \\
&\leq \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q^-}\int_\Omega \mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x - \int_\Omega F(x,v)\,\mathrm{d}x \\
&\quad + \int_{\{v<v_*\}} [F(x,v) - F(x,v_*)]\,\mathrm{d}x + \int_{\{u_*<v\}} [F(x,v) - F(x,u_*)]\,\mathrm{d}x.
\end{aligned}$$

Now, as f is odd and thanks to hypothesis (H2) (iii), we know that, for each $\eta > 0$, it is possible to find $\delta \in (0, \min\{\frac{\eta_0}{2}, 1\})$ such that

$$F(x,s) \geq \frac{\eta}{p}|s|^p \quad \text{for all } |s| \leq \delta.$$

Consequently, choosing r_V small enough so that

$$\int_{\{v<v_*\}} [F(x,v) - F(x,v_*)]\,\mathrm{d}x + \int_{\{u_*<v\}} [F(x,v) - F(x,u_*)]\,\mathrm{d}x < \delta^p,$$

we then get

$$\Psi_*(v) \le \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q^-}\int_\Omega \mu(x)|\nabla v|^{q(x)} - \frac{\eta}{p}\int_\Omega |v|^p\,\mathrm{d}x + \delta^p.$$

Next, we remark that

$$\int_\Omega \mu(x)|\nabla v|^{q(x)}\,\mathrm{d}x \le \rho_{\mathcal{H}}(\nabla v) \le \max\{\|v\|^p, \|v\|^{q^+}\}$$

due to Proposition 2.2 (ii), (iii). Also, we recall again that V is finite-dimensional, and so all the norms on V are equivalent. On account of this, we know that there exist positive constants c_1, c_2, c_3, independent of δ, such that

$$\Psi_*(v) \le c_1\|v\|_\infty^p + c_2\max\{\|v\|_\infty^p, \|v\|_\infty^{q^+}\} - \eta\, c_3\|v\|_\infty^p + \delta^p.$$

Further, for $v \in V$ with $\|v\| = r_V$, again using the equivalence of the norms, we get

$$\begin{aligned}\Psi_*(v) &\le c_1\delta^p + c_2\max\{\delta^p, \delta^{q^+}\} - \eta c_3\delta^p + \delta^p\\ &= c_1\delta^p + (c_2 - \eta c_3 + 1)\delta^p\end{aligned}$$

as $\delta < 1$. Therefore, if we choose $\eta > \frac{c_1+c_2+1}{c_3}$, then we have that $\Psi_*(v) < 0$ for all $v \in V$ with $\|v\| = r_V$. This gives the assertion of the proposition. ∎

Now, using Proposition 4.1, we can apply a generalized version of the symmetric mountain pass theorem due to Kajikiya [18, Theorem 1] in order to give the proof of Theorem 1.2.

Proof of Theorem 1.2. It is easy to see that the truncated functional $\Psi_*\colon W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ is even and coercive. This implies, in particular, that Ψ_* is bounded from below. In addition, we recall that Ψ_* satisfies the PS-condition; see Papageorgiou–Radulescu–Repovs [26, Proposition 5.1.15]. On account of this and thanks to Proposition 4.1, we can apply [18, Theorem 1] by Kajikiya, which implies the existence of a sequence $\{w_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ satisfying the following properties:

$$w_n \in K_{\Psi_*} \subseteq [v_*, u_*], \quad w_n \ne 0, \quad \Psi_*(w_n) \le 0 \quad \text{for all } n \in \mathbb{N}$$

and

$$\|w_n\| \to 0 \quad \text{as } n \to +\infty.$$

We point out that v_* and u_* are extremal solutions for problem (3.4). Thus, from $w_n \in K_{\Psi_*} \subseteq [v_*, u_*]$ and $w_n \ne 0$ for all $n \in \mathbb{N}$, we deduce that w_n is a nodal solution of problem (3.4) for all $n \in \mathbb{N}$. In addition, we recall the following estimate already mentioned in the proof of Proposition 3.2:

$$\|w_n\|_\infty \leq d\,\|w_n\|_{s(\tau-1)}^{\frac{\tau-1}{p-1}}$$

for some $d > 0$ and for all $n \in \mathbb{N}$ with $s > \frac{N}{p}$ as well as $s(\tau-1) < p^*$. Therefore, since $\|w_n\| \to 0$, we can deduce that $\|w_n\|_\infty \to 0$ as $n \to +\infty$. Furthermore, we can find $n_0 \in \mathbb{N}$ such that $|w_n(x)| \leq \frac{\eta_0}{2}$ for a.a. $x \in \Omega$ and for all $n \geq n_0$. This implies that $\theta(w_n(x)) = 1$ for a.a. $x \in \Omega$ and for all $n \geq n_0$. Hence, we conclude that w_n is a sign changing solution of problem (1.1) for all $n > n_0$. ∎

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