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The effect of the weight function on the number of solutions for double phase problems in $\mathbb{R}^N$

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Abstract
In this paper we deal with quasilinear elliptic equations of the form

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(\varepsilon x)|\nabla u|^{q-2}\nabla u\right) + |u|^{p-2}u + a(\varepsilon x)|u|^{q-2}u = f(u)$$

in $\mathbb{R}^N$, where $0 \leq a(\cdot) \in C\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right)$, $1 < p < N$, $p < q < p^* = \frac{Np}{N-p}$, $\varepsilon > 0$ is a parameter, and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function that grows superlinearly and subcritically which does not need to fulfill the Ambrosetti-Rabinowitz condition. Based on the Lusternik-Schnirelmann category we prove several existence results of constant-sign and sign-changing solutions to the problem above provided the parameter $\varepsilon > 0$ is sufficiently small.

Keywords Double phase operator · genus · multiple solutions · Lusternik-Schnirelmann category · Nehari manifold · unbalanced growth · unbounded domain · weight function

Mathematics Subject Classification 35J15 · 35J62 · 35J92 · 35P30

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1 Introduction and main result

In this paper we study quasilinear elliptic equations with unbalanced growth in the whole $\mathbb{R}^N$ given by

$$T_\varepsilon(u) + |u|^{p-2}u + a(\varepsilon x)|u|^{q-2}u = f(u) \quad \text{in } \mathbb{R}^N, \qquad u \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N), \tag{1.1}$$

where $T_\varepsilon(u)$ is the double phase operator given by

$$T_\varepsilon(u) = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(\varepsilon x)|\nabla u|^{q-2}\nabla u\right) \tag{1.2}$$

with $\varepsilon > 0$ being a parameter, $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ is the related Musielak-Orlicz Sobolev space depending on ε and we suppose the following assumptions:

(H0) $0 \le a(\cdot) \in C\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right)$, $1 < p < N$ and $p < q < p^* = \frac{Np}{N-p}$ with the critical exponent p^* of p.

(H1) The weight function $a(\cdot)$ satisfies the following conditions:

(i) $\inf\limits_{x \in \mathbb{R}^N} a(x) = 0$;
(ii) there exists an open bounded set $\Omega \subset \mathbb{R}^N$ such that $0 < \min\limits_{x \in \partial\Omega} a(x)$;
(iii) $\inf\limits_{x \in \Omega} a(x) = 0$ with Ω from (ii);
(iv) $a(\cdot)$ is radially symmetric, that is, $a(x) = a(|x|)$ for a.a. $x \in \mathbb{R}^N$.

Remark 1.1 Let $A = \{x \in \Omega : a(x) = 0\}$ with Ω from (H1)(ii). Then (H1)(iii) implies that $A \neq \emptyset$.

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a continuous and odd function satisfying the following conditions:

(i) there exist $r \in (q, p^*)$ and a constant $C > 0$ such that

$$|f(s)| \le C\left(1 + |s|^{r-1}\right) \quad \text{for all } s > 0;$$

(ii) $\lim\limits_{s \to 0} \dfrac{f(s)}{|s|^{p-2}s} = 0$;
(iii) $\lim\limits_{|s| \to +\infty} \dfrac{f(s)}{|s|^{q-2}s} = +\infty$;
(iv) $\dfrac{f(s)}{|s|^{q-1}}$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

The corresponding energy functional $E_\varepsilon : W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \to \mathbb{R}$ for problem (1.1) is given by

$$E_\varepsilon(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\int_{\mathbb{R}^N} a(\varepsilon x)\left(|\nabla u|^q + |u|^q\right) \mathrm{d}x - \int_\Omega F(u)\, \mathrm{d}x,$$

where $F(s)=\int_0^s f(t)\,\mathrm{d}t$. A function $u\in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ is said to be a weak solution of (1.1) if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + a(\varepsilon x)|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v\,\mathrm{d}x + \int_\Omega \left(|u|^{p-2}u + a(\varepsilon x)|u|^{q-2}u\right)v\,\mathrm{d}x - \int_\Omega f(u)v\,\mathrm{d}x = 0$$

is satisfied for all $v\in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$.

Our first result reads as follows. Note that γ stands for the genus, see its Definition in Section 2.

Theorem 1.2 *Let hypotheses (H0), (H1) and (H2) be satisfied and let A be given as in Remark 1.1. Then there exists $\tilde{\varepsilon}>0$ such that, for any $0<\varepsilon\le\tilde{\varepsilon}$, problem (1.1) has at least*

(i) $\gamma(A\setminus\{0\})$ pairs $\left(u^+,(-u)^+\right)$ of positive weak solutions;
(ii) $\gamma(A\setminus\{0\})$ pairs $\left(u^-,(-u)^-\right)$ of negative weak solutions;
(iii) $\gamma(A\setminus\{0\})$ pairs $\left(u^++u^-,(-u)^++(-u)^-\right)$ of odd weak solutions with precisely two nodal domains.

Furthermore, for $\varepsilon_n\to 0$, if u_{ε_n} is one of these solutions and $\tilde{p}_n\in\mathbb{R}^N$ is a global maximum point of u_{ε_n}, then we have

$$\lim_{\varepsilon_n\to 0} a\left(\varepsilon_n\tilde{p}_n\right)=0.$$

Next, we are interested in positive solutions of problem (1.1) under the following hypotheses on the right-hand side:

(H3) $f:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the following conditions:

(i) there exist $r\in(q,p^*)$ and a constant $C>0$ such that

$$|f(s)|\le C\left(1+|s|^{r-1}\right)\quad\text{for all } s>0;$$

(ii) $\displaystyle\lim_{s\to 0}\frac{f(s)}{|s|^{p-2}s}=0$;

(iii) $\displaystyle\lim_{s\to+\infty}\frac{f(s)}{|s|^{q-2}s}=+\infty$;

(iv) $\dfrac{f(s)}{s^{q-1}}$ is strictly increasing on $(0,\infty)$.

(v) $f(s)=0$ for $s\le 0$.

The second result in this paper is given as follows, whereby cat stands for the category of a set, see its precise Definition in Section 2.

Theorem 1.3 *Let hypotheses (H0), (H1)(i)–(iii) and (H3) be satisfied. Then there exists $\hat{\varepsilon}>0$ such that for every $0<\varepsilon<\hat{\varepsilon}$ problem (1.1) has at least* $\operatorname{cat}(A)$ *positive* *solutions. Furthermore, for $\varepsilon_n \to 0$, if u_{ε_n} is one of these solutions and $\hat{p}_n \in \mathbb{R}^N$ is a global maximum point of u_{ε_n}, then we have*

$$\lim_{\varepsilon_n \to 0} a\left(\varepsilon_n \hat{p}_n\right) = 0.$$

The proofs of Theorems 1.2 and 1.3 are mainly based on the Lusternik-Schnirelmann category theory along with appropriate subsets of the Nehari manifold. In particular, the proof of Theorem 1.2 relies on the properties of the odd symmetry invariant Nehari submanifold. To the best of our knowledge, the result of Theorem 1.2 is new in the literature and has not been published before. The main novelties in our work is the combination of an elliptic equation with unbalanced growth on the whole of $\mathbb{R}^N$ and a parameter ε inside of the weight function in order to control the number of solutions of problem (1.1).

The application of the Lusternik-Schnirelmann category to elliptic equations began with the work of Benci-Cerami [11], who studied the existence of positive solution of the problem

$$-\Delta u + \lambda u = u^{p-1} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad p \in (2, 2^*). \tag{1.3}$$

The authors proved that if p is close to 2^*, problem (1.3) has at least $\operatorname{cat}(\Omega)$ solutions, where $\operatorname{cat}(\Omega)$ denotes the Lusternik-Schnirelmann category of Ω. In 2000, Bartsch-Wang [9] considered nonlinear Schrödinger equations defined by

$$-\Delta u + (\lambda a(x) + 1)u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N, \quad 1 < p < 2^* - 1 \tag{1.4}$$

and proved existence of at least $\operatorname{cat}(\Omega)$ solutions of (1.4) provided $\lambda > 0$ is sufficiently large. We also refer to the paper by Bartsch-Wang [8]. Note that Theorem 1.3 is motivated by the works of Figueiredo-Furtado [26] and Alves-Figueiredo-Furtado [3]. Indeed, in [26] the authors studied the multiplicity of positive solutions for the equation

$$-\varepsilon^p \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + u^{p-1} = f(u) \quad \text{in } \mathbb{R}^N, \quad u \in W^{1,p}(\mathbb{R}^N),$$

while in [26] the existence of nontrivial solutions of

$$\left(\frac{\varepsilon}{i}\nabla - A(z)\right)^2 u + V(z)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N$$

has been shown. In both papers the number of solutions depend on the Lusternik-Schnirelmann category theory provided the parameter is sufficiently small. In general, the Lusternik-Schnirelmann category became a very powerful tool over the years and has been used in different models and equations to get multiplicity of solutions. We refer, for example, to the papers of Alves [1], Alves-Ding [2], Benci-Bonanno-Micheletti [10], Cingolani [15], Cingolani-Lazzo [16], Figueiredo-Pimenta-Siciliano [27], Figueiredo-Siciliano [28], see also the references therein.

In all of the aforementioned works, the existence of constant sign solutions has been demonstrated. In 2003, Castro-Clapp [14] studied the problem

$$\Delta u + \lambda u + |u|^{2^*-2}u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(\tau x) = -u(x)$$

for all $x \in \Omega$ with τ being a nontrivial orthogonal involution and proved the existence of pairs of sign-changing solutions provided $\lambda > 0$ is small enough. An improvement of their results has been done in the work of Cano-Clapp [13]. Recently, Liu-Dai-Winkert [37] obtained $\gamma(\Omega_\lambda \setminus \{0\})$ pairs $(\pm u)$ of odd weak solutions with precisely two nodal domains for the (p, q)-problem

$$-\Delta_p u - \mu\Delta_q u = f(u) - |u|^{p-2}u \quad \text{in } \Omega_\lambda, \quad u = 0 \quad \text{on } \partial\Omega_\lambda, \quad u(-x) = -u(x)$$

for a. a. $x \in \Omega_\lambda$ provided $\lambda > 0$ is sufficiently small, where $\Omega_\lambda := \lambda\Omega$ is an expanding domain for $\Omega \subseteq \mathbb{R}^N$ to be bounded and symmetric.

In our paper we extend some of the results of [37] to parameter dependent weight functions of double phase type as given in (1.1) and (1.2). It is worth noting that the issue addressed in problem (1.1) arises in the context of the study of certain non-Newtonian fluids, where $|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}$ stands for the viscosity coefficient of the fluid and $f(u) - |u|^{p-2}u - a|u|^{q-2}u$ is the divergence of shear stress. Then the solutions of (1.1) denote the speed of the fluid, see Liu-Dai [34]. Note that the operator in (1.2) is related to the energy functional

$$\mathcal{R}(u) = \int \left(|\nabla u|^p + a(x)|\nabla u|^q\right) \mathrm{d}x, \tag{1.5}$$

which was first introduced by Zhikov [48] in order to describe models for strongly anisotropic materials in the context of homogenization and elasticity. In fact, the hardening properties of strongly anisotropic materials change point by point and the modulating coefficient $a(\cdot)$ helps to describe the mixture of two different materials with hardening powers p and q. We point out that functionals of the form (1.5) belong to the class of the integral functionals with nonstandard growth condition according to Marcellini's terminology [39, 40]. Over the past 10 years several regularity results for local minimizers of (1.5) have been developed, we mention just the most famous ones by Baroni-Colombo-Mingione [5–7], De Filippis-Mingione [22] and Colombo-Mingione [18, 19], see also the references therein. Concerning existence and multiplicity results of double phase problems, lots of works for bounded or unbounded domains with different right-hand sides and various techniques have been published in the last decade. We mention the papers of Biagi-Esposito-Vecchi [12], Colasuonno-Squassina [17], Crespo-Blanco-Gasiński-Winkert [21], Farkas-Winkert [25], Gasiński-Papageorgiou [29], Gasiński-Winkert [30, 31], Liu-Dai [33–35], Liu-Papageorgiou [36], Papageorgiou-Rădulescu-Repovš [41, 42] Perera-Squassina [43] and Zeng-Bai-Gasiński-Winkert [46], see also the references therein.

As far as we know the only papers for double phase problems using the Lusternik-Schnirelmann category have been published by Liu-Dai-Winkert-Zeng [38] and Zhang-Zuo-Rădulescu [47]. In [38] the authors prove the existence of at least

$\operatorname{cat}(\Omega_\lambda) + 1$ positive solutions for problems as in (1.1) with $\varepsilon = 1$ where $\Omega_\lambda := \lambda\Omega$ is an expanding domain with λ to be positive. In [47] only the existence of nonnegative solutions to problem (1.1) has been shown for small values of ε in the situation of an unbounded potential V and under stronger assumptions as in our paper, for example, their nonlinearity has to fulfill the Ambrosetti-Rabinowitz condition. Since working on weighted Musielak-Orlicz-Sobolev spaces which are different from ours, there is no need to suppose condition (H1) (iv). We emphasize that we obtain the positive solutions of problem (1.1) as stated in Theorem 1.3 without relying on the unbounded potential V and without assuming condition (H1) (iv). To the best of our knowledge, no papers exist which prove the existence of sign-changing solutions for problem (1.1) depending on the weight function $a(\cdot)$.

The paper is organized as follows. In Section 2 we present the involved function space, recall a penalization technique due to del Pino-Felmer and introduce two auxiliary problems. Section 3 presents the mappings between the unit sphere and related Nehari manifolds while Section 4 discussed the limit problem when ε goes to zero. In Section 5 we give existence results for our auxiliary problems introduced in Section 2 and finally, Section 6 gives the proofs of our main Theorems 1.2 and 1.3.

2 Preliminaries and the penalization method

In this section we first recall some facts about the underlying function spaces and the properties of the operator. Then we introduce a penalization method due to del Pino-Felmer [23].

To this end, for $1 \le r < \infty$, by $L^r(\Omega)$ and $L^r(\mathbb{R}^N; \mathbb{R}^N)$ we denote the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$ and $W^{1,r}(\mathbb{R}^N)$ $(1 < r < \infty)$ stands for the usual Sobolev space equipped with the norm

$$\|u\|_{1,r} = \left(\|\nabla u\|_r^r + \|u\|_r^r\right)^{\frac{1}{r}}.$$

Let hypothesis (H0) be satisfied, $\varepsilon > 0$ and let $M(\mathbb{R}^N)$ be the set of all measurable functions $u\colon \mathbb{R}^N \to \mathbb{R}$. We define the nonlinear mapping $\mathcal{H}_\varepsilon\colon \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ by

$$\mathcal{H}_\varepsilon(x,t) = t^p + a(\varepsilon x)t^q.$$

Then, by $L^{\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ we denote the Musielak-Orlicz Lebesgue space given by

$$L^{\mathcal{H}_\varepsilon}(\mathbb{R}^N) = \left\{u \in M(\mathbb{R}^N)\colon \int_\Omega \mathcal{H}_\varepsilon(x,|u|)\,\mathrm{d}x < +\infty\right\},$$

which is endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}_\varepsilon} = \inf\left\{\tau > 0\colon \int_\Omega \mathcal{H}_\varepsilon\left(x, \frac{|u|}{\tau}\right)\mathrm{d}x \le 1\right\}.$$

From Liu-Dai [34, Theorem 2.7 (i)] we know that the space $L^{\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ is a reflexive Banach space. The Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ is defined by

$$W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) = \left\{ u \in L^{\mathcal{H}_\varepsilon}(\mathbb{R}^N) \colon |\nabla u| \in L^{\mathcal{H}_\varepsilon}(\mathbb{R}^N) \right\}$$

equipped with the norm

$$\|u\|_\varepsilon = \|\nabla u\|_{\mathcal{H}_\varepsilon} + \|u\|_{\mathcal{H}_\varepsilon},$$

where $\|\nabla u\|_{\mathcal{H}_\varepsilon} = \| \, |\nabla u| \, \|_{\mathcal{H}_\varepsilon}$. As before, $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ is a reflexive Banach space, see Liu-Dai [34, Theorem 2.7 (ii)]. Write

$$A_\varepsilon = \left\{ x \in \mathbb{R}^N \colon \varepsilon x \in A \right\}$$

with A given in Remark 1.1. Note that if $x \in A_\varepsilon$ then $a(\varepsilon x) = 0$. Consequently $W^{1,\mathcal{H}_\varepsilon}(A_\varepsilon)$ coincides with $W^{1,p}(A_\varepsilon)$. If $x \in \mathbb{R}^N \setminus A_\varepsilon$ then $a(\varepsilon x) > 0$. In this case, we know that the embedding

$$W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N \setminus A_\varepsilon) \hookrightarrow W^{1,p}(\mathbb{R}^N \setminus A_\varepsilon)$$

is continuous. Therefore, we have

$$\begin{aligned} &W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N) \quad \text{continuously for all } s \in [p, p^*]; \\ &W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \hookrightarrow L^s_{\text{loc}}(\mathbb{R}^N) \quad \text{compactly for all } s \in (p, p^*). \end{aligned}$$

For more details on the spaces, we refer to the papers of Crespo-Blanco-Gasiński-Harjulehto-Winkert [20], Liu-Dai [34] and Perera-Squassina [43].

Let

$$\varrho_\varepsilon(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^p + a(\varepsilon x)|\nabla u|^q + |u|^p + a(\varepsilon x)|u|^q \right) \mathrm{d}x. \tag{2.1}$$

It is easy to see that

$$\varrho_\varepsilon(u) = \|u\|_{1,p}^p + \int_{\mathbb{R}^N} \left(a(\varepsilon x) \left(|\nabla u|^q + |u|^q \right) \right) \mathrm{d}x \ge \|u\|_{1,p}^p.$$

The norm $\|\cdot\|_\varepsilon$ and the modular function ϱ_ε are related as follows, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [20, Proposition 2.15] or Liu-Dai [33, Proposition 2.1].

Proposition 2.1 *Let (H0) be satisfied, let $y \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ and let ϱ_ε be defined by (2.1). Then the following hold:*

(i) If $y \neq 0$, then $\|y\|_\varepsilon = \lambda$ if and only if $\varrho_\varepsilon(\frac{y}{\lambda}) = 1$;
(ii) $\|y\|_\varepsilon < 1$ (resp. > 1, $= 1$) if and only if $\varrho_\varepsilon(y) < 1$ (resp. > 1, $= 1$);
(iii) If $\|y\|_\varepsilon < 1$, then $\|y\|_\varepsilon^q \le \varrho_\varepsilon(y) \le \|y\|_\varepsilon^p$;

(iv) *If* $\|y\|_\varepsilon > 1$*, then* $\|y\|_\varepsilon^p \leq \varrho_\varepsilon(y) \leq \|y\|_\varepsilon^q$*;*
(v) $\|y\|_\varepsilon \to 0$ *if and only if* $\varrho_\varepsilon(y) \to 0$*;*
(vi) $\|y\|_\varepsilon \to +\infty$ *if and only if* $\varrho_\varepsilon(y) \to +\infty$.

Moreover, let $B_\varepsilon \colon W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \to W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^*$ be the nonlinear operator given by

$$\begin{aligned}\langle B_\varepsilon(u), v\rangle_{\mathcal{H}_\varepsilon} = &\int_\Omega \left(|\nabla u|^{p-2}\nabla u + a(\varepsilon x)|\nabla u|^{q-2}\nabla u\right) \cdot \nabla v \, \mathrm{d}x \\ &+ \int_\Omega \left(|u|^{p-2}u + a(\varepsilon x)|u|^{q-2}u\right) v \, \mathrm{d}x\end{aligned} \tag{2.2}$$

for all $u, v \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ where $\langle \cdot, \cdot\rangle_{\mathcal{H}_\varepsilon}$ is the duality pairing between $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ and its dual space $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^*$. The operator $B_\varepsilon \colon W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \to W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^*$ has the following properties, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [20, Proposition 3.4].

Proposition 2.2 *The operator* B_ε *defined by (2.2) is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone (hence maximal monotone) and it is of type* (S_+).

Let X be a Banach space and let $\mathcal{A}$ be the class of all closed subsets B of $X \setminus \{0\}$ which are symmetric, that is, $u \in B$ implies $-u \in B$.

Definition 2.3 Let $B \in \mathcal{A}$. The genus $\gamma(B)$ of B is defined as the least integer n such that there exists $\varphi \in C(X, \mathbb{R}^n)$ such that φ is odd and $\varphi(x) \neq 0$ for all $x \in B$. We set $\gamma(B) = +\infty$ if there are no integers with the above property and $\gamma(\emptyset) = 0$.

Remark 2.4 An equivalent way to define $\gamma(B)$ is to take the minimal integer n such that there exists an odd map $\varphi \in C(B, \mathbb{R}^n \setminus \{0\})$.

We denote by $\operatorname{cat}_B(A)$ the category of A with respect to B, namely the least integer k such that $A \subseteq A_1 \cup \cdots \cup A_k$ with A_i $(i = 1, \cdots, k)$ being closed and contractible in B. We set $\operatorname{cat}_B(\emptyset) = 0$ and $\operatorname{cat}_B(A) = +\infty$ if there is no integer with the above property. Furthermore, we set $\operatorname{cat}(B) := \operatorname{cat}_B(B)$.

In the second part of this section we construct an auxiliary problem for which we use the construction idea due to del Pino-Felmer [23], who found a positive standing wave solution for the classical Schrödinger equation under local condition of potential. The auxiliary problem is used to overcome the lack of compactness of problem (1.1).

First, we suppose that f fulfills (H2). We set $k > 0$ with $k > q$ and take $\tau > 0$ such that $f(\tau)/\tau^{p-1} = 1/k$. We define

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } |s| \leq \tau, \\ \frac{1}{k}s^{p-1} & \text{if } s > \tau, \\ -\frac{1}{k}\,|s|^{p-1} & \text{if } s < -\tau, \end{cases}$$

and

$$\tilde{g}(x,s) = \chi_\Omega(x) f(s) + (1 - \chi_\Omega(x))\, \tilde{f}(s),$$

where Ω is given in the assumption (H1)(ii) and χ_Ω is its characteristic function, that is

$$\chi_\Omega(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}$$

By hypothesis (H2), it is clear that $\tilde{g}$ has the following properties:

(H4) $\tilde{g} \colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous and odd function with respect to s, satisfying the following conditions:

(i) there exist $r \in (q, p^*)$ and a constant $C > 0$ such that

$$|\tilde{g}(x,s)| \le C \left(1 + |s|^{r-1}\right) \quad \text{for all } s > 0;$$

(ii) $\lim\limits_{s \to 0} \dfrac{\tilde{g}(x,s)}{|s|^{p-2}\, s} = 0$ uniformly in $x \in \mathbb{R}^N$;

(iii) (a) $\lim\limits_{|s| \to +\infty} \dfrac{\tilde{g}(x,s)}{|s|^{q-2}\, s} = +\infty$ uniformly in $x \in \Omega$;

(b) $0 \le \left|\tilde{G}(x,s)\right| \le |s|^p / k$ and $0 \le |\tilde{g}(x,s)| \le |s|^{p-1} / k$ for all $|s| > 0$ and $x \in \Omega^c$, where $\tilde{G}(x,s) = \int_0^s \tilde{g}(x,t)\, \mathrm{d}t$.

(iv) (a) $\dfrac{\tilde{g}(x,s)}{|s|^{q-1}}$ is strictly increasing for all $|s| > 0$ and $x \in \Omega$ or $|s| \le \tau$ and $x \in \Omega^c$;

(b) $\dfrac{\tilde{g}(x,s)}{|s|^{p-2}\, s} = \dfrac{1}{k}$ for all $|s| > \tau$ and $x \in \Omega^c$.

Next, we suppose that hypothesis (H3) holds and define

$$\hat{f}(s) = \begin{cases} f(s) & \text{if } 0 < s \le \tau, \\ \frac{1}{k} s^{p-1} & \text{if } s > \tau, \end{cases}$$

and

$$\hat{g}(x,s) = \chi_\Omega(x) f(s) + (1 - \chi_\Omega(x))\, \hat{f}(s).$$

Then, due to (H3), the function $\hat{g}$ fulfills the following conditions:

(H5) $\hat{g} \colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with primitive $\hat{G}(x,s) = \int_0^s \hat{g}(x,t)\, \mathrm{d}t$ satisfying the following assumptions:

(i) there exist $r \in (q, p^*)$ and a constant $C > 0$ such that
$$|\hat{g}(x,s)| \leq C\left(1+|s|^{r-1}\right) \quad \text{for all } s > 0;$$

(ii) $\displaystyle\lim_{s\to 0} \frac{\hat{g}(x,s)}{|s|^{p-2}s} = 0$ uniformly in $x \in \mathbb{R}^N$;

(iii) (a) $\displaystyle\lim_{s\to +\infty} \frac{\hat{g}(x,s)}{|s|^{q-2}s} = +\infty$ uniformly in $x \in \Omega$;

(b) $0 \leq \hat{G}(x,s) \leq s^p/k$ and $0 \leq \hat{g}(x,s) \leq s^{p-1}/k$ for all $s > 0$ and $x \in \Omega^c$.

(iv) (a) $\displaystyle\frac{\hat{g}(x,s)}{|s|^{q-2}s}$ is strictly increasing for all $s > 0$ and $x \in \Omega$ or $s \leq \tau$ and $x \in \Omega^c$;

(b) $\displaystyle\frac{\hat{g}(x,s)}{|s|^{p-2}s} = \frac{1}{k}$ for all $s > \tau$ and $x \in \Omega^c$.

(v) $\hat{g}(x,s) = 0$ for $s \leq 0$.

By (H4) (i), (ii) and (H5) (i), (ii), we can find for any $\xi > 0$ a number $C_\xi > 0$ such that
$$\begin{aligned} \left|\tilde{G}(x,s)\right| &\leq \xi\,|s|^p + C_\xi\,|s|^r \quad \text{for all } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R},\\ \left|\hat{G}(x,s)\right| &\leq \xi\,|s|^p + C_\xi\,|s|^r \quad \text{for all } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R}. \end{aligned} \tag{2.3}$$

Now we consider the auxiliary problems
$$\begin{aligned} T_\varepsilon(u) + |u|^{p-2}u + a(\varepsilon x)|u|^{q-2}u &= \tilde{g}(\varepsilon x, u) \quad \text{in } \mathbb{R}^N,\\ u &\in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \end{aligned} \tag{2.4}$$

and
$$\begin{aligned} T_\varepsilon(u) + |u|^{p-2}u + a(\varepsilon x)|u|^{q-2}u &= \hat{g}(\varepsilon x, u) \quad \text{in } \mathbb{R}^N,\\ u &\in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N). \end{aligned} \tag{2.5}$$

It is easy to see that, if u_ε is a solution of the auxiliary problem (2.4) (resp. (2.5)) such that $u_\varepsilon \leq \tau$ for $x \in \Omega_\varepsilon^c := \left\{x \in \mathbb{R}^N : \varepsilon x \in \Omega\right\}$, then $\tilde{g}(\varepsilon x, u_\varepsilon) = f(u_\varepsilon)$ (resp. $\hat{g}(\varepsilon x, u_\varepsilon) = f(u_\varepsilon)$) and consequently u_ε is also a solution of (1.1). Therefore, we will look for solutions u_ε of the problems (2.4) and (2.5) satisfying
$$u_\varepsilon \leq \tau \quad \text{for all } x \in \Omega_\varepsilon^c.$$

Finally, we denote the corresponding energy functional $\tilde{E}_\varepsilon : W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \to \mathbb{R}$ for problem (2.4) by
$$\tilde{E}_\varepsilon(u) = \frac{1}{p}\,\|u\|_{1,p}^p + \frac{1}{q}\int_{\mathbb{R}^N} a(\varepsilon x)\left(|\nabla u|^q + |u|^q\right)\,\mathrm{d}x - \int_{\mathbb{R}^N} \tilde{G}(\varepsilon x, u)\,\mathrm{d}x$$

and the energy functional for (2.5) by $\hat{E}_\varepsilon \colon W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\hat{E}_\varepsilon(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\int_{\mathbb{R}^N} a(\varepsilon x)\left(|\nabla u|^q + |u|^q\right)\,\mathrm{d}x - \int_{\mathbb{R}^N} \hat{G}(\varepsilon x, u)\,\mathrm{d}x.$$

3 The mapping between the unit sphere and the Nehari manifold

From now on, for a function $u\colon \mathbb{R}^N \to \mathbb{R}$, we denote by u^+ and u^- the positive and negative part of u, respectively, that is

$$u^+ = \max(u,0), \quad u^- = \min(u,0).$$

Let

$$W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ := \left\{u \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)\colon u(-x) = -u(x)\right\}.$$

The Nehari manifold corresponding to (2.4) is defined by

$$\tilde{\mathcal{N}}_\varepsilon := \left\{u \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)\setminus\{0\}\colon \left\langle \tilde{E}_\varepsilon'(u), u\right\rangle = 0\right\}$$

while the odd symmetry invariant Nehari submanifold is given by

$$\tilde{\mathcal{N}}_\varepsilon^\circ := \left\{u \in \tilde{\mathcal{N}}_\varepsilon\colon u(-x) = -u(x)\right\}.$$

Note that

$$\tilde{\mathcal{N}}_\varepsilon^\circ = \tilde{\mathcal{N}}_\varepsilon \cap W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ.$$

We point out that $\tilde{E}_\varepsilon\colon W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ \to \mathbb{R}$ is an even functional with $(\tilde{E}_\varepsilon(-u))' = -\tilde{E}_\varepsilon'(u)$. Hence, if $\tilde{E}_\varepsilon \in C^2$, then the nontrivial solutions of (2.4) are the critical points of the restriction of the functional $\tilde{E}_\varepsilon$ to the odd symmetry invariant Nehari submanifold $\tilde{\mathcal{N}}_\varepsilon^\circ$. But we only suppose that $\tilde{g}$ is continuous and so we just have $\tilde{E}_\varepsilon \in C^1$ which implies, in general, the nondifferentiability of $\tilde{\mathcal{N}}_\varepsilon^\circ$. The same holds for the auxiliary problem in (2.5) with $\hat{g}$ instead of $\tilde{g}$, respectively. The next results will overcome these difficulties.

We write

$$\mathcal{S}^\circ = \left\{u \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ\colon \|u\|_\varepsilon = 1\right\}$$

and

$$\mathcal{S}_+^\circ = \left\{u^+\colon u \in \mathcal{S}^\circ\right\}, \qquad \mathcal{N}_+^\circ = \left\{u^+\colon u \in \tilde{\mathcal{N}}_\varepsilon^\circ\right\}.$$

In the next lemma we can define a one-to-one correspondence between $\mathcal{S}_+^\circ$ and $\mathcal{N}_+^\circ$.

Proposition 3.1 *Let hypotheses (H0), (H1)(i)–(iii) and (H4) be satisfied. Then the following hold:*

(i) For each $w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ \setminus \{0\}$, set $\tilde{\varphi}_{w^+}(t) = \tilde{E}_\varepsilon(tw^+)$ for $t \geq 0$. Then there exists a unique $t_{w^+} > 0$ such that $\tilde{\varphi}'_{w^+}(t) > 0$ if $0 < t < t_{w^+}$ and $\tilde{\varphi}'_{w^+}(t) < 0$ if $t > t_{w^+}$, that is, $\max\limits_{t\in[0,+\infty)} \tilde{\varphi}_{w^+}(t)$ is achieved at $t = t_{w^+}$ and $t_{w^+}w^+ \in \mathcal{N}_+^\circ$.

(ii) There exists $\delta > 0$ such that $t_{w^+} \geq \delta$ for $w^+ \in \mathcal{S}_+^\circ$ and for each compact subset $\mathcal{W}_+^\circ \subseteq \mathcal{S}_+^\circ$ there exists a constant $C_{\mathcal{W}_+^\circ}$ such that $t_{w^+} \leq C_{\mathcal{W}_+^\circ}$ for all $w \in \mathcal{W}_+^\circ$.

(iii) Let us denote by

$$\begin{aligned}&\tilde{m}_+^\circ\colon \left\{w^+\colon w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ \setminus \{0\}\right\} \to \mathcal{N}_+^\circ,\\&w^+ \mapsto \tilde{m}_+^\circ(w^+) := t_{w^+}w^+.\end{aligned}$$

Then the mapping $\tilde{m}_+^\circ$ is continuous.

(iv) Let $m_+^\circ := \tilde{m}_+^\circ|_{\mathcal{S}_+^\circ}$. Then m_+° is a homeomorphism between $\mathcal{S}_+^\circ$ and $\mathcal{N}_+^\circ$ and the inverse of m_+° is given by

$$\left(m_+^\circ\right)^{-1}(u^+) = \frac{u^+}{\|u^+\|_\varepsilon} \quad \text{for all } u \in \mathcal{N}_+^\circ.$$

Proof (i) It is clear that $\tilde{\varphi}_{w^+}(0) = 0$. We deduce from (2.3) that

$$\begin{aligned}\tilde{\varphi}_{w^+}(t) &\geq \frac{t^p}{p}\left\|w^+\right\|_{1,p}^p + \frac{t^q}{q}\int_{\mathbb{R}^N} a(\varepsilon x)\left(\left|\nabla w^+\right|^q + \left|w^+\right|^q\right)\,\mathrm{d}x\\&\quad - \int_{\mathbb{R}^N}\left(\frac{1}{2p}t^p\left|w^+\right|^p + C_{\frac{1}{2p}}t^r\left|w^+\right|^r\right)\mathrm{d}x\\&\geq \frac{t^p}{2p}\left\|w^+\right\|_{1,p}^p + \frac{t^q}{q}\int_{\mathbb{R}^N} a(\varepsilon x)\left(\left|\nabla w^+\right|^q + \left|w^+\right|^q\right)\mathrm{d}x - C_{\frac{1}{2p}}t^r\int_{\mathbb{R}^N}\left|w^+\right|^r\,\mathrm{d}x\\&= C_1t^p + C_2t^q - C_3t^r,\end{aligned}$$

which implies that $\tilde{\varphi}_{w^+}(t) > 0$ for t small enough. It follows from (H4)(iii) that, for any $M > 0$, there exists $T_M > 0$ such that $\tilde{G}(\varepsilon x, t) \geq M|t|^q$ for $|t| > T_M$ and $x \in \Omega_\varepsilon$. Thus

$$\begin{aligned}\tilde{\varphi}_{w^+}(t) &\leq \frac{t^p}{p}\left\|w^+\right\|_{1,p}^p + \frac{t^q}{q}\int_{\mathbb{R}^N} a(\varepsilon x)\left(\left|\nabla w^+\right|^q + \left|w^+\right|^q\right)\mathrm{d}x\\&\quad - Mt^q\int_{\Omega_\varepsilon}\left|w^+\right|^q\,\mathrm{d}x + \frac{1}{k}t^p\int_{\Omega_\varepsilon^c}\left|w^+\right|^p\,\mathrm{d}x\\&= C_1t^p + C_2t^q - C_3Mt^q\\&\leq C_1t^p - C_2t^q \quad \text{when } M \geq \frac{2C_2}{C_3},\end{aligned}$$

which implies that $\tilde{\varphi}_{w^+}(t) < 0$ for t large enough. Hence there exists $t_{w^+} > 0$ such that $\tilde{\varphi}'_{w^+}(t_{w^+}) = 0$. We also note that

$$\begin{aligned}0 = \tilde{\varphi}'_{w^+}(t) = &\int_{\mathbb{R}^N} \left(t^{p-1}\left(\left|\nabla w^+\right|^p + \left|w^+\right|^p\right) + a(\varepsilon x)t^{q-1}\left(\left|\nabla w^+\right|^q + \left|w^+\right|^q\right)\right) \mathrm{d}x\\ &- \int_{\mathbb{R}^N} \tilde{g}(\varepsilon x, tw^+)w^+ \,\mathrm{d}x\end{aligned}$$

implies $tw^+ \in \mathcal{N}_+^\circ$.

We claim that $E := \{x \in \Omega_\varepsilon^c \colon tw^+ > \tau \text{ for a.a. } x \in \mathbb{R}^N\} = \emptyset$. Suppose $E \neq \emptyset$. Then $\left\langle \tilde{E}'_\varepsilon(tw^+), tw^+\chi_E\right\rangle = 0$, where χ_E is the characteristic function of E. However, we have

$$\begin{aligned}&\left\langle \tilde{E}'_\varepsilon(tw^+), tw^+\chi_E\right\rangle\\ &= \int_E \left(t^{p-1}\left(\left|\nabla w^+\right|^p + \left|w^+\right|^p\right) + a(\varepsilon x)t^{q-1}\left(\left|\nabla w^+\right|^q + \left|w^+\right|^q\right)\right) \mathrm{d}x\\ &\quad - \int_E \tilde{g}(\varepsilon x, tw^+)w^+ \,\mathrm{d}x\\ &\geq \int_E \left(t^{p-1}\left(\left|\nabla w^+\right|^p + \left|w^+\right|^p\right) + a(\varepsilon x)t^{q-1}\left(\left|\nabla w^+\right|^q + \left|w^+\right|^q\right)\right) \mathrm{d}x\\ &\quad - \frac{1}{k}t^{p-1}\int_E \left|w^+\right|^p \,\mathrm{d}x\\ &\geq \left(1 - \frac{1}{k}\right)t^{p-1}\int_E \left|w^+\right|^p \,\mathrm{d}x \geq \sigma > 0,\end{aligned}$$

for some positive constant σ which is a contradiction and so the claim holds true. Consequently, we deduce from $tw^+ \in \mathcal{N}_+^\circ$ that

$$\begin{aligned}&\int_{\mathbb{R}^N} a(\varepsilon x)\left(\left|\nabla w^+\right|^q + \left|w^+\right|^q\right) \mathrm{d}x\\ &= \int_{\mathbb{R}^N} \frac{\tilde{g}(\varepsilon x, tw^+)w^+}{t^{q-1}} \,\mathrm{d}x - \frac{1}{t^{q-p}}\int_{\mathbb{R}^N} \left(\left|\nabla w^+\right|^p + \left|w^+\right|^p\right) \mathrm{d}x\\ &= \int_{\Omega_\varepsilon} \frac{\tilde{g}(\varepsilon x, tw^+)w^+}{t^{q-1}} \,\mathrm{d}x + \int_{\{\Omega_\varepsilon^c,\, tw^+ \leq \tau\}} \frac{\tilde{g}(\varepsilon x, tw^+)w^+}{t^{q-1}} \,\mathrm{d}x\\ &\quad - \frac{1}{t^{q-p}}\int_{\mathbb{R}^N} \left(\left|\nabla w^+\right|^p + \left|w^+\right|^p\right) \mathrm{d}x.\end{aligned}$$

By (H4)(iv), the right-hand side of the last equality is strictly increasing in t. It follows that $\tilde{\varphi}_{w^+}(t)$ has a unique critical point. Therefore $\max_{t\in[0,+\infty)} \tilde{\varphi}_{w^+}(t)$ is achieved at a unique $t = t_{w^+} > 0$ so that $h'_{w^+}(t_{w^+}) = 0$ and $t_{w^+}w^+ \in \mathcal{N}_+^\circ$.

(ii) First, we prove that there exists $\delta > 0$ such that $t_{w^+} > \delta$ for $w^+ \in \mathcal{S}_+^\circ$. If $t_{w^+} \geq 1$ we are done. If $t_{w^+} < 1$, we deduce from $t_{w^+} w^+ \in \mathcal{N}_+^\circ$ and (2.3) that

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(t_{w^+}^p \left(\left|\nabla w^+\right|^p + \left|w^+\right|^p\right) + t_{w^+}^q a(x) \left(\left|\nabla w^+\right|^q + \left|w^+\right|^q\right)\right) \mathrm{d}x \\ &\leq \frac{1}{2} t_{w^+}^p \int_{\mathbb{R}^N} \left|w^+\right|^p \,\mathrm{d}x + C_{\frac{1}{2}} t_{w^+}^r \int_{\mathbb{R}^N} \left|w^+\right|^r \,\mathrm{d}x\end{aligned}$$

or

$$\frac{1}{2} t_{w^+}^q \leq C_{\frac{1}{2}} t_{w^+}^r.$$

Clearly, we can take $\delta = \left(\frac{1}{2C_{1/2}}\right)^{\frac{1}{r-q}} > 0$ in this case.

Next, if $\mathcal{W}_+^\circ \subseteq \mathcal{S}_+^\circ$ is compact, and suppose by contradiction that there is $\left\{w_n^+\right\}_{n\in\mathbb{N}} \subset \mathcal{W}_+^\circ$ with $t_n := t_{w_n^+} \to +\infty$. By (i), we see that

$$\tilde{E}_\varepsilon(t_n w_n^+) = \max_{t\in[0,+\infty)} \tilde{E}_\varepsilon(t w_n^+) \geq 0.$$

On the other hand, by (H4)(iii), we deduce that

$$0 \leq \frac{\tilde{E}_\varepsilon(t_n w_n^+)}{t_n^q} \leq \frac{1}{p} + \frac{1}{k} - \int_{\Omega_\varepsilon} \frac{\tilde{G}(\varepsilon x, t_n w_n^+)}{t_n^q} \,\mathrm{d}x \to -\infty \quad \text{as } n \to \infty,$$

which yields a contradiction. Thus there exists $C_{\mathcal{W}_+^\circ}$ such that $t_{w^+} \leq C_{\mathcal{W}_+^\circ}$.

(iii) Suppose that $w_n^+ \to w^+$ in $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\}$. It follows from (ii) that $\{t_{w_n^+}\}_{n\in\mathbb{N}}$ is uniformly bounded. Therefore, there exist a subsequence of $\{t_{w_n^+}\}_{n\in\mathbb{N}}$, which we still denote by $\{t_{w_n^+}\}_{n\in\mathbb{N}}$, converging to a limit t_0. It follows from the uniqueness of t_{w^+} that $t_0 = t_{w^+}$. But then $t_{w_n} \to t_{w^+}$. Thus $\tilde{m}_+^\circ$ is continuous.

(iv) By (i), we can easily see that $m_+^\circ(\mathcal{S}_+^\circ)$ is a bounded set in $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ and for any $w^+ \in m_+^\circ(\mathcal{S}_+^\circ)$, there exists $\delta > 0$ such that $\left\|w^+\right\|_\varepsilon \geq \delta$, that is, for any $w^+ \in \mathcal{N}_+^\circ$, we can find $\delta > 0$ such that $\left\|w^+\right\|_\varepsilon \geq \delta$. The argument is similar to the proof of (ii). By the continuity of $\tilde{m}_+^\circ$ and its definition, we know that the map $m_+^\circ \colon \mathcal{S}_+^\circ \to \mathcal{N}_+^\circ$ is continuous and one-to-one. Clearly, the inverse function of m_+° is $(m_+^\circ)^{-1}(w^+) = \frac{w^+}{\|w^+\|_\varepsilon}$ for any $w^+ \in \mathcal{N}_+^\circ$. We only have to prove that $\left(m_+^\circ\right)^{-1}$ is continuous. Indeed, it holds

$$\begin{aligned}&\left\|\left(m_+^\circ\right)^{-1}(w^+) - \left(m_+^\circ\right)^{-1}(v^+)\right\|_\varepsilon \\ &= \left\|\frac{w^+}{\left\|w^+\right\|_\varepsilon} - \frac{v^+}{\left\|v^+\right\|_\varepsilon}\right\|_\varepsilon = \left\|\frac{w^+ - v^+}{\left\|w^+\right\|_\varepsilon} + \frac{v^+\left(\left\|v^+\right\|_\varepsilon - \left\|w^+\right\|_\varepsilon\right)}{\left\|w^+\right\|_\varepsilon \left\|v^+\right\|_\varepsilon}\right\|_\varepsilon \leq \frac{2\left\|w^+ - v^+\right\|_\varepsilon}{\left\|w^+\right\|_\varepsilon} \\ &\leq \frac{2}{\delta}\left\|w^+ - v^+\right\|_\varepsilon,\end{aligned}$$

which shows that $(m_+^\circ)^{-1}$ is Lipschitz continuous. $\square$

Now we can define

$$\begin{aligned} &\tilde{J}_+^\circ\colon \left\{w^+ : w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ \setminus \{0\}\right\} \to \mathbb{R}^N, \\ &w^+ \mapsto \tilde{J}_+^\circ(w^+) = \tilde{E}_\varepsilon(\tilde{m}_+^\circ(w^+)), \\ &\tilde{J}_+ := \tilde{J}_+^\circ|_{\mathcal{S}_+^\circ}. \end{aligned} \tag{3.1}$$

A direct consequence of Proposition 3.1 and by Szulkin-Weth [45, Proposition 9 and Corollary 10] is the following proposition.

Proposition 3.2 *Let hypotheses (H0), (H1)(i)–(iii) and (H4) be satisfied. Then the following hold:*

(i) $\tilde{J}_+ \in C^1\left(\mathcal{S}_+^\circ, \mathbb{R}\right)$ *and*

$$\left\langle \tilde{J}_+'(w^+), z\right\rangle = \left\langle \tilde{E}_\varepsilon'(m_+^\circ(w^+)), z\|m_+^\circ(w^+)\|_\varepsilon\right\rangle$$

for all $w^+ \in \mathcal{S}_+^\circ$ *and for all* $z \in T_{w^+}(\mathcal{S}_+^\circ)$*, where* $T_{w^+}(\mathcal{S}_+^\circ)$ *denotes the tangent space to* $\mathcal{S}_+^\circ$ *at* w^+*.*

(ii) If $\{w_n^+\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+^\circ$ *is a* $(\mathrm{PS})_c$*-sequence for* $\tilde{J}_+$*, then* $\{m_+^\circ(w_n^+)\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_+^\circ$ *is a* $(\mathrm{PS})_c$*-sequence for* $\tilde{E}_\varepsilon$*. If* $\{u_n^+\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_+^\circ$ *is a bounded* $(\mathrm{PS})_c$*-sequence for* $\tilde{E}_\varepsilon$*, then* $\{(m_+^\circ)^{-1}(u_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+^\circ$ *is a* $(\mathrm{PS})_c$*-sequence for* $\tilde{J}_+$*.*

(iii) $w^+ \in \mathcal{S}_+^\circ$ *is a critical point of* $\tilde{J}_+$ *if and only if* $m_+^\circ(w^+) \in \mathcal{N}_+^\circ$ *is a nontrivial critical point of* $\tilde{E}_\varepsilon$*. Moreover,* $\inf_{\mathcal{S}_+^\circ} \tilde{J}_+ = \inf_{\mathcal{N}_+^\circ} \tilde{E}_\varepsilon$*.*

(iv) If $\tilde{E}_\varepsilon$ *is even, then so is* $\tilde{J}_+$*.*

Next, we write

$$\mathcal{S}_-^\circ = \left\{u^- : u \in \mathcal{S}^\circ\right\}, \qquad \mathcal{N}_-^\circ = \left\{u^- : u \in \tilde{\mathcal{N}}_\varepsilon^\circ\right\}.$$

Then we can set up a one-to-one correspondence between $\mathcal{S}_-^\circ$ and $\mathcal{N}_-^\circ$ as follows.

Proposition 3.3 *Let hypotheses (H0), (H1)(i)–(iii) and (H4) be satisfied. Then the following hold:*

(i) For each $w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ \setminus \{0\}$*, set* $\tilde{\varphi}_{w^-}(t) = \tilde{E}_\varepsilon(tw^-)$ *for* $t \geq 0$*. Then there exists a unique* $t_{w^-} > 0$ *such that* $\tilde{\varphi}_{w^-}'(t) > 0$ *if* $0 < t < t_{w^-}$ *and* $\tilde{\varphi}_{w^-}'(t) < 0$ *if* $t > t_{w^-}$*, that is,* $\max\limits_{t\in[0,+\infty)} \tilde{\varphi}_{w^-}(t)$ *is achieved at* $t = t_{w^-}$ *and* $t_{w^-}w^- \in \mathcal{N}_-^\circ$*.*

(ii) There exists $\delta > 0$ *such that* $t_{w^-} \geq \delta$ *for* $w^- \in \mathcal{S}_-^\circ$ *and for each compact subset* $\mathcal{W}_-^\circ \subseteq \mathcal{S}_-^\circ$ *there exists a constant* $C_{\mathcal{W}_-^\circ}$ *such that* $t_{w^-} \leq C_{\mathcal{W}_-^\circ}$ *for all* $w \in \mathcal{W}_-^\circ$*.*

(iii) Let us denote by

$$\begin{aligned} &\tilde{m}_-^\circ\colon \left\{w^- : w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ \setminus \{0\}\right\} \to \mathcal{N}_-^\circ, \\ &w^- \mapsto \tilde{m}_-^\circ(w^-) := t_{w^-}w^-. \end{aligned}$$

Then the mapping $\tilde{m}_-^\circ$ *is continuous.*

(iv) *Let* $m_-^\circ := \tilde{m}_-^\circ|_{\mathcal{S}_-^\circ}$. *Then* m_-° *is a homeomorphism between* $\mathcal{S}_-^\circ$ *and* $\mathcal{N}_-^\circ$ *and the inverse of* m_-° *is given by*

$$\left(m_-^\circ\right)^{-1}(u^-) = \frac{u^-}{\|u^-\|_\varepsilon} \quad \text{for all } u^- \in \mathcal{N}_-^\circ.$$

Proof The proof can be done as the proof of Proposition 3.1. □

Now we can define

$$\begin{aligned} &\tilde{J}_-^\circ \colon \left\{w^- \colon w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)^\circ \setminus \{0\}\right\} \to \mathbb{R}^N, \\ &w^- \mapsto \tilde{J}_-^\circ(w^-) = E_\varepsilon(\tilde{m}_-^\circ(w^-)), \\ &\tilde{J}_- := \tilde{J}_-^\circ|_{\mathcal{S}_-^\circ}. \end{aligned} \tag{3.2}$$

As before, as a consequence of Proposition 3.3 and of Szulkin-Weth [45, Proposition 9 and Corollary 10] we have the following proposition.

Proposition 3.4 *Let hypotheses* (H0), (H1)(i)–(iii) *and* (H4) *be satisfied. Then the following hold:*

(i) $\tilde{J}_- \in C^1\left(\mathcal{S}_-^\circ, \mathbb{R}\right)$ *and*

$$\left\langle \tilde{J}_-'(w^-), z\right\rangle = \left\langle \tilde{E}_\varepsilon'(m_-^\circ(w^-)), z\|m_-^\circ(w^-)\|_\varepsilon\right\rangle$$

for all $w^- \in \mathcal{S}_-^\circ$ *for all and* $z \in T_{w^-}(\mathcal{S}_-^\circ)$, *where* $T_{w^-}(\mathcal{S}_-^\circ)$ *stands for the tangent space to* $\mathcal{S}_-^\circ$ *at* w^-.

(ii) *If* $\{w_n^-\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_-^\circ$ *is a* $(\mathrm{PS})_c$*-sequence for* $\tilde{J}_-$, *then* $\{m_-^\circ(w_n^-)\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_-^\circ$ *is a* $(\mathrm{PS})_c$*-sequence for* $\tilde{E}_\varepsilon$. *If* $\{u_n^-\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_-^\circ$ *is a bounded* $(\mathrm{PS})_c$*-sequence for* $\tilde{E}_\varepsilon$, *then* $\{(m_-^\circ)^{-1}(u_n^-)\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_-^\circ$ *is a* $(\mathrm{PS})_c$*-sequence for* $\tilde{J}_-$.

(iii) $w^- \in \mathcal{S}_-^\circ$ *is a critical point of* $\tilde{J}_-$ *if and only if* $m_-^\circ(w^-) \in \mathcal{N}_-^\circ$ *is a nontrivial critical point of* $\tilde{E}_\varepsilon$. *Moreover,* $\inf_{\mathcal{S}_-^\circ} \tilde{J}_- = \inf_{\mathcal{N}_-^\circ} \tilde{E}_\varepsilon$.

(iv) *If* $\tilde{E}_\varepsilon$ *is even, then so is* $\tilde{J}_-$.

Now, we write

$$\begin{aligned} \hat{\mathcal{N}}_\varepsilon :&= \left\{u \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\} \colon \left\langle \hat{E}_\varepsilon'(u), u\right\rangle = 0\right\} \\ \mathcal{S} &= \left\{u \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \colon \|u\|_\varepsilon = 1\right\}, \\ \mathcal{S}_+ &= \left\{u^+ \colon u \in \mathcal{S}\right\}, \\ \mathcal{N}_+ &= \left\{u^+ \colon u \in \hat{\mathcal{N}}_\varepsilon\right\}. \end{aligned}$$

Then we can set up a one-to-one correspondence between $\mathcal{S}_+$ and $\mathcal{N}_+$ in the following way.

Proposition 3.5 *Let hypotheses (H0), (H1)(i)–(iii) and (H5) be satisfied. Then the following hold:*

(i) *For each* $w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\}$, *set* $\hat{\varphi}_{w^+}(t) = \hat{E}_\varepsilon(tw^+)$ *for* $t \geq 0$. *Then there exists a unique* $t_{w^+} > 0$ *such that* $\hat{\varphi}'_{w^+}(t) > 0$ *if* $0 < t < t_{w^+}$ *and* $\hat{\varphi}'_{w^+}(t) < 0$ *if* $t > t_{w^+}$, *that is,* $\max\limits_{t \in [0,+\infty)} \hat{\varphi}_{w^+}(t)$ *is achieved at* $t = t_{w^+}$ *and* $t_{w^+}w^+ \in \mathcal{N}_+$.

(ii) *There exists* $\delta > 0$ *such that* $t_{w^+} \geq \delta$ *for* $w^+ \in \mathcal{S}_+$ *and for each compact subset* $\mathcal{W}_+ \subseteq \mathcal{S}_+$ *there exists a constant* $C_{\mathcal{W}_+}$ *such that* $t_{w^+} \leq C_{\mathcal{W}_+}$ *for all* $w \in \mathcal{W}_+$.

(iii) *Let us denote by*

$$
\begin{aligned}
&\hat{m}_+ \colon \left\{ w^+ \colon w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\} \right\} \to \mathcal{N}_+,\\
&w^+ \mapsto \hat{m}_+(w^+) := t_{w^+}w^+.
\end{aligned}
$$

Then the mapping $\hat{m}_+$ *is continuous.*

(iv) *Let* $m := \hat{m}_+|_{\mathcal{S}_+}$. *Then* m *is a homeomorphism between* $\mathcal{S}_+$ *and* $\mathcal{N}_+$ *and the inverse of* m *is given by*

$$
m^{-1}(u^+) = \frac{u^+}{\|u^+\|_\varepsilon} \quad \text{for all } u^+ \in \mathcal{N}_+.
$$

Now we can define

$$
\begin{aligned}
&\hat{J}_+ \colon \left\{ w^+ \colon w \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\} \right\} \to \mathbb{R}^N,\\
&w^+ \mapsto \hat{J}_+(w^+) = \hat{E}_\varepsilon(\hat{m}_+(w^+)),\\
&\hat{J} = \hat{J}_+|_{\mathcal{S}_+}.
\end{aligned}
\tag{3.3}
$$

Proposition 3.6 *Let hypotheses (H0), (H1)(i)–(iii) and (H5) be satisfied. Then the following hold:*

(i) $\hat{J} \in C^1(\mathcal{S}_+, \mathbb{R})$ *and*

$$
\left\langle \hat{J}'(w^+), z \right\rangle = \left\langle \hat{E}'_\varepsilon(m(w^+)), z\|m(w^+)\|_\varepsilon \right\rangle
$$

for all $w^+ \in \mathcal{S}_+$ *and for all* $z \in T_{w^+}(\mathcal{S}_+)$, *with* $T_{w^+}(\mathcal{S}_+)$ *being the tangent space to* $\mathcal{S}_+$ *at* w^+.

(ii) *If* $\{w_n^+\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ *is a* $(\mathrm{PS})_c$*-sequence for* $\hat{J}$, *then* $\{m(w_n^+)\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_+$ *is a* $(\mathrm{PS})_c$*-sequence for* $\hat{E}_\varepsilon$. *If* $\{u_n^+\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_+$ *is a bounded* $(\mathrm{PS})_c$*-sequence for* $\hat{E}_\varepsilon$, *then* $\{m^{-1}(u_n^+)\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ *is a* $(\mathrm{PS})_c$*-sequence for* $\hat{J}$.

(iii) $w^+ \in \mathcal{S}_+$ *is a critical point of* $\hat{J}$ *if and only if* $m(w^+) \in \mathcal{N}_+$ *is a nontrivial critical point of* $\hat{E}_\varepsilon$. *Moreover,* $\inf_{\mathcal{S}_+} \hat{J} = \inf_{\mathcal{N}_+} \hat{E}_\varepsilon$.

Remark 3.7

(i) If we set

$$c_+^\circ = \inf_{u^+ \in \mathcal{N}_+^\circ} \tilde{E}_\varepsilon(u^+),$$

then it follows from Proposition 3.2 (iii) that

$$c_+^\circ = \inf_{w^+ \in \mathcal{S}_+^\circ} \tilde{J}_+(w^+).$$

From Proposition 3.1 it is easy to see that c_+° has the following minimax characterization:

$$c_+^\circ = \inf_{w \in W_0^{1,p}(\Omega)^\circ \setminus \{0\}} \max_{t>0} \tilde{E}_\varepsilon(tw^+) = \inf_{w^+ \in \mathcal{S}_+^\circ} \max_{t>0} \tilde{E}_\varepsilon(tw^+).$$

We know from the proof of Proposition 3.1 that there exists a unique $t_{w^+} > 0$ such that $\max_{t>0} \tilde{E}_\varepsilon(tw^+) = \tilde{E}_\varepsilon\left(t_{w^+} w^+\right)$ for $w^+ \in \mathcal{S}_+^\circ$. Proposition 3.1 (ii) implies that there exists $\delta > 0$ such that $t_{w^+} \geq \delta$ uniformly for $w^+ \in \mathcal{S}_+^\circ$. Thus, for any $w^+ \in \mathcal{S}_+^\circ$, we have

$$\tilde{E}_\varepsilon\left(t_{w^+} w^+\right) = \max_{t>0} \tilde{E}_\varepsilon(tw^+) \geq \sigma,$$

for some $\sigma > 0$ independent of w^+ and consequently

$$\inf_{w^+ \in \mathcal{S}_+^\circ} \max_{t>0} \tilde{E}_\varepsilon(tw^+) \geq \sigma,$$

that is

$$c_+^\circ \geq \sigma > 0.$$

If we set

$$c_-^\circ = \inf_{u^- \in \mathcal{N}_-^\circ} \tilde{E}_\varepsilon(u^-),$$

then, similarly, From Proposition 3.3, It can show that

$$c_-^\circ > 0.$$

We also note that $\tilde{E}_\varepsilon(u) = \tilde{E}_\varepsilon(u^+) + \tilde{E}_\varepsilon(u^-)$. If we set

$$c^\circ = \inf_{u \in \tilde{\mathcal{N}}_\varepsilon^\circ} \tilde{E}_\varepsilon(u),$$

then it is clear that $c^\circ \geq c_+^\circ + c_-^\circ$. In our case, $c_+^\circ = c_-^\circ$ since u is an odd function.

(ii) Set

$$c = \inf_{u^+ \in \mathcal{N}_+} \hat{E}_\varepsilon(u^+).$$

By an argument similar to that of (i), we can show that $c > 0$ and $c^\circ \geq 2c$.

4 Limiting problem

We consider the limiting problem associated to (1.1), that is, the following p-Laplacian problem:

$$\begin{aligned} -\Delta_p u + |u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N). \end{aligned} \tag{4.1}$$

Since we are interested in the existence of positive solutions, we consider the functional

$$E_0(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_{\mathbb{R}^N} F(u^+)\ \mathrm{d}x.$$

First, we consider the radially symmetric ground state solutions of (4.1). It is similar to the proof of Liu-Dai [34, Theorem 1.9] and we can show that there exists a positive radially symmetric ground state solution ω of (4.1). Moreover, we define

$$\mathcal{N}_0^r := \left\{u \in W_r^{1,p}(\mathbb{R}^N) \setminus \{0\} : \left\langle E_0'(u), u\right\rangle = 0\right\} \quad \text{and} \quad c_0^r = \inf_{u \in \mathcal{N}_0^r} E_0(u),$$

where $W_r^{1,p}(\mathbb{R}^N) := \left\{u \in W^{1,p}(\mathbb{R}^N) \colon u \text{ is radially symmetric}\right\}$. Then, we have

$$E_0(\omega) = c_0^r.$$

Next, we consider positive ground state solutions of (4.1), not necessarily radially symmetric. For this purpose, as in Section 3, we define:

$$\begin{aligned}
&\mathcal{N}_0 = \left\{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} : \left\langle E_0'(u), u\right\rangle = 0,\ u^+ \neq 0\right\}, \\
&\mathcal{S}_0 = \left\{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} : \|u\|_{1,p} = 1,\ u^+ \neq 0\right\}, \\
&m_0 \colon \mathcal{S}_0 \to \mathcal{N}_0, \quad \omega_0 \mapsto m_0(\omega_0), \\
&J_0(\omega_0) = E_0\left(m_0(\omega_0)\right), \quad 0 < c_0 = \inf_{u \in \mathcal{N}_0} E_0(u).
\end{aligned}$$

Similarly, we also know that for each $w_0 \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ there exists a unique $t_0 := t_{w_0}$ such that $t_0 w_0 \in \mathcal{N}_0$.

Lemma 4.1 *Let $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_0$ be such that $J_0(\omega_n) \to c_0$ and $\omega_n \rightharpoonup \omega_0$ in $W^{1,p}(\mathbb{R}^N)$. Then there exists a sequence $\{\overline{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n := \omega_n(\cdot + \overline{y}_n) \to v_0 \in \mathcal{S}_0$ with $J_0(v_0) = c_0$. Moreover, if $\omega_0 \neq 0$, then $\{\overline{y}_n\}_{n\in\mathbb{N}}$ can be taken identically zero and thus $\omega_n \to \omega_0$ in $W^{1,p}(\mathbb{R}^N)$.*

Proof If $\omega_0 = 0$, then there exist $R, \sigma > 0$ and $\{\overline{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B_R(\overline{y}_n)} |\omega_n|^p \, \mathrm{d}x \geq \sigma.$$

Suppose by contradiction that

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |\omega_n|^p \, \mathrm{d}x = 0.$$

Then it follows from Lemma I.1 of Lions [32] that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\omega_n|^\alpha \, \mathrm{d}x = 0 \quad \text{for all } \alpha \in (p, p^*).$$

Consequently

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |m_0(\omega_n)|^\alpha \, \mathrm{d}x = 0 \quad \text{for all } \alpha \in (p, p^*).$$

By (H3) (i) and (H3) (ii), we have

$$|f(m_0(\omega_n))| \leq \xi \, |m_0(\omega_n)|^{p-1} + C_\xi \, |m_0(\omega_n)|^{r-1}$$

and

$$|F(m_0(\omega_n))| \leq \xi \, |m_0(\omega_n)|^p + C_\xi \, |m_0(\omega_n)|^r .$$

Thus

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(m_0(\omega_n))\, m_0(\omega_n) \, \mathrm{d}x = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(m_0(\omega_n)) \, \mathrm{d}x = 0.$$

Therefore,

$$\lim_{n\to\infty} \|m_0(\omega_n)\|_{1,p} = 0$$

and consequently

$$\lim_{n\to\infty} J_0(\omega_n) = 0,$$

which is a contradiction to $J_0(\omega_n) \to c_0 > 0$ as $n \to \infty$.

Now we define $v_n(x) = \omega_n(x + \overline{y}_n)$, then $J_0(v_n) \to c_0$ and there exists $0 \neq v_0 \in W^{1,p}(\mathbb{R}^N)$ such that $v_n(x) \rightharpoonup v_0$. By the Sobolev embedding theorem, we have that $|\overline{y}_n| \to \infty$. Note that $m_0(v_n) \rightharpoonup m_0(v_0)$ in $W^{1,p}(\mathbb{R}^N)$. For any $s \in [p, p^*)$ and $R > 0$, we have that

$$\begin{aligned}\lim_{R\to+\infty} \int_{B_R^c(0)} |m_0(v_n)|^s \,\mathrm{d}x &= \lim_{R\to+\infty} \int_{\mathbb{R}^N\setminus B_R(0)} |m_0(v_n)|^s \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |m_0(v_n)|^s \,\mathrm{d}x - \lim_{R\to+\infty} \int_{B_R(0)} |m_0(v_n)|^s \,\mathrm{d}x \\ &= 0.\end{aligned}$$

Thus there exists $R_1 > 0$ large enough such that

$$\int_{B_{R_1}^c(0)} |m_0(v_n)|^s \,\mathrm{d}x = o_n(1).$$

By (H3) (i) and (H3) (ii), we know that

$$\left| \int_{B_{R_1}^c(0)} f(m_0(v_n))\, m_0(v_n) \,\mathrm{d}x \right| \le o_n(1). \tag{4.2}$$

From the compact embedding $W^{1,p}\left(B_{R_1}(0)\right) \hookrightarrow L^s\left(B_{R_1}(0)\right)$ and the subcritical growth of f, we deduce that

$$\int_{B_{R_1}(0)} f(m_0(v_n))\, m_0(v_n) \,\mathrm{d}x \to \int_{B_{R_1}(0)} f(m_0(v_0))\, m_0(v_0) \,\mathrm{d}x \tag{4.3}$$

as $n \to +\infty$. Combining (4.2) with (4.3) yields

$$\int_{\mathbb{R}^N} f(m_0(v_n))\, m_0(v_n) \,\mathrm{d}x \to \int_{\mathbb{R}^N} f(m_0(v_0))\, m_0(v_0) \,\mathrm{d}x \tag{4.4}$$

as $n \to +\infty$. By definition of m_0 and (4.4), we conclude that $\|m_0(v_n)\|_{1,p} \to \|m_0(v_0)\|_{1,p}$. And consequently $\|m_0(v_n) - m_0(v_0)\|_{1,p} \to 0$ since $W^{1,p}(\mathbb{R}^N)$ is uniformly convex. Therefore, $v_n \to v_0$ in $W^{1,p}(\mathbb{R}^N)$ and $v_0 \in \mathcal{S}_0$ with $J_0(v_0) = c_0$.

If $\omega_0 \neq 0$, the proof is similar to the proof of $v_n \to v_0$. We omit it here. □

Theorem 4.2 *Problem (4.1) has a positive ground state solution.*

Proof Let $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_0$ be a minimizing sequence for J_0, that is, $J_0(\omega_n) \to c_0$. By Ekeland's variational principle (see Ekeland [24]), we may assume that $J_0'(\omega_n) \to 0$. Then $\{u_n := m_0(\omega_n)\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ is a $(\mathrm{PS})_c$-sequence for E_0. First we claim that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Suppose not, then there exists a subsequence (still denoted by $\{u_n\}_{n\in\mathbb{N}}$) such that $\|u_n\|_{1,p} \to +\infty$. Set $v_n = u_n/\|u_n\|_{1,p}$, then $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Thus, after passing to a subsequence if necessary, we may assume that $v_n \rightharpoonup v_0$ in $W^{1,p}(\mathbb{R}^N)$ as $n \to +\infty$. If $v_0 = 0$, then, by an argument similar to that of Proposition 3.5 and Remark 3.7, for any $t > 0$, we have

$$c_0 + o(1) \geq E_0(u_n) = E_0(t_{v_n} v_n) \geq E_0(t v_n)$$

and

$$E_0(t v_n) \geq \frac{1}{p} t^p - \int_{\mathbb{R}^N} F(t v_n)\, \mathrm{d}x \geq \frac{1}{p} t^p.$$

This yields a contradiction by choosing $t > \max\left\{1,\ 2\,(pc_0)^{\frac{1}{p}}\right\}$. If $v_0 \neq 0$, then we know from (H3) (iii) that

$$0 \leq \frac{E_0(u_n)}{\|u_n\|_{1,p}^p} \leq \frac{1}{p} - \int_{\mathbb{R}^N} \frac{F(\|u_n\| v_n)}{\|u_n\|_{1,p}^p}\, \mathrm{d}x \to -\infty$$

as $n \to \infty$, again a contradiction. Hence $\{u_n\}_{n\in\mathbb{N}}$ is bounded and so $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded as well. Therefore, we may assume that $\omega_n \rightharpoonup \omega_0$ for some $\omega_0 \in W^{1,p}(\mathbb{R}^N)$. From Lemma 4.1 it follows that there exists $\omega \in \mathcal{S}_0$ such that $J_0(\omega) = c_0$ and $J_0'(\omega) = 0$. Consequently $u := m_0(\omega)$ satisfies $E_0(u) = c_0$ and $E_0'(u) = 0$, which is our desired ground state solution. It is standard to prove that u is positive, we omit it. $\square$

5 Multiple solutions of the auxiliary problem

In this section we are going to solve our auxiliary problems (2.4) and (2.5), respectively. We start with some important lemmas in order to get the desired results.

Lemma 5.1 *Let hypotheses (H0), (H1)(i)–(iii) and (H4) be satisfied and let $\tilde{J}_+$ be given in (3.1). Then the following hold:*

(i) If $\left\{w_n^+\right\}_{n\in\mathbb{N}} \subset \mathcal{S}_+^\circ$ is a sequence such that $\operatorname{dist}\left(w_n^+,\ \partial\mathcal{S}_+^\circ\right) \to 0$ as $n \to +\infty$, then $\left\|m_+^\circ\left(w_n^+\right)\right\|_\varepsilon \to +\infty$ and $\tilde{J}_+\left(w_n^+\right) \to +\infty$ as $n \to +\infty$.

(ii) $\tilde{J}_+$ satisfies the (PS)-condition on $\mathcal{S}_+^\circ$, i.e. every sequence $\left\{w_n^+\right\}_{n\in\mathbb{N}}$ in $\mathcal{S}_+^\circ$ such that, for any $c > 0$, $\tilde{J}_+(w_n^+) \to c$ and $\tilde{J}_+'(w_n^+) \to 0$ as $n \to +\infty$ contains a subsequence which converges strongly to some $w^+ \in \mathcal{S}_+^\circ$ and $\operatorname{dist}\left(w^+,\ \partial\mathcal{S}_+^\circ\right) > 0$.

Proof (i) Let $\{w_n^+\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+^\circ$ be a sequence such that $\operatorname{dist}(w_n^+, \partial\mathcal{S}_+^\circ) \to 0$ as $n \to +\infty$. Then, for any $v \in \partial\mathcal{S}_+^\circ$ and $n \in \mathbb{N}$, it holds $w_n^+ \leq |w_n^+ - v|$ a.e. in $\mathbb{R}^N$. From the embedding theorem, for any $\gamma \in [p, p^*]$, it follows

$$\|w_n^+\|_\gamma \leq \inf_{v\in\partial\mathcal{S}_+^\circ} \|w_n^+ - v\|_\gamma \leq C_\gamma \inf_{v\in\partial\mathcal{S}_+^\circ} \|w_n^+ - v\|_\varepsilon = C_\gamma \operatorname{dist}(w_n^+, \partial\mathcal{S}_+^\circ)$$

for all $n \in \mathbb{N}$. Moreover, for every $t > 0$, by (2.3), we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \tilde{G}(\varepsilon x, t w_n^+)\, \mathrm{d}x\right| &\leq \xi t^p \int_{\mathbb{R}^N} |w_n^+|^p \,\mathrm{d}x + C_\xi t^r \int_{\mathbb{R}^N} |w_n^+|^r \,\mathrm{d}x \\ &\leq C\left(t^p \operatorname{dist}^p(w_n^+, \partial\mathcal{S}_+^\circ) + t^r \operatorname{dist}^r(w_n^+,\ \partial\mathcal{S}_+^\circ)\right) \to 0\end{aligned}$$

as $n \to +\infty$. Note that for any $t > 1$, we have

$$\frac{1}{p}\|tw_n\|_\varepsilon^q + \left|\int_{\mathbb{R}^N} \tilde{G}(\varepsilon x, t w_n^+)\,\mathrm{d}x\right| \geq \tilde{E}_\varepsilon(tw_n^+) \geq \frac{1}{q}\|tw_n\|_\varepsilon^p - \left|\int_{\mathbb{R}^N} \tilde{G}(\varepsilon x, t w_n^+)\,\mathrm{d}x\right|.$$

Therefore, we obtain

$$\liminf_{n\to+\infty} \frac{1}{p}\|m_+^\circ(w_n^+)\|_\varepsilon^q \geq \liminf_{n\to+\infty} \tilde{J}_+(w_n^+) \geq \liminf_{n\to+\infty} \tilde{E}_\varepsilon(tw_n^+) \geq \frac{C_1 t^p}{q},$$

for every $t > 1$, and hence $\|m_+^\circ(w_n^+)\|_\varepsilon \to +\infty$ and $\tilde{J}_+(w_n^+) \to +\infty$ as $n \to +\infty$.

(ii) For any $c > 0$, let $\{w_n^+\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+^\circ$ be a $(\mathrm{PS})_c$-sequence for $\tilde{J}_+$. It follows from Proposition 3.2 that $\{u_n^+ := m_+^\circ(w_n^+)\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_+^\circ$ is a $(\mathrm{PS})_c$-sequence for $\tilde{E}_\varepsilon$. First we will prove that $\{u_n^+\}_{n\in\mathbb{N}}$ is a bounded sequence. Assuming not, we can find a subsequence of $\{u_n^+\}_{n\in\mathbb{N}}$, not relabeled, such that $\|u_n^+\|_\varepsilon \to +\infty$. Set $v_n^+ = u_n^+/\|u_n^+\|_\varepsilon$, then $\{v_n^+\}_{n\in\mathbb{N}}$ is bounded. Thus, after passing to a subsequence if necessary, we may assume that $v_n^+ \rightharpoonup v^+$ in $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ as $n \to +\infty$. If $v^+ = 0$, from Proposition 3.1, we get

$$c + o(1) \geq \tilde{E}_\varepsilon(u_n^+) = \tilde{E}_\varepsilon(t_{v_n^+} v_n^+) \geq \tilde{E}_\varepsilon(tv_n^+) \quad \text{for all } t > 0.$$

In case $t > 1$, we have

$$\begin{aligned}\tilde{E}_\varepsilon(tv_n^+) &\geq \frac{1}{q}t^p - \int_{\mathbb{R}^N} \tilde{G}(\varepsilon x, tv_n^+)\,\mathrm{d}x = \frac{1}{q}t^p - \int_{\Omega_\varepsilon} \tilde{G}(\varepsilon x, tv_n^+)\,\mathrm{d}x - \int_{\Omega_\varepsilon^c} \tilde{G}(\varepsilon x, tv_n^+)\,\mathrm{d}x \\ &\geq \frac{1}{q}t^p - \int_{\Omega_\varepsilon} \tilde{G}(\varepsilon x, tv_n^+)\,\mathrm{d}x - \frac{1}{k}t^p \int_{\Omega_\varepsilon^c} |v_n^+|^p\,\mathrm{d}x \\ &\geq \left(\frac{1}{q} - \frac{1}{k}\right)t^p - \int_{\Omega_\varepsilon} \tilde{G}(\varepsilon x, tv_n^+)\,\mathrm{d}x \to \left(\frac{1}{q} - \frac{1}{k}\right)t^p,\end{aligned}$$

which is a contradiction if we take $t > \max\left\{1,\ 2\left(\frac{cqk}{k-q}\right)^{\frac{1}{p}}\right\}$. If $v^+ \neq 0$, then by (H4) (iii), one has

$$
\begin{aligned}
0 \leq \frac{\tilde{E}_\varepsilon(u_n^+)}{\|u_n^+\|_\varepsilon^q} &\leq \frac{C}{p} - \int_{\mathbb{R}^N} \frac{\tilde{G}(\varepsilon x, \|u_n^+\|_\varepsilon v_n^+)}{\|u_n^+\|_\varepsilon^q}\, \mathrm{d}x \\
&= \frac{C}{p} - \int_{\Omega_\varepsilon} \frac{\tilde{G}(\varepsilon x, \|u_n^+\|_\varepsilon v_n^+)}{\|u_n^+\|_\varepsilon^q}\, \mathrm{d}x - \int_{\Omega_\varepsilon^c} \frac{\tilde{G}(\varepsilon x, \|u_n^+\|_\varepsilon v_n^+)}{\|u_n^+\|_\varepsilon^q}\, \mathrm{d}x \to -\infty
\end{aligned}
$$

as $n \to \infty$. This is again a contradiction. Thus, the sequence $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded and so we can find a subsequence of $\{u_n^+\}_{n\in\mathbb{N}}$, not relabeled, such that $u_n^+ \rightharpoonup u^+$ in $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$. Note that there exists $R_0 > 0$ such that $\Omega_\varepsilon \subset B_{R_0}(0)$. Then, applying hypothesis (H4) (iii), for any $R \geq R_0$, we obtain that

$$
\int_{B_R^c(0)} \tilde{g}\left(\varepsilon x, u_n^+\right) u_n^+\, \mathrm{d}x \leq \frac{1}{k} \int_{B_R^c(0)} \left|u_n^+\right|^p\, \mathrm{d}x. \tag{5.1}
$$

Obviously, we have that

$$
\begin{aligned}
\lim_{r\to+\infty} \int_{B_r^c(0)} \left|u_n^+\right|^p\, \mathrm{d}x &= \lim_{r\to+\infty} \int_{\mathbb{R}^N\setminus B_r(0)} \left|u_n^+\right|^p\, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left|u_n^+\right|^p\, \mathrm{d}x - \lim_{r\to+\infty} \int_{B_r(0)} \left|u_n^+\right|^p\, \mathrm{d}x \\
&= 0.
\end{aligned}
$$

So there exists $R_1 \geq R_0$ such that for any $R \geq R_1$

$$
\int_{B_R^c(0)} \left|u_n^+\right|^p\, \mathrm{d}x = o_n(1), \tag{5.2}
$$

that is,

$$
\int_{B_R^c(0)} \tilde{g}\left(\varepsilon x, u_n^+\right) u_n^+\, \mathrm{d}x \leq o_n(1).
$$

From the compact embedding $W^{1,\mathcal{H}_\varepsilon}(B_R(0)) \hookrightarrow L^p(B_R(0))$ and (H4) (i), we deduce that

$$
\int_{B_R(0)} \tilde{g}\left(\varepsilon x, u_n^+\right) u_n^+\, \mathrm{d}x \to \int_{B_R(0)} \tilde{g}(\varepsilon x, u^+) u^+\, \mathrm{d}x \tag{5.3}
$$

as $n \to +\infty$. Combining (5.3) with (5.1) and (5.2) yields

$$
\int_{\mathbb{R}^N} \tilde{g}\left(\varepsilon x, u_n^+\right) u_n^+\, \mathrm{d}x =: \left\langle \tilde{K}_\varepsilon'(u_n^+), u_n^+ \right\rangle \to \left\langle \tilde{K}_\varepsilon'(u^+), u^+ \right\rangle := \int_{\mathbb{R}^N} \tilde{g}(\varepsilon x, u^+) u^+\, \mathrm{d}x
$$

as $n \to +\infty$. Similarly, we can obtain that $\tilde{K}'_\varepsilon(u_n^+) \to \tilde{K}'_\varepsilon(u^+)$. Since $\tilde{E}'_\varepsilon(u_n^+) = B_\varepsilon(u_n^+) - \tilde{K}'_\varepsilon(u_n^+) \to 0$, one has that $B_\varepsilon(u_n^+) \to \tilde{K}'_\varepsilon(u^+)$ as $n \to +\infty$, where B_ε is given in (2.2). Therefore, we conclude that $u_n^+ \to u^+$ in $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ as $n \to +\infty$, since B_ε is a mapping of type (S_+) (see Proposition 2.2). Consequently $(m_+^\circ)^{-1}(u_n^+) \to (m_+^\circ)^{-1}(u^+)$ by Proposition 3.2, that is, $w_n^+ \to w^+$. Therefore, $\tilde{E}_\varepsilon$ satisfies the (PS)-condition on $\mathcal{S}_+^\circ$. □

The next lemmas can be shown in a similar way as Lemma 5.1.

Lemma 5.2 *Let hypotheses (H0), (H1)(i)–(iii) and (H4) be satisfied and let $\tilde{J}_-$ be given in (3.2). Then the following hold:*

(i) If $\{w_n^-\}_{n\in\mathbb{N}} \subset \mathcal{S}_-^\circ$ is a sequence such that $\operatorname{dist}\left(w_n^-, \partial\mathcal{S}_-^\circ\right) \to 0$ as $n \to +\infty$. Then $\left\|m_-\left(w_n^-\right)\right\|_\varepsilon \to +\infty$ and $\tilde{J}_-\left(w_n^-\right) \to +\infty$ as $n \to +\infty$.

(ii) $\tilde{J}_-$ satisfies the (PS)-condition on $\mathcal{S}_-^\circ$, i.e. every sequence $\{w_n^-\}_{n\in\mathbb{N}}$ in $\mathcal{S}_-^\circ$ such that, for any $c > 0$, $\tilde{J}_-(w_n^-) \to c$ and $\tilde{J}'_-(w_n^-) \to 0$ as $n \to +\infty$ contains a subsequence which converges strongly to some $w^- \in \mathcal{S}_-^\circ$ and $\operatorname{dist}\left(w^-, \partial\mathcal{S}_-^\circ\right) > 0$.

Lemma 5.3 *Let hypotheses (H0), (H1)(i)–(iii) and (H5) be satisfied and let $\hat{J}$ be given in (3.3). Then the following hold:*

(i) If $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_+$ is a sequence such that $\operatorname{dist}(w_n, \partial\mathcal{S}_+) \to 0$ as $n \to +\infty$. Then $\|m(w_n)\|_\varepsilon \to +\infty$ and $\hat{J}(w_n) \to +\infty$ as $n \to +\infty$.

(ii) $\hat{J}$ satisfies the (PS)-condition on $\mathcal{S}_+$, that is, every sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}_+$ such that, for any $c > 0$, $\hat{J}(w_n) \to c$ and $\hat{J}'(w_n) \to 0$ as $n \to +\infty$ contains a subsequence which converges strongly to some $w \in \mathcal{S}_+$ and $\operatorname{dist}(w, \partial\mathcal{S}_+) > 0$.

In what follows, without any loss of generality, we shall assume that $0 \in A$, where A is given in Remark 1.1. Moreover, we choose $\delta > 0$ such that the set

$$A_\delta^- := \{x \in A : \operatorname{dist}(x, \partial A \cup \{0\}) \geq \delta\}$$

is homotopically equivalent to A. Next, we choose a function $\zeta \in C_c^\infty\left(\mathbb{R}^+\right)$ such that $0 \leq \zeta \leq 1$ and

$$\zeta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta/2, \\ 0, & \text{if } s \geq \delta. \end{cases}$$

For each $y \in A_\delta^-$ and $\varepsilon > 0$, we define the function

$$[\Psi_\varepsilon(y)](x) = \zeta(|\varepsilon x - y|)\, \omega\left(\frac{|\varepsilon x - y|}{\varepsilon}\right),$$

where ω is the positive radially symmetric ground state solution of equation (4.1). It can be proved that $[\Psi_\varepsilon(y)](\cdot) \in W^{1,p}(\mathbb{R}^N)$. By definition of ζ and A_δ^-, we also know that $[\Psi_\varepsilon(y)](\cdot) \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$. We define $\Phi_\varepsilon\colon A_\delta^- \to \tilde{\mathcal{N}}_\varepsilon^\circ$ by

$$[\Phi_\varepsilon(y)](x) = t_\varepsilon \left\{[\Psi_\varepsilon(y)](x) - [\Psi_\varepsilon(-y)](x)\right\},$$

where $t_\varepsilon > 0$ is such that $\Phi_\varepsilon(y) \in \tilde{\mathcal{N}}_\varepsilon^\circ$. Propositions 3.1 and 3.3 show that $\Phi_\varepsilon(y)$ is well defined. Note that

$$[\Phi_\varepsilon(y)](-x) = -[\Phi_\varepsilon(y)](x) \quad \text{and} \quad \Phi_\varepsilon(-y) = -\Phi_\varepsilon(y).$$

Hence $\Phi_\varepsilon(y)^+ \in \mathcal{N}_+^\circ$ and $\Phi_\varepsilon(y)^- \in \mathcal{N}_-^\circ$.

Then we have the following lemmas:

Lemma 5.4 *Let hypotheses (H0), (H1) and (H4) be satisfied. Then we have*

$$\lim_{\varepsilon\to 0^+} \tilde{E}_\varepsilon\left(\Phi_\varepsilon(y)^+\right) = c_0^r \quad \textit{uniformly in } y \in A_\delta^-.$$

Proof First, we note that $\Phi_\varepsilon(y)^+ = t_\varepsilon \Psi_\varepsilon(y)$. We argue by contradiction and assume that there exist $\sigma > 0$, $\{y_n\}_{n\in\mathbb{N}} \subset A_\delta^-$ and $\varepsilon_n \to 0^+$ such that

$$\left|\tilde{E}_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y)^+\right) - c_0^r\right| \geq \sigma > 0. \tag{5.4}$$

By changing the variables $z = (\varepsilon_n x - y)/\varepsilon_n$, we deduce from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
&\left\|\Psi_{\varepsilon_n}(y)\right\|_{1,p}^p\\
&= \int_{\mathbb{R}^N} \left(\left|\nabla\Psi_{\varepsilon_n}(y)\right|^p + \left|\Psi_{\varepsilon_n}(y)\right|^p\right) \mathrm{d}x\\
&= \int_{\mathbb{R}^N} \left(\left|\nabla\left(\zeta\left(|\varepsilon_n x - y|\right)\omega\left(\frac{|\varepsilon_n x - y|}{\varepsilon_n}\right)\right)\right|^p\right) \mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N} \left(\left|\zeta\left(|\varepsilon_n x - y|\right)\omega\left(\frac{|\varepsilon_n x - y|}{\varepsilon_n}\right)\right|^p\right) \mathrm{d}x\\
&= \int_{\mathbb{R}^N} \left(\left|\varepsilon_n \omega(|z|)\nabla\zeta(|\varepsilon_n z|) + \zeta(|\varepsilon_n z|)\nabla\omega(|z|)\right|^p + \left|\zeta(|\varepsilon_n z|)\omega(|z|)\right|^p\right) \mathrm{d}z\\
&\to \|\omega(|z|)\|_{1,p}^p.
\end{aligned} \tag{5.5}$$

Similarly, we can check that

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(\varepsilon_n x)\left(\left|\nabla\Psi_{\varepsilon_n}(y)\right|^q + \left|\Psi_{\varepsilon_n}(y)\right|^q\right) \mathrm{d}x\\
&\to \int_{\mathbb{R}^N} a(y)\left(\left|\nabla\omega(|z|)\right|^q + \left|\omega(|z|)\right|^q\right) \mathrm{d}z = 0
\end{aligned}$$

since $y \in A_\delta^- \subset A$ and so $a(y) = 0$. Consequently

$$\begin{aligned}&\varrho_{\varepsilon_n}\left(\Psi_{\varepsilon_n}(y)\right)\\&=\left\|\Psi_{\varepsilon_n}(y)\right\|_{1,p}^p+\int_{\mathbb{R}^N} a(\varepsilon_n x)\left(\left|\nabla\Psi_{\varepsilon_n}(y)\right|^q+\left|\Psi_{\varepsilon_n}(y)\right|^q\right)\,\mathrm{d}x \to \|\omega(|z|)\|_{1,p}^p.\end{aligned} \tag{5.6}$$

By the definition of t_{ε_n} and the change of variables $z = (\varepsilon_n x - y)/\varepsilon_n$, we get

$$\begin{aligned}0&=\left\langle \tilde{E}'_{\varepsilon_n}\left(t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right), t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right\rangle\\&=\varrho_{\varepsilon_n}\left(t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right)-\int_{\mathbb{R}^N}\tilde{g}\left(\varepsilon_n x, t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right)t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\,\mathrm{d}x\\&=\varrho_{\varepsilon_n}\left(t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right)-\int_{\mathbb{R}^N}\tilde{g}\left(\varepsilon_n z+y, t_{\varepsilon_n}\zeta\left(|\varepsilon_n z|\right)\omega\left(|z|\right)\right)t_{\varepsilon_n}\zeta\left(|\varepsilon_n z|\right)\omega\left(|z|\right)\,\mathrm{d}z.\end{aligned}$$

Note that if $\varepsilon_n z \in B_\delta(0)$ then $\varepsilon_n z + y \in B_\delta(y) \subset A \subset \Omega$. If $t_{\varepsilon_n} \to +\infty$, it follows from the above expression that

$$\left\|\Psi_{\varepsilon_n}(y)\right\|_{\varepsilon_n}^q \geq \int_{\mathbb{R}^N}\frac{\tilde{g}\left(\varepsilon_n z+y, t_{\varepsilon_n}\zeta\left(|\varepsilon_n z|\right)\omega\left(|z|\right)\right)}{\left(t_{\varepsilon_n}\zeta\left(|\varepsilon_n z|\right)\omega\left(|z|\right)\right)^{q-1}}\left|\zeta\left(|\varepsilon_n z|\right)\omega\left(|z|\right)\right|^q\,\mathrm{d}z$$

since

$$\varrho_{\varepsilon_n}\left(t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right)\leq\left\|t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right\|_{\varepsilon_n}^q=t_{\varepsilon_n}^q\left\|\Psi_{\varepsilon_n}(y)\right\|_{\varepsilon_n}^q.$$

Then from (H4)(iii) we deduce that $\left\|\Psi_{\varepsilon_n}(y)\right\|_{\varepsilon_n}^q \to +\infty$ and so $\varrho_{\varepsilon_n}\left(\Psi_{\varepsilon_n}(y)\right) \to +\infty$ by Proposition 2.1 (vi), which contradicts (5.6). Thus, we conclude that $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded. Then there exists a subsequence $\{t_{\varepsilon_{n_k}}\}_{k\in\mathbb{N}}$ such that $t_{\varepsilon_{n_k}} \to t_0 \geq 0$. We claim that $t_0 > 0$. Indeed, if $t_0 = 0$, then we can use (2.3) and

$$\left\langle \tilde{E}'_{\varepsilon_{n_k}}\left(t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right), t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right\rangle = 0$$

to get that, for any $\xi > 0$,

$$\begin{aligned}&\left\|t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right\|_{1,p}^p \leq \varrho_{\varepsilon_{n_k}}\left(t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right)\\&=\int_{\mathbb{R}^N}\tilde{g}\left(\varepsilon_{n_k}x, t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right)t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\,\mathrm{d}x\\&\leq\xi\int_{\mathbb{R}^N}\left|t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right|^p\,\mathrm{d}x+C_\xi\int_{\mathbb{R}^N}\left|t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right|^r\,\mathrm{d}x,\end{aligned}$$

that is,

$$\left\|\Psi_{\varepsilon_{n_k}}(y)\right\|_{1,p}^{p} \leq \xi \int_{\mathbb{R}^N} \left|\Psi_{\varepsilon_{n_k}}(y)\right|^p \,\mathrm{d}x + C_\xi t_{\varepsilon_{n_k}}^{r-p} \int_{\mathbb{R}^N} \left|\Psi_{\varepsilon_{n_k}}(y)\right|^r \,\mathrm{d}x.$$

Similar to the above proof, we can deduce that $\left\|\Psi_{\varepsilon_{n_k}}(y)\right\|_{1,p}^{p} \to 0$, contradicting (5.5). Thus $t_0 > 0$. Letting $\varepsilon_{n_k} \to 0^+$ in the following equality

$$\varrho_{\varepsilon_{n_k}}\left(t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right) = \int_{\mathbb{R}^N} \tilde{g}\left(\varepsilon_{n_k}x, t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right) t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\,\mathrm{d}x,$$

similar to above again, we can obtain that

$$\|t_0\omega(|z|)\|_{1,p}^{p} = \int_{\mathbb{R}^N} f\left(t_0\omega\left(|z|\right)\right) t_0\omega\left(|z|\right)\,\mathrm{d}z,$$

from which we conclude that $t_0\omega \in \mathcal{N}_0^r$. Therefore, it follows from the uniqueness of t_0 and $\omega \in \mathcal{N}_0^r$ that $t_0 = 1$. Finally, letting $\varepsilon_{n_k} \to 0^+$ in

$$\begin{aligned}
&\tilde{E}_{\varepsilon_{n_k}}\left(\Phi_{\varepsilon_{n_k}}(y)^+\right)\\
&= \frac{t_{\varepsilon_{n_k}}^p}{p}\left\|\Psi_{\varepsilon_n}(y)\right\|_{1,p}^p + \frac{t_{\varepsilon_{n_k}}^q}{q}\int_{\mathbb{R}^N} a(\varepsilon_{n_k}x)\left(\left|\nabla\Psi_{\varepsilon_{n_k}}(y)\right|^q + \left|\Psi_{\varepsilon_{n_k}}(y)\right|^q\right)\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N} \tilde{G}\left(\varepsilon_{n_k}x, t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right)\mathrm{d}x,
\end{aligned}$$

together with

$$\int_{\mathbb{R}^N} \tilde{G}\left(\varepsilon_{n_k}x, t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right)\mathrm{d}x \to \int_{\mathbb{R}^N} F(\omega)\,\mathrm{d}z,$$

we obtain that

$$\tilde{E}_{\varepsilon_{n_k}}\left(\Phi_{\varepsilon_{n_k}}(y)^+\right) \to \frac{1}{p}\|\omega\|_{1,p}^p - \int_{\mathbb{R}^N} F(\omega)\,\mathrm{d}z = E_0(\omega) = c_0^r,$$

which contradicts (5.4). This shows the assertion of the lemma. □

Lemma 5.5 *Let hypotheses (H0), (H1) and (H4) be satisfied.*

$$\lim_{\varepsilon\to 0^+} \tilde{E}_\varepsilon\left(\Phi_\varepsilon(y)^-\right) = c_0^r \quad \textit{uniformly in } y \in A_\delta^-.$$

Proof By the definition of $\Phi_\varepsilon(y)$, we know that $\Phi_\varepsilon(y)^- = -t_\varepsilon\Psi_\varepsilon(-y)$. Suppose there exist $\sigma > 0$, $\{y_n\}_{n\in\mathbb{N}} \subset A_\delta^-$ and $\varepsilon_n \to 0^+$ such that

$$\left|\tilde{E}_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y)^-\right) - c_0^r\right| \geq \sigma > 0. \tag{5.7}$$

Applying Lebesgue's dominated convergence theorem along with changing the variables with $z = (\varepsilon_n x + y)/\varepsilon_n$ yields

$$\begin{aligned}
&\left\| \Psi_{\varepsilon_n}(-y) \right\|_{1,p}^{p} \\
&= \int_{\mathbb{R}^N} \left(\left| \nabla \Psi_{\varepsilon_n}(-y) \right|^p + \left| \Psi_{\varepsilon_n}(-y) \right|^p \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(\left| \nabla \left(\zeta \left(|\varepsilon_n x + y| \right) \omega \left(\frac{|\varepsilon_n x + y|}{\varepsilon_n} \right) \right) \right|^p \right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left(\left| \zeta \left(|\varepsilon_n x + y| \right) \omega \left(\frac{|\varepsilon_n x + y|}{\varepsilon_n} \right) \right|^p \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(\left| \varepsilon_n \omega \left(|z| \right) \nabla \zeta \left(|\varepsilon_n z| \right) + \zeta \left(|\varepsilon_n z| \right) \nabla \omega \left(|z| \right) \right|^p + \left| \zeta \left(|\varepsilon_n z| \right) \omega \left(|z| \right) \right|^q \right) \mathrm{d}z \\
&\to \| \omega(|z|) \|_{1,p}^{p} .
\end{aligned} \tag{5.8}$$

Since $a(\cdot)$ is radially symmetric (see (H1)(iv)), that is, $a(x) = a(|x|)$ for a.a. $x \in \mathbb{R}^N$, the set A_δ^- is invariant to rotation. In particular, A is symmetric with respect to the origin, that is, $A_\delta^- = -A_\delta^-$. Hence, if $y \in A_\delta^-$, then $-y \in A_\delta^-$ as well. Similar to (5.8), we can check that

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(\varepsilon_n x) \left(\left| \nabla \Psi_{\varepsilon_n}(-y) \right|^q + \left| \Psi_{\varepsilon_n}(-y) \right|^q \right) \mathrm{d}x \\
&\quad \to \int_{\mathbb{R}^N} a(-y) \left(|\nabla \omega \left(|z| \right)|^q + |\omega \left(|z| \right)|^q \right) \mathrm{d}z = 0
\end{aligned}$$

since $-y \in A_\delta^-$ and so $a(-y) = 0$. Consequently

$$\begin{aligned}
&\varrho_{\varepsilon_n} \left(\Psi_{\varepsilon_n}(-y) \right) \\
&\quad = \left\| \Psi_{\varepsilon_n}(-y) \right\|_{1,p}^{p} + \int_{\mathbb{R}^N} a(\varepsilon_n x) \left(\left| \nabla \Psi_{\varepsilon_n}(-y) \right|^q + \left| \Psi_{\varepsilon_n}(-y) \right|^q \right) \mathrm{d}x \\
&\quad \to \| \omega(|z|) \|_{1,p}^{p} .
\end{aligned} \tag{5.9}$$

Changing again the variables $z = (\varepsilon_n x + y)/\varepsilon_n$ together with the definition of t_{ε_n} it follows that

$$\begin{aligned}
0 &= \left\langle \tilde{E}'_{\varepsilon_n} \left(-t_{\varepsilon_n} \Psi_{\varepsilon_n}(-y) \right), -t_{\varepsilon_n} \Psi_{\varepsilon_n}(-y) \right\rangle \\
&= \varrho_{\varepsilon_n} \left(-t_{\varepsilon_n} \Psi_{\varepsilon_n}(-y) \right) - \int_{\mathbb{R}^N} \tilde{g} \left(\varepsilon_n x, -t_{\varepsilon_n} \Psi_{\varepsilon_n}(-y) \right) \left(-t_{\varepsilon_n} \Psi_{\varepsilon_n}(-y) \right) \mathrm{d}x \\
&= \varrho_{\varepsilon_n} \left(-t_{\varepsilon_n} \Psi_{\varepsilon_n}(-y) \right) - \int_{\mathbb{R}^N} \tilde{g} \left(\varepsilon_n z - y, t_{\varepsilon_n} \zeta \left(|\varepsilon_n z| \right) \omega \left(|z| \right) \right) t_{\varepsilon_n} \zeta \left(|\varepsilon_n z| \right) \omega \left(|z| \right) \mathrm{d}z.
\end{aligned}$$

As before, if $\varepsilon_n z \in B_\delta(0)$ then $\varepsilon_n z - y \in B_\delta(-y) \subset A \subset \Omega$. Letting $t_{\varepsilon_n} \to +\infty$ gives

$$\left\| \Psi_{\varepsilon_n}(-y) \right\|_{\varepsilon_n}^q \geq \int_{\mathbb{R}^N} \frac{\tilde{g}\left(\varepsilon_n z - y, t_{\varepsilon_n} \zeta\left(|\varepsilon_n z|\right) \omega\left(|z|\right)\right)}{\left(t_{\varepsilon_n} \zeta\left(|\varepsilon_n z|\right) \omega\left(|z|\right)\right)^{q-1}} \left|\zeta\left(|\varepsilon_n z|\right) \omega\left(|z|\right)\right|^q \,\mathrm{d}z,$$

because

$$\varrho_{\varepsilon_n}\left(-t_{\varepsilon_n} \Psi_{\varepsilon_n}(-y)\right) \leq \left\| -t_{\varepsilon_n} \Psi_{\varepsilon_n}(-y) \right\|_{\varepsilon_n}^q = t_{\varepsilon_n}^q \left\| \Psi_{\varepsilon_n}(y) \right\|_{\varepsilon_n}^q.$$

From (H4)(iii) it follows that $\left\| \Psi_{\varepsilon_n}(-y) \right\|_{\varepsilon_n}^q \to +\infty$ and so $\varrho_{\varepsilon_n}\left(\Psi_{\varepsilon_n}(-y)\right) \to +\infty$ due to Proposition 2.1 (vi), this contradicts (5.9). Hence, we see that the sequence $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded and so there exists a subsequence $\{t_{\varepsilon_{n_k}}\}_{k\in\mathbb{N}}$ of $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ such that $t_{\varepsilon_{n_k}} \to t_0 \geq 0$. Let us show that $t_0 > 0$ and suppose that $t_0 = 0$. Using (2.3) and

$$\left\langle \tilde{E}'_{\varepsilon_{n_k}}\left(-t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y)\right), -t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y) \right\rangle = 0$$

yield that, for any $\xi > 0$,

$$\begin{aligned}
&\left\| -t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y) \right\|_{1,p}^p \\
&\leq \varrho_{\varepsilon_{n_k}}\left(-t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y)\right) \\
&= \int_{\mathbb{R}^N} \tilde{g}\left(\varepsilon_{n_k} x, -t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y)\right) \left(-t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y)\right) \mathrm{d}x \\
&\leq \xi \int_{\mathbb{R}^N} \left| -t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y) \right|^p \mathrm{d}x + C_\xi \int_{\mathbb{R}^N} \left| -t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y) \right|^r \mathrm{d}x.
\end{aligned}$$

Hence

$$\left\| \Psi_{\varepsilon_{n_k}}(-y) \right\|_{1,p}^p \leq \xi \int_{\mathbb{R}^N} \left| \Psi_{\varepsilon_{n_k}}(-y) \right|^p \mathrm{d}x + C_\xi t_{\varepsilon_{n_k}}^{r-p} \int_{\mathbb{R}^N} \left| \Psi_{\varepsilon_{n_k}}(-y) \right|^r \mathrm{d}x.$$

In the same way, we can prove that $\left\| \Psi_{\varepsilon_{n_k}}(-y) \right\|_{1,p}^p \to 0$ which contradicts (5.8). Then we have $t_0 > 0$. Next, letting $\varepsilon_{n_k} \to 0^+$ in the equality

$$\varrho_{\varepsilon_{n_k}}\left(-t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y)\right) = \int_{\mathbb{R}^N} \tilde{g}\left(\varepsilon_{n_k} x, -t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y)\right) \left(-t_{\varepsilon_{n_k}} \Psi_{\varepsilon_{n_k}}(-y)\right) \mathrm{d}x,$$

gives

$$\|t_0 \omega(|z|)\|_{1,p}^p = \int_{\mathbb{R}^N} f\left(t_0 \omega\left(|z|\right)\right) t_0 \omega\left(|z|\right) \mathrm{d}z.$$

This implies that $t_0\omega \in \mathcal{N}_0^r$ and so, from the uniqueness of t_0 and $\omega \in \mathcal{N}_0^r$, we obtain $t_0 = 1$. Then, for $\varepsilon_{n_k} \to 0^+$ in

$$\begin{aligned}
&\tilde{E}_{\varepsilon_{n_k}}\left(\Phi_{\varepsilon_{n_k}}(y)^-\right)\\
&= \frac{t^p_{\varepsilon_{n_k}}}{p}\left\|\Psi_{\varepsilon_n}(-y)\right\|^p_{1,p} + \frac{t^q_{\varepsilon_{n_k}}}{q}\int_{\mathbb{R}^N} a(\varepsilon_{n_k}x)\left(\left|\nabla\Psi_{\varepsilon_{n_k}}(-y)\right|^q + \left|\Psi_{\varepsilon_{n_k}}(-y)\right|^q\right)\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N}\tilde{G}\left(\varepsilon_{n_k}x, -t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(-y)\right)\mathrm{d}x,
\end{aligned}$$

along with

$$\int_{\mathbb{R}^N}\tilde{G}\left(\varepsilon_{n_k}x, -t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(-y)\right)\mathrm{d}x \to \int_{\mathbb{R}^N}F(\omega)\,\mathrm{d}z,$$

we arrive at

$$\tilde{E}_{\varepsilon_{n_k}}\left(\Phi_{\varepsilon_{n_k}}(y)^-\right) \to \frac{1}{p}\|\omega\|^p_{1,p} - \int_{\mathbb{R}^N}F(\omega)\,\mathrm{d}z = E_0(\omega) = c_0^r,$$

contradicting (5.7). □

Now we can prove our existence result for problem (2.4).

Theorem 5.6 *Let hypotheses (H0), (H1) and (H4) be satisfied. Then there exists $\tilde{\varepsilon} > 0$ such that, for any $0 < \varepsilon \leq \tilde{\varepsilon}$, problem (2.4) has at least $\gamma(A \setminus \{0\})$ pairs $\left(u^+, (-u)^+\right)$ of positive weak solutions.*

Proof Taking Lemma 5.4 and Proposition 3.1 into account we have

$$\lim_{\varepsilon\to 0^+}\tilde{J}_+\left((m_+^\circ)^{-1}\left(\Phi_\varepsilon(y)^+\right)\right) = \lim_{\varepsilon\to 0^+}\tilde{E}_\varepsilon\left(\Phi_\varepsilon(y)^+\right) = c_0^r$$

uniformly in $y \in A_\delta^-$. For each $y \in A_\delta^-$, we set

$$h(\varepsilon) := \left|\tilde{E}_\varepsilon\left(\Phi_\varepsilon(y)^+\right) - c_0^r\right|.$$

Then $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Now we write

$$\widetilde{\mathcal{S}_+^\circ} := \left\{u^+ \in \mathcal{S}_+^\circ : \tilde{J}_+(u^+) \leq c_0^r + h(\varepsilon)\right\}.$$

It is clear that $\widetilde{\mathcal{S}_+^\circ} \neq \emptyset$ since $(m_+^\circ)^{-1}(\Phi_\varepsilon(y)^+) \in \widetilde{\mathcal{S}_+^\circ}$. Then, by Lemma 5.1 and Krasnosel'skii's genus theory (see Ambrosetti-Malchiodi [4, Theorem 10.9]), we know that $\tilde{J}_+$ has at least $\gamma(\widetilde{\mathcal{S}_+^\circ})$ pairs $\left(u^+, (-u)^+\right)$ of critical points on $\widetilde{\mathcal{S}_+^\circ}$.
Claim: $\gamma(\widetilde{\mathcal{S}_+^\circ}) \geq \gamma(A \setminus \{0\})$.

Assume that $\gamma(\widetilde{\mathcal{S}^\circ_+}) = n$ and note that for a set $\mathcal{A}$ we write $\mathcal{A}^* = \{(x, -x) : x \in \mathcal{A}\}$. We deduce that

$$\gamma(\widetilde{\mathcal{S}^\circ_+}) = \operatorname{cat}_{(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)\setminus\{0\})^*}\widetilde{\mathcal{S}^\circ_+}^*,$$

see Rabinowitz [44, Theorem 3.9]. Hence, we can find a smallest positive integer n such that

$$\widetilde{\mathcal{S}^\circ_+}^* \subseteq \mathcal{D}_1^* \cup \mathcal{D}_2^* \cup \cdots \cup \mathcal{D}_n^*,$$

where $\mathcal{D}_i^*, i = 1, 2, \cdots, n$ are closed and contractible in $(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\})^*$, which means that there are

$$h_i^* \in C\left([0,1] \times \mathcal{D}_i^*, \left(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\}\right)^*\right) \quad \text{for } i = 1, 2, \cdots, n$$

such that

$$\begin{aligned}
&h_i^*(0, u^+) = (u^+, (-u)^+) \quad \text{for all } (u^+, (-u)^+) \in \mathcal{D}_i^*,\\
&h_i^*(1, u^+) = (\omega_i, -\omega_i) \in \left(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\}\right)^* \quad \text{for all } (u^+, (-u)^+) \in \mathcal{D}_i^*.
\end{aligned}$$

Let

$$\mathcal{D}_i = \left\{u^+ \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) : (u^+, (-u)^+) \in \mathcal{D}_i^*\right\}.$$

Then there exists a homotopy

$$h_i \in C\left([0,1] \times \mathcal{D}_i, \left(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\}\right)\right)$$

such that $h_i(0,\cdot) = \mathrm{id}$, $h_i(1,\cdot) = \omega_i$ or $-\omega_i$ and $h_i(t, u^+) = -h_i(t, (-u)^+)$.

We define

$$\begin{aligned}
&\Phi_\varepsilon^* = (\Phi_\varepsilon^+, (-\Phi_\varepsilon)^+) \colon \left(A_\delta^-\right)^* \to \left(\mathcal{N}_+^\circ\right)^*,\\
&\left[\Phi_\varepsilon^*(y, -y)\right](x) = \left([\Phi_\varepsilon(y)]^+(x), [\Phi_\varepsilon(-y)]^+(x)\right).
\end{aligned}$$

Now we choose $R \geq \operatorname{diam}(A)$, where $\operatorname{diam}(A)$ denotes the diameter of A. For $u \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ with compact support in $B_R(0)$, we define the barycenter map

$$\beta_+ \colon W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N, \quad \beta_+(u) = \frac{\displaystyle\int_{\mathbb{R}^N} x|u^+(x)|^p \,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u^+(x)|^p \,\mathrm{d}x}.$$

We observe that for any $(y,-y) \in \left(A_\delta^-\right)^*$ we have

$$\beta_+\left(\Phi_\varepsilon(y)^+\right) = y \quad \text{and} \quad \beta_+\left(\Phi_\varepsilon(-y)^+\right) = -y.$$

Next, we write $\beta^*(\cdot,\cdot) = (\beta_+(\cdot), \beta_+(\cdot))$ and obtain

$$\beta^*\left(\Phi_\varepsilon(y)^+, [-\Phi_\varepsilon(y)]^+\right) = \left(\beta_+\left(\Phi_\varepsilon(y)^+\right), \beta_+\left[\Phi_\varepsilon(-y)\right]^+\right) = (y,-y).$$

Let

$$\mathcal{K}_i^* = \left(\Phi_\varepsilon^*\right)^{-1}\left(m^*\left(\mathcal{D}_i^*\right)\right),$$

where $m^*(\cdot,\cdot) = (m_+^\circ(\cdot), m_+^\circ(\cdot))$. Obviously the sets $\mathcal{K}_i^*$ are closed subsets of $\left(A_\delta^-\right)^*$ and $\left(A_\delta^-\right)^* \subseteq \mathcal{K}_1^* \cup \cdots \cup \mathcal{K}_n^*$. Defining the deformation $\mathfrak{h}_i : [0,1] \times \mathcal{K}_i^* \to \left(\mathbb{R}^N \setminus \{0\}\right)^*$ by

$$\mathfrak{h}_i(t,x) = \left(\beta^* \circ h_i^*\right)\left(t, \left(m^*\right)^{-1}\left(\Phi_\varepsilon^*(y,-y)\right)\right),$$

we see that $\mathcal{K}_i^*$ is contractible in $\left(\mathbb{R}^N \setminus \{0\}\right)^*$. Indeed, we have

$$\begin{aligned}
\mathfrak{h}_i &\in C\left([0,1] \times \mathcal{K}_i^*, \left(\mathbb{R}^N \setminus \{0\}\right)^*\right),\\
\mathfrak{h}_i(0,x) &= \left(\beta^* \circ h_i^*\right)\left(0, \left(m^*\right)^{-1}\left(\Phi_\varepsilon^*(y,-y)\right)\right) = (y,-y) \quad \text{for all } (y,-y) \in \mathcal{K}_i^*,\\
\mathfrak{h}_i(1,x) &= \left(\beta^* \circ h_i^*\right)\left(1, \left(m^*\right)^{-1}\left(\Phi_\varepsilon^*(y,-y)\right)\right)\\
&= \beta^*\left(\omega_i, -\omega_i\right) = \left(y_i^0, -y_i^0\right) \in \left(\mathbb{R}^N \setminus \{0\}\right)^* \quad \text{for all } (y,-y) \in \mathcal{K}_i^*.
\end{aligned}$$

Thus

$$\gamma\left(A \setminus \{0\}\right) = \operatorname{cat}_{(\mathbb{R}^N\setminus\{0\})^*}\left(A \setminus \{0\}\right)^* = \operatorname{cat}_{(\mathbb{R}^N\setminus\{0\})^*}\left(A_\delta^-\right)^* \le n,$$

which implies that $\widetilde{\mathcal{S}_+^\circ}$ contains at least $\gamma(A \setminus \{0\})$ pairs of critical points of $\tilde{J}_+$. Thus we conclude from Proposition 3.2 that there exist at least $\gamma(A \setminus \{0\})$ pairs $(u^+, (-u)^+)$ of critical points of $\tilde{E}_\varepsilon$, that is, problem (2.4) has at least $\gamma(A \setminus \{0\})$ pairs of positive weak solutions. □

Next, we are going to prove the existence of negative solutions for problem (2.4).

Theorem 5.7 *Let hypotheses (H0), (H1) and (H4) be satisfied. Then there exists $\tilde{\varepsilon} > 0$ such that, for any $0 < \varepsilon \le \tilde{\varepsilon}$, problem (2.4) has at least $\gamma(A \setminus \{0\})$ pairs $\left(u^-, (-u)^-\right)$ of negative weak solutions.*

Proof As before, using Lemma 5.5 and Proposition 3.3, we know that

$$\lim_{\varepsilon\to 0^+} \tilde{J}_-\left((m_-^\circ)^{-1}\left(\Phi_\varepsilon(y)^-\right)\right) = \lim_{\varepsilon\to 0^+} \tilde{E}_\varepsilon\left(\Phi_\varepsilon(y)^-\right) = c_0^r$$

uniformly in $y \in A_\delta^-$. For each $y \in A_\delta^-$, we set

$$h(\varepsilon) := \left|\tilde{E}_\varepsilon\left(\Phi_\varepsilon(y)^-\right) - c_0^r\right|.$$

This gives $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Setting

$$\widetilde{\mathcal{S}_-^\circ} := \left\{u^- \in \mathcal{S}_-^\circ : \tilde{J}_-(u^-) \le c_0^r + h(\varepsilon)\right\}.$$

We easily see that $\widetilde{\mathcal{S}_-^\circ} \neq \emptyset$ because $(m_-^\circ)^{-1}(\Phi_\varepsilon(y)^-) \in \widetilde{\mathcal{S}_-^\circ}$. Then, from Lemma 5.2 and Ambrosetti-Malchiodi [4, Theorem 10.9], it follows that $\tilde{J}_-$ has at least $\gamma(\widetilde{\mathcal{S}_-^\circ})$ pairs $\left(u^-, (-u)^-\right)$ of critical points on $\widetilde{\mathcal{S}_-^\circ}$.

Claim: $\gamma(\widetilde{\mathcal{S}_-^\circ}) \ge \gamma(A \setminus \{0\})$.

Suppose that $\gamma(\widetilde{\mathcal{S}_-^\circ}) = n$ and recall that we write $\mathcal{A}^* = \{(x, -x) : x \in \mathcal{A}\}$ for a set $\mathcal{A}$. From [44, Theorem 3.9] it follows that

$$\gamma(\widetilde{\mathcal{S}_-^\circ}) = \operatorname{cat}_{(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)\setminus\{0\})^*} \widetilde{\mathcal{S}_-^\circ}^*,$$

which guarantees the existence of a smallest positive integer n such that

$$\widetilde{\mathcal{S}_-^\circ}^* \subseteq \mathcal{D}_1^* \cup \mathcal{D}_2^* \cup \cdots \cup \mathcal{D}_n^*,$$

with $\mathcal{D}_i^*$, $i = 1, 2, \cdots, n$ being closed and contractible in $(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\})^*$, e.g., there exist

$$h_i^* \in C\left([0,1] \times \mathcal{D}_i^*, \left(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\}\right)^*\right) \quad \text{for } i = 1, 2, \cdots, n$$

such that

$$\begin{aligned} &h_i^*(0, u^-) = (u^-, (-u)^-) \quad \text{for all } (u^-, (-u)^-) \in \mathcal{D}_i^*,\\ &h_i^*(1, u^-) = (\omega_i, -\omega_i) \in \left(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\}\right)^* \quad \text{for all } (u^-, (-u)^-) \in \mathcal{D}_i^*. \end{aligned}$$

We define

$$\mathcal{D}_i = \left\{u^- \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) : (u^-, (-u)^-) \in \mathcal{D}_i^*\right\}.$$

Then we can find a homotopy

$$h_i \in C\left([0,1] \times \mathcal{D}_i, \left(W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\}\right)\right)$$

satisfying $h_i(0,\cdot) = \mathrm{id}$, $h_i(1,\cdot) = \omega_i$ or $-\omega_i$ and $h_i(t,u^-) = -h_i(t,(-u)^-)$. Next we define

$$\begin{aligned}&\Phi_\varepsilon^* = (\Phi_\varepsilon^-, (-\Phi_\varepsilon)^-) \colon \left(A_\delta^-\right)^* \to \left(\mathcal{N}_-^\circ\right)^*,\\ &\left[\Phi_\varepsilon^*(y,-y)\right](x) = \left([\Phi_\varepsilon(y)]^-(x), [\Phi_\varepsilon(-y)]^-(x)\right).\end{aligned}$$

Taking $R \geq \mathrm{diam}(A)$, we define the barycenter map, for $u \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ with compact support in $B_R(0)$, by

$$\beta_- \colon W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N, \quad \beta_-(u) = \frac{\displaystyle\int_{\mathbb{R}^N} x|u^-(x)|^p \,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u^-(x)|^p \,\mathrm{d}x}.$$

Clearly, for any $(y,-y) \in \left(A_\delta^-\right)^*$, we have

$$\beta_-\left(\Phi_\varepsilon(y)^-\right) = y \quad \text{and} \quad \beta_-\left(\Phi_\varepsilon(-y)^-\right) = -y.$$

As before, we write $\beta^*(\cdot,\cdot) = (\beta_-(\cdot), \beta_-(\cdot))$ and get

$$\beta^*\left(\Phi_\varepsilon(y)^-, [-\Phi_\varepsilon(y)]^-\right) = \left(\beta_-\left(\Phi_\varepsilon(y)^-\right), \beta_-\left[\Phi_\varepsilon(-y)\right]^-\right) = (y,-y).$$

Note that the sets

$$\mathcal{K}_i^* = \left(\Phi_\varepsilon^*\right)^{-1}\left(m^*\left(\mathcal{D}_i^*\right)\right),$$

are closed subsets of $\left(A_\delta^-\right)^*$ and it holds $\left(A_\delta^-\right)^* \subseteq \mathcal{K}_1^* \cup \cdots \cup \mathcal{K}_n^*$, where $m^*(\cdot,\cdot) = (m_-^\circ(\cdot), m_-^\circ(\cdot))$. Also, the sets $\mathcal{K}_i^*$, $i = 1,\ldots,n$, are contractible in $\left(\mathbb{R}^N \setminus \{0\}\right)^*$ due to the deformation $\mathfrak{h}_i \colon [0,1] \times \mathcal{K}_i^* \to \left(\mathbb{R}^N \setminus \{0\}\right)^*$ defined by

$$\mathfrak{h}_i(t,x) = \left(\beta^* \circ h_i^*\right)\left(t, \left(m^*\right)^{-1}\left(\Phi_\varepsilon^*(y,-y)\right)\right).$$

Indeed, we have

$$\begin{aligned}&\mathfrak{h}_i \in C\left([0,1] \times \mathcal{K}_i^*, \left(\mathbb{R}^N \setminus \{0\}\right)^*\right),\\ &\mathfrak{h}_i(0,x) = \left(\beta^* \circ h_i^*\right)\left(0, \left(m^*\right)^{-1}\left(\Phi_\varepsilon^*(y,-y)\right)\right) = (y,-y) \quad \text{for all } (y,-y) \in \mathcal{K}_i^*,\\ &\mathfrak{h}_i(1,x) = \left(\beta^* \circ h_i^*\right)\left(1, \left(m^*\right)^{-1}\left(\Phi_\varepsilon^*(y,-y)\right)\right)\\ &\qquad = \beta^*\left(\omega_i, -\omega_i\right) = \left(y_i^0, -y_i^0\right) \in \left(\mathbb{R}^N \setminus \{0\}\right)^* \quad \text{for all } (y,-y) \in \mathcal{K}_i^*,\end{aligned}$$

which implies

$$\gamma\left(A \setminus \{0\}\right) = \mathrm{cat}_{(\mathbb{R}^N \setminus \{0\})^*}\left(A \setminus \{0\}\right)^* = \mathrm{cat}_{(\mathbb{R}^N \setminus \{0\})^*}\left(A_\delta^- \setminus \{0\}\right)^* \leq n.$$

Hence, $\widetilde{\mathcal{S}_-^\circ}$ contains at least $\gamma(A \setminus \{0\})$ pairs of critical points of $\tilde{J}_-$. From Proposition 3.4 we deduce that there are at least $\gamma(A \setminus \{0\})$ pairs $(u^-, (-u)^-)$ of critical points of $\tilde{E}_\varepsilon$. This means that problem (2.4) has at least $\gamma(A \setminus \{0\})$ pairs of negative weak solutions. □

Finally we give the existence result for odd weak solutions with precisely two nodal domains for (2.4).

Theorem 5.8 *Let hypotheses (H0), (H1) and (H4) be satisfied. Then there exists $\tilde{\varepsilon} > 0$ such that, for any $0 < \varepsilon \leq \tilde{\varepsilon}$, problem (2.4) has at least $\gamma(A \setminus \{0\})$ pairs $\left(u^+ + u^-, (-u)^+ + (-u)^-\right)$ of odd weak solutions with precisely two nodal domains.*

Proof Note that $\tilde{E}_\varepsilon(u) = \tilde{E}_\varepsilon\left(u^+ + u^-\right) = \tilde{E}_\varepsilon\left(u^+\right) + \tilde{E}_\varepsilon\left(u^-\right)$. Hence if u^+ and u^- are the critical points of $\tilde{E}_\varepsilon$, then is so $u = u^+ + u^-$ as well. Consequently, Theorem 5.8 follows from Theorems 5.6 and 5.7. □

Now we will prove an existence result for problem (2.5). We choose $\delta > 0$ such that $A_\delta = \left\{x \in \mathbb{R}^N : \operatorname{dist}(x, A) < \delta\right\}$ is homotopically equivalent to A and $A_\delta \subset \Omega$. Define a function $\zeta \in C_c^\infty\left(\mathbb{R}^+\right)$ such that $0 \leq \zeta \leq 1$ and

$$\zeta(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq \delta/2, \\ 0, & \text{if } t \geq \delta. \end{cases}$$

For each $y \in A$ and $\varepsilon > 0$, we define the function

$$\Psi_{\varepsilon,y}(x) = \zeta\left(|\varepsilon x - y|\right) \omega\left(\frac{\varepsilon x - y}{\varepsilon}\right),$$

with ω being the positive ground state solution of equation (4.1). We define

$$\Phi_\varepsilon \colon A \to \mathcal{N}_\varepsilon, \quad \Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y},$$

where t_ε is the unique positive number such that

$$\max_{t \geq 0} \hat{E}_\varepsilon\left(t\Psi_{\varepsilon,y}(x)\right) = \hat{E}_\varepsilon\left(t_\varepsilon \Psi_{\varepsilon,y}(x)\right),$$

that is,

$$t_\varepsilon \Psi_{\varepsilon,\xi} \in \mathcal{N}_\varepsilon.$$

It follows from Proposition 3.5 that $\Phi_\varepsilon(y)$ is well defined since $\zeta\left(|\varepsilon x - y|\right) = 1$ for all $x \in B_{\delta/2\varepsilon}(y/\varepsilon)$ and $y/\varepsilon \in A_\varepsilon := \left\{x \in \mathbb{R}^N : \varepsilon x \in A\right\}$.

Lemma 5.9 *Let hypotheses (H0), (H1)(i)–(iii) and (H5) be satisfied. Then we have*

$$\lim_{\varepsilon \to 0^+} \hat{E}_\varepsilon\left(\Phi_\varepsilon(y)\right) = c_0 \quad \textit{uniformly in } y \in A.$$

Proof We argue by contradiction and assume that there exist $\sigma > 0$, $\{y_n\}_{n\in\mathbb{N}} \subset A$ and $\varepsilon_n \to 0^+$ such that

$$\left|\hat{E}_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y)\right) - c_0\right| \geq \sigma > 0. \tag{5.10}$$

Using Lebesgue's dominated convergence theorem with a change of variables via $z = (\varepsilon_n x - y)/\varepsilon_n$, it follows that

$$\begin{aligned}
&\left\|\Psi_{\varepsilon_n}(y)\right\|_{1,p}^p \\
&= \int_{\mathbb{R}^N} \left(\left|\nabla\Psi_{\varepsilon_n}(y)\right|^p + \left|\Psi_{\varepsilon_n}(y)\right|^p\right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(\left|\nabla\left(\zeta\left(|\varepsilon_n x - y|\right)\omega\left(\frac{\varepsilon_n x - y}{\varepsilon_n}\right)\right)\right|^p\right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left(\left|\zeta\left(|\varepsilon_n x - y|\right)\omega\left(\frac{\varepsilon_n x - y}{\varepsilon_n}\right)\right|^p\right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(\left|\varepsilon_n\omega(z)\nabla\zeta\left(|\varepsilon_n z|\right) + \zeta\left(|\varepsilon_n z|\right)\nabla\omega(z)\right|^p + \left|\zeta\left(|\varepsilon_n z|\right)\omega(z)\right|^q\right) \mathrm{d}z \\
&\to \|\omega(z)\|_{1,p}^p .
\end{aligned} \tag{5.11}$$

In a similar way, we can obtain that

$$\int_{\mathbb{R}^N} a(\varepsilon_n x)\left(\left|\nabla\Psi_{\varepsilon_n}(y)\right|^q + \left|\Psi_{\varepsilon_n}(y)\right|^q\right) \mathrm{d}x \to \int_{\mathbb{R}^N} a(y)\left(|\nabla\omega(z)|^q + |\omega(z)|^q\right) \mathrm{d}z = 0$$

as $y \in A$ and so $a(y) = 0$. Therefore,

$$\begin{aligned}
&\varrho_{\varepsilon_n}\left(\Psi_{\varepsilon_n}(y)\right) \\
&\quad = \left\|\Psi_{\varepsilon_n}(y)\right\|_{1,p}^p + \int_{\mathbb{R}^N} a(\varepsilon_n x)\left(\left|\nabla\Psi_{\varepsilon_n}(y)\right|^q + \left|\Psi_{\varepsilon_n}(y)\right|^q\right) \mathrm{d}x \to \|\omega(z)\|_{1,p}^p .
\end{aligned} \tag{5.12}$$

Again by changing the variables by $z = (\varepsilon_n x - y)/\varepsilon_n$ and the definition of t_{ε_n} leads to

$$\begin{aligned}
0 &= \left\langle \hat{E}'_{\varepsilon_n}\left(t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right), t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right\rangle \\
&\quad = \varrho_{\varepsilon_n}\left(t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right) - \int_{\mathbb{R}^N} \hat{g}\left(\varepsilon_n x, t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right) t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\ \mathrm{d}x \\
&\quad = \varrho_{\varepsilon_n}\left(t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right) - \int_{\mathbb{R}^N} \hat{g}\left(\varepsilon_n z + y, t_{\varepsilon_n}\zeta\left(|\varepsilon_n z|\right)\omega(z)\right) t_{\varepsilon_n}\zeta\left(|\varepsilon_n z|\right)\omega(z)\ \mathrm{d}z.
\end{aligned}$$

If $\varepsilon_n z \in B_\delta(0)$ then $\varepsilon_n z + y \in B_\delta(y) \subset A_\delta \subset \Omega$. Letting $t_{\varepsilon_n} \to +\infty$ gives

$$\left\|\Psi_{\varepsilon_n}(y)\right\|_{\varepsilon_n}^q \geq \int_{\mathbb{R}^N} \frac{\hat{g}\left(\varepsilon_n z + y, t_{\varepsilon_n}\zeta\left(|\varepsilon_n z|\right)\omega(z)\right)}{\left(t_{\varepsilon_n}\zeta\left(|\varepsilon_n z|\right)\omega(z)\right)^{q-1}} \left|\zeta\left(|\varepsilon_n z|\right)\omega(z)\right|^q \,\mathrm{d}z,$$

due to the fact that

$$\varrho_{\varepsilon_n}\left(t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right) \leq \left\|t_{\varepsilon_n}\Psi_{\varepsilon_n}(y)\right\|_{\varepsilon_n}^q = t_{\varepsilon_n}^q \left\|\Psi_{\varepsilon_n}(y)\right\|_{\varepsilon_n}^q.$$

From (H5)(ii) we deduce that $\left\|\Psi_{\varepsilon_n}(y)\right\|_{\varepsilon_n}^q \to +\infty$ and so $\varrho_{\varepsilon_n}\left(\Psi_{\varepsilon_n}(y)\right) \to +\infty$ by Proposition 2.1 (vi), which contradicts (5.12). Hence, $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded and so we can find a subsequence $\{t_{\varepsilon_{n_k}}\}_{k\in\mathbb{N}}$ such that $t_{\varepsilon_{n_k}} \to t_0 \geq 0$. Suppose that $t_0 = 0$, then we get from (2.3) and

$$\left\langle \hat{E}'_{\varepsilon_{n_k}}\left(t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right), t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right\rangle = 0,$$

for any $\xi > 0$,

$$\begin{aligned}
&\left\|t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right\|_{1,p}^p \leq \varrho_{\varepsilon_{n_k}}\left(t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right)\\
&= \int_{\mathbb{R}^N} \hat{g}\left(\varepsilon_{n_k}x, t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right) t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\,\mathrm{d}x\\
&\leq \xi \int_{\mathbb{R}^N} \left|t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right|^p \mathrm{d}x + C_\xi \int_{\mathbb{R}^N} \left|t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right|^r \mathrm{d}x,
\end{aligned}$$

which results in

$$\left\|\Psi_{\varepsilon_{n_k}}(y)\right\|_{1,p}^p \leq \xi \int_{\mathbb{R}^N} \left|\Psi_{\varepsilon_{n_k}}(y)\right|^p \mathrm{d}x + C_\xi t_{\varepsilon_{n_k}}^{r-p} \int_{\mathbb{R}^N} \left|\Psi_{\varepsilon_{n_k}}(y)\right|^r \mathrm{d}x.$$

Using similar arguments, we are able to show that $\left\|\Psi_{\varepsilon_{n_k}}(y)\right\|_{1,p}^p \to 0$, contradicting (5.11). We conclude that $t_0 > 0$. Letting $\varepsilon_{n_k} \to 0^+$ in

$$\varrho_{\varepsilon_{n_k}}\left(t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right) = \int_{\mathbb{R}^N} \hat{g}\left(\varepsilon_{n_k}x, t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right) t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\,\mathrm{d}x,$$

it follows that

$$\|t_0\omega(z)\|_{1,p}^p = \int_{\mathbb{R}^N} f\left(t_0\omega(z)\right) t_0\omega(z)\,\mathrm{d}z.$$

This yields $t_0\omega \in \mathcal{N}_0$ and so from the uniqueness of t_0 as well as $\omega \in \mathcal{N}_0$ we arrive at $t_0 = 1$. Finally, letting $\varepsilon_{n_k} \to 0^+$ in

$$\begin{aligned}
&\hat{E}_{\varepsilon_{n_k}}\left(\Phi_{\varepsilon_{n_k}}(y)\right)\\
&= \frac{t^p_{\varepsilon_{n_k}}}{p}\left\|\Psi_{\varepsilon_n}(y)\right\|^p_{1,p} + \frac{t^q_{\varepsilon_{n_k}}}{q}\int_{\mathbb{R}^N} a(\varepsilon_{n_k}x)\left(\left|\nabla\Psi_{\varepsilon_{n_k}}(y)\right|^q + \left|\Psi_{\varepsilon_{n_k}}(y)\right|^q\right)\,\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^N}\hat{G}\left(\varepsilon_{n_k}x, t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right)\,\mathrm{d}x,
\end{aligned}$$

by using

$$\int_{\mathbb{R}^N}\hat{G}\left(\varepsilon_{n_k}x, t_{\varepsilon_{n_k}}\Psi_{\varepsilon_{n_k}}(y)\right)\,\mathrm{d}x \to \int_{\mathbb{R}^N} F(\omega)\,\mathrm{d}z,$$

this leads to

$$\hat{E}_{\varepsilon_{n_k}}\left(\Phi_{\varepsilon_{n_k}}(y)\right) \to \frac{1}{p}\|\omega\|^p_{1,p} - \int_{\mathbb{R}^N} F(\omega)\,\mathrm{d}z = E_0(\omega) = c_0,$$

which contradicts (5.10). □

Now, we choose $R > 0$ such that $A_\delta \subset B_R(0)$ and let $\kappa\colon \mathbb{R}^N \to \mathbb{R}^N$ be defined by

$$\kappa(x) = \begin{cases} x, & \text{if } |x| < R,\\ \frac{Rx}{|x|}, & \text{if } |x| \geq R.\end{cases}$$

Next, we define $\beta_\varepsilon\colon \mathcal{N}_\varepsilon \to \mathbb{R}^N$ by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N}\kappa(\varepsilon x)|u(x)|^p\,\mathrm{d}x}{\int_{\mathbb{R}^N}|u(x)|^p\,\mathrm{d}x}.$$

Since $A \subset A_\delta \subset B_R(0)$ we have that

$$\begin{aligned}
\beta_\varepsilon\left(\Phi_\varepsilon(y)\right) &= \frac{\int_{\mathbb{R}^N}\kappa(\varepsilon x)\left|\Phi_\varepsilon(y)\right|^p\,\mathrm{d}x}{\int_{\mathbb{R}^N}\left|\Phi_\varepsilon(y)\right|^p\,\mathrm{d}x}\\
&= \frac{\int_{\mathbb{R}^N}\kappa(\varepsilon x)\left|t_\varepsilon\zeta\left(|\varepsilon x - y|\right)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right)\right|^p\,\mathrm{d}x}{\int_{\mathbb{R}^N}\left|t_\varepsilon\zeta\left(|\varepsilon x - y|\right)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right)\right|^p\,\mathrm{d}x}\\
&= \frac{\int_{\mathbb{R}^N}\kappa\left(\varepsilon z + y\right)\left|\zeta\left(|\varepsilon z|\right)\omega(z)\right|^p\,\mathrm{d}z}{\int_{\mathbb{R}^N}\left|\zeta\left(|\varepsilon z|\right)\omega(z)\right|^p\,\mathrm{d}z}\\
&= y + \frac{\int_{\mathbb{R}^N}\left(\kappa\left(\varepsilon z + y\right) - y\right)\left|\zeta\left(|\varepsilon z|\right)\omega(z)\right|^p\,\mathrm{d}z}{\int_{\mathbb{R}^N}\left|\zeta\left(|\varepsilon z|\right)\omega(z)\right|^p\,\mathrm{d}z}\\
&= y + o(1),
\end{aligned}$$

as $\varepsilon \to 0$, uniformly for $y \in A$.

Lemma 5.10 *Let hypotheses (H0), (H1)(i)–(iii) and (H4) be satisfied and let* $\varepsilon_n \to 0$ *as* $n \to +\infty$ *and* $\{u_n\}_{n\in\mathbb{N}} \subset \tilde{\mathcal{N}}_{\varepsilon_n}$ *be such that* $\tilde{E}_{\varepsilon_n}(u_n) \to c_0^r$ *as* $n \to +\infty$. *Then there exists a subsequence* $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ *such that* $\varepsilon_n \tilde{y}_n =: y_n \to y \in A$ *as* $n \to +\infty$. *Moreover, up to a subsequence,* $v_n(\cdot) := u_n(\cdot + \tilde{y}_n)$ *converges strongly in* $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$.

The proof of Lemma 5.10 is similar to the proof of the following lemma.

Lemma 5.11 *Let hypotheses (H0), (H1)(i)–(iii) and (H5) be satisfied and let* $\varepsilon_n \to 0$ *as* $n \to +\infty$ *and* $\{u_n\}_{n\in\mathbb{N}} \subset \hat{\mathcal{N}}_{\varepsilon_n}$ *be such that* $\hat{E}_{\varepsilon_n}(u_n) \to c_0$ *as* $n \to +\infty$. *Then there exists a subsequence* $\left\{\hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ *such that* $\varepsilon_n \hat{y}_n =: y_n \to y \in A$ *as* $n \to +\infty$. *Moreover, up to a subsequence,* $v_n(\cdot) := u_n\left(\cdot + \hat{y}_n\right)$ *converges strongly in* $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$.

Proof As in the proof of Lemma 5.1 we can show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. We first claim that there is a sequence $\left\{\hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R,\ \sigma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\hat{y}_n)} |u_n|^p \,\mathrm{d}x \geq \sigma > 0. \tag{5.13}$$

Suppose this is not true. Then the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ together with Lemma I.1 of Lions [32] imply that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $p < s < p^*$. Since $\{u_n\}_{n\in\mathbb{N}} \subset \hat{\mathcal{N}}_{\varepsilon_n}$ and due to (2.3) we have

$$\begin{aligned}
\|u_n\|_{\varepsilon_n}^q &\leq \int_{\mathbb{R}^N} \left(|\nabla u_n|^p + a(\varepsilon_n x)|\nabla u_n|^q + |u_n|^p + a(\varepsilon_n x)|u_n|^q\right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \hat{g}(x, u_n) u_n \,\mathrm{d}x \\
&\leq \xi \int_{\mathbb{R}^N} |u_n|^p \,\mathrm{d}x + C_\xi \int_{\mathbb{R}^N} |u_n|^r \,\mathrm{d}x.
\end{aligned}$$

We conclude that $\|u_n\|_{\varepsilon_n} \to 0$ due to the arbitrariness of ξ and $u_n \to 0$ in $L^r(\mathbb{R}^N)$. We also know that $\int_{\mathbb{R}^N} \hat{G}(x, u_n)\,\mathrm{d}x \to 0$. Therefore, $\hat{E}_{\varepsilon_n}(u_n) \to 0$, which contradicts $c_0 > 0$, and (5.13) is proved.

Let $v_n = u_n\left(\cdot + \hat{y}_n\right)$. Up to a subsequence, we can assume that $v_n \rightharpoonup v \neq 0$ in $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$. Since $W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N) \hookrightarrow W^{1,p}\left(\mathbb{R}^N\right)$, we can choose $t_{v_n} > 0$ to be such that $w_n := t_{v_n} v_n \in \mathcal{N}_0$.

Note that $\max_{t\geq 0} \hat{E}_{\varepsilon_n}(tu_n)$ is obtained at $t = 1$. Using the translation invariance of the Lebesgue integral and $u_n \in \hat{\mathcal{N}}_{\varepsilon_n}$, we have

$$\begin{aligned} c_0 &\leq \hat{E}_0(w_n) \\ &= \frac{1}{p}\|w_n\|_{1,p}^p - \int_{\mathbb{R}^N} F(w_n)\,\mathrm{d}x \\ &= \frac{t_n^p}{p}\|v_n\|_{1,p}^p - \int_{\mathbb{R}^N} F(t_n v_n)\,\mathrm{d}x \\ &= \frac{t_n^p}{p}\|u_n\|_{1,p}^p - \int_{\mathbb{R}^N} F(t_n u_n)\,\mathrm{d}x \\ &\leq \frac{t_n^p}{p}\|u_n\|_{1,p}^p + \frac{t_n^q}{q}\int_{\mathbb{R}^N} a(\varepsilon x)\left(|\nabla u|^q + |u|^q\right)\mathrm{d}x - \int_{\mathbb{R}^N} \hat{G}(\varepsilon x, t_n u_n)\,\mathrm{d}x \\ &\leq \hat{E}_{\varepsilon_n}(t_n u_n) \leq \max_{t\geq 0} \hat{E}_{\varepsilon_n}(tu_n) = \hat{E}_{\varepsilon_n}(u_n) = c_0 + o(1), \end{aligned} \tag{5.14}$$

which implies that $\lim\limits_{n\to\infty} \hat{E}_0(w_n) = c_0$. As in the proof of Theorem 4.2 we can show that $\{w_n\}_{n\in\mathbb{N}}$ is bounded. This together with the boundedness of $\{v_n\}_{n\in\mathbb{N}}$ yields that $\{t_{v_n}\}_{n\in\mathbb{N}}$ is bounded as well. Thus, up to a subsequence, we can assume that $t_{v_n} \to t_0 \geq 0$ as $n \to +\infty$.

If $t_0 = 0$, then $\|w_n\|_{1,p} \to 0$, and consequently $\hat{E}_0(w_n) \to 0$, which contradicts that $c_0 > 0$. Therefore $t_0 > 0$, and $\{w_n\}_{n\in\mathbb{N}}$ satisfies

$$\hat{E}_0(w_n) \to c_0, \quad w_n \rightharpoonup w := t_0 v \not\equiv 0.$$

Similar to the argument in the proof of Lemma 4.1 we can show that $w_n \to w$ as $n \to +\infty$ which implies $v_n \to v$ as $n \to +\infty$.

We claim now that $\left\{y_n := \varepsilon_n \hat{y}_n\right\}_{n\in\mathbb{N}}$ is bounded. Suppose this is not the case, then there is a subsequence of $\{y_n\}_{n\in\mathbb{N}}$, not relabeled, such that $|y_n| \to +\infty$ as $n \to +\infty$. We take $R > 0$ such that $\Omega \subset B_R(0)$. Suppose $|y_n| > 2R$. Then, for any $x \in B_{R/\varepsilon_n}(0)$, we have

$$|\varepsilon_n x + y_n| \geq |y_n| - |\varepsilon_n x| > R.$$

Because of $\{u_n\}_{n\in\mathbb{N}} \subset \hat{\mathcal{N}}_{\varepsilon_n}$, (H1) (i), the definition of $\hat{g}$, after the change of variable $x = z + \hat{y}_n$ we get that

$$\begin{aligned} \|v_n\|_{1,p}^p &\leq \int_{\mathbb{R}^N} \hat{g}(\varepsilon_n z + y_n, v_n)\, v_n\,\mathrm{d}z \\ &= \int_{B_{R/\varepsilon_n}(0)} \hat{g}(\varepsilon_n z + y_n, v_n)\, v_n\,\mathrm{d}z + \int_{B^c_{R/\varepsilon_n}(0)} \hat{g}(\varepsilon_n z + y_n, v_n)\, v_n\,\mathrm{d}z \\ &\leq \int_{B_{R/\varepsilon_n}(0)} \hat{f}(v_n)\, v_n\,\mathrm{d}z + \int_{B^c_{R/\varepsilon_n}(0)} f(v_n)\, v_n\,\mathrm{d}z. \end{aligned}$$

From $v_n \to v$ and the definition of $\hat{f}$ we conclude that

$$\left(1-\frac{1}{k}\right)\|v_n\|_{1,p}^p \le \int_{B^c_{R/\varepsilon_n}(0)} f(v_n)\, v_n \, \mathrm{d}z = o_n(1).$$

Letting $n \to +\infty$ we deduce that $v \equiv 0$, which contradicts $v \not\equiv 0$. Therefore $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Up to a subsequence, we may assume that $y_n \to y \in \mathbb{R}^N$ as $n \to +\infty$. If $y \notin \overline{\Omega}$, then we can apply the above argument again to obtain a contradiction. Hence we have $y \in \overline{\Omega}$.

It remains to check that $y \in A$, that is, we should prove $a(y) = 0$. Suppose by contradiction that $a(y) > 0$. Then we have

$$\begin{aligned}
c_0 &= E_0(w)\\
&< \frac{1}{p}\|w\|_{1,p}^p + \frac{1}{q}\int_{\mathbb{R}^N} a(y)\,|\nabla w|^q \, \mathrm{d}x\\
&\quad + \frac{1}{q}\int_{\Omega} a(y)\,|w|^q \, \mathrm{d}x - \int_{\mathbb{R}^N} F(w)\, \mathrm{d}x\\
&\le \liminf_{n\to+\infty}\left[\frac{1}{p}\|w_n\|_{1,p}^p + \frac{1}{q}\int_{\mathbb{R}^N} a(\varepsilon_n z + y_n)\,|\nabla w_n|^q \, \mathrm{d}x\right.\\
&\qquad\qquad \left. + \frac{1}{q}\int_{\mathbb{R}^N} a(\varepsilon_n z + y_n)\,|w_n|^q \, \mathrm{d}x - \int_{\mathbb{R}^N} F(w_n)\, \mathrm{d}x\right]\\
&\le \liminf_{n\to+\infty} \hat{E}_{\varepsilon_n}\left(t_{v_n} v_n\right) \le \liminf_{n\to\infty} \hat{E}_{\varepsilon_n}(u_n) = c_0,
\end{aligned}$$

a contradiction, and thus $a(y) = 0$. The condition (H1)(ii) implies $y \notin \partial\Omega$. Hence $y \in A$. □

For each $y \in A$, we set

$$h(\varepsilon) := \left|\hat{E}_\varepsilon(\Phi_\varepsilon(y)) - c_0\right|.$$

Then we deduce from Lemma 5.9 that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. We define the sublevel set

$$\widehat{\mathcal{N}_\varepsilon} = \left\{u \in \hat{\mathcal{N}_\varepsilon} \colon \hat{E}_\varepsilon(u) \le c_0 + h(\varepsilon)\right\}.$$

Note that $\Phi_\varepsilon(y) \in \widehat{\mathcal{N}_\varepsilon}$, and so $\widehat{\mathcal{N}_\varepsilon} \ne \emptyset$ for any $\varepsilon > 0$.

Lemma 5.12 *Let hypotheses (H0), (H1)(i)–(iii) and (H5) be satisfied. Then we have*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\widehat{\mathcal{N}_\varepsilon}} \operatorname{dist}(\beta_\varepsilon(u), A_\delta) = 0.$$

Proof Let $\varepsilon_n \to 0$ as $n \to +\infty$. By the definition of the supremum, there exists $u_n \in \widehat{\mathcal{N}_{\varepsilon_n}}$ such that

$$\operatorname{dist}\left(\beta_{\varepsilon_n}(u_n), A_\delta\right) = \sup_{u \in \widehat{\mathcal{N}_{\varepsilon_n}}} \operatorname{dist}\left(\beta_{\varepsilon_n}(u), A_\delta\right) + o_n(1),$$

where we denote by $o_n(1)$ the quantity that tends to 0 as $n \to \infty$. Therefore, it is sufficient to prove that there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset A_\delta$ such that

$$\lim_{n\to+\infty} \left|\beta_{\varepsilon_n}(u_n) - y_n\right| = 0. \tag{5.15}$$

Since $\{u_n\}_{n\in\mathbb{N}} \subset \widehat{\mathcal{N}_{\varepsilon_n}} \subset \hat{\mathcal{N}}_{\varepsilon_n}$, we note that

$$c_0 \le \max_{t\ge 0} \hat{E}_0(tu_n) \le \max_{t\ge 0} \hat{E}_{\varepsilon_n}(tu_n) = \hat{E}_{\varepsilon_n}(u_n) \le c_0 + h(\varepsilon_n),$$

which implies that $\hat{E}_{\varepsilon_n}(u_n) \to c_0$. Then, from Lemma 5.11, it follows that there exists a sequence $\left\{\hat{y}_n\right\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\varepsilon_n \hat{y}_n =: y_n \to y \in A$ as $n \to +\infty$. Hence

$$\begin{aligned}
\beta_{\varepsilon_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \kappa(\varepsilon_n x)\, |u_n(x)|^p \,\mathrm{d}x}{\int_{\mathbb{R}^N} |u_n(x)|^p \,\mathrm{d}x} \\
&= \frac{\int_{\mathbb{R}^N} \kappa(\varepsilon_n z + y_n) \left|u_n\left(z + \hat{y}_n\right)\right|^p \mathrm{d}z}{\int_{\mathbb{R}^N} \left|u_n\left(z + \hat{y}_n\right)\right|^p \mathrm{d}z} \\
&= y_n + \frac{\int_{\mathbb{R}^N} (\kappa(\varepsilon_n z + y_n) - y_n)\, |v_n(z)|^p \,\mathrm{d}z}{\int_{\mathbb{R}^N} |v_n(z)|^p \,\mathrm{d}z}.
\end{aligned}$$

Note that $\varepsilon_n z + y_n \to y \in A$, and so $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$, that is, $\{y_n\}_{n\in\mathbb{N}}$ satisfies (5.15) and the lemma is proved. □

Now we can state and prove our existence result for problem (2.5).

Theorem 5.13 *Let hypotheses (H0), (H1)(i)–(iii) and (H5) be satisfied. Then there exists a small positive number $\hat{\varepsilon}$ such that for every $0 < \varepsilon < \hat{\varepsilon}$ problem (2.5) has at least* $\operatorname{cat}(A)$ *positive solutions.*

Proof From Lemma 5.9 and Proposition 3.5 we conclude that

$$\lim_{\varepsilon\to 0^+} \hat{J}(m^{-1}(\Phi_\varepsilon(y))) = \lim_{\varepsilon\to 0^+} \hat{E}_\varepsilon(\Phi_\varepsilon(y)) = c_0$$

uniformly in $y \in A$. For each $y \in A$, we set

$$h(\varepsilon) := \left|\hat{E}_\varepsilon\left(\Phi_\varepsilon(y)\right) - c_0\right|.$$

Then $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Now we write

$$\widehat{\mathcal{S}_+} = \left\{ u^+ \in \mathcal{S}_+ : \hat{J}(u^+) \le c_0 + h(\varepsilon) \right\}.$$

It is clear that $\widehat{\mathcal{S}_+} \neq \emptyset$, since $m^{-1}(\Phi_\varepsilon(y)) \in \widehat{\mathcal{S}_+}$. From Lemma 5.3 and the Lusternik-Schnirelmann theory (see Szulkin-Weth [45, Theorem 27]), it follows that $\hat{J}$ has at least cat $\left(\widehat{\mathcal{S}_+}\right)$ critical points on $\widehat{\mathcal{S}_+}$. Lemmas 5.9 and 5.12 imply that there exists $\hat{\varepsilon} > 0$ such that, for any $0 < \varepsilon < \hat{\varepsilon}$, the diagram

$$A \xrightarrow{\Phi_\varepsilon} \widehat{\mathcal{N}_\varepsilon} \xrightarrow{m^{-1}} \widehat{\mathcal{S}_+} \xrightarrow{m} \widehat{\mathcal{N}_\varepsilon} \xrightarrow{\beta_\varepsilon} A_\delta$$

is well defined and $\beta_\varepsilon \circ m \circ m^{-1} \circ \Phi_\varepsilon$ is homotopic to the inclusion $\mathrm{id}\colon A \to A_\delta$.

We claim that

$$\operatorname{cat}\left(\widehat{\mathcal{S}_+}\right) \ge \operatorname{cat}_{A_\delta}(A). \tag{5.16}$$

We assume that $\operatorname{cat}(\widehat{\mathcal{S}_+}) = n$, that is, there exists a smallest positive integer n such that

$$\widehat{\mathcal{S}_+} \subseteq \mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_n,$$

where $\mathcal{D}_i$, $i = 1, 2, \cdots, n$ are closed and contractible in $\widehat{\mathcal{S}_+}$, that is, there exist

$$h_i \in C([0,1] \times \mathcal{D}_i, \widehat{\mathcal{S}_+}), \quad i = 1, 2, \cdots, n$$

such that

$$\begin{aligned} &h_i(0,u) = u \quad \text{for all } u \in \mathcal{D}_i,\\ &h_i(1,u) = \omega_i \in \widehat{\mathcal{S}_+} \quad \text{for all } u \in \mathcal{D}_i. \end{aligned}$$

We set

$$\mathcal{K}_i = \Phi_\varepsilon^{-1}(m(\mathcal{D}_i)).$$

As before, $\mathcal{K}_i$ are closed subsets of A and $A \subseteq \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_n$. Furthermore, $\mathcal{K}_i$, $i = 1, \cdots, n$ are contractible in A using the deformation $\mathfrak{h}_i\colon [0,1] \times \mathcal{K}_i \to A_\delta$ defined by

$$\mathfrak{h}_i(t,x) = (\beta \circ m \circ h_i)(t, m^{-1}(\Phi_\varepsilon(x))).$$

Indeed, we conclude from Lemmas 5.9 and 5.12 that

$$\begin{aligned}
&\mathfrak{h}_i \in C([0,1]\times \mathcal{K}_i, A_\delta),\\
&\mathfrak{h}_i(0,x) = (\beta\circ m\circ h_i)(0, m^{-1}(\Phi_\varepsilon(x))) = x \quad \text{for all } x\in\mathcal{K}_i,\\
&\mathfrak{h}_i(1,x) = (\beta\circ m\circ h_i)(1, m^{-1}(\Phi_\lambda(x))) = \beta(m(\omega_i)) = x_i \in A_\delta \quad \text{for all } x\in\mathcal{K}_i.
\end{aligned}$$

Hence

$$\operatorname{cat}_{A_\delta}(A) \le n,$$

that is, (5.16) holds. We also note that

$$\operatorname{cat}_{A_\delta}(A) = \operatorname{cat}(A),$$

since A_δ is homotopically equivalent to A. Thus, $\widehat{\mathcal{S}_+}$ contains at least $\operatorname{cat}(A)$ critical points of $\hat{J}$. Proposition 3.6 implies that these critical points are also the critical points of the functional $\hat{E}_\varepsilon$. Thus we show that the problem (2.5) has $\operatorname{cat}(A)$ positive solutions. $\square$

6 Proof of the main results

In this section we are going to proof our main results in this paper. A key lemma in our proofs is the following one.

Lemma 6.1 *Let hypotheses (H0), (H1) and (H4) be satisfied and $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be a positive weak solution of (2.4). Then $\tilde{E}_{\varepsilon_n}(u_n) \to c_0^r$ and for any $\sigma > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n)^c)} < \sigma \quad \textit{for all } n \ge n_0,$$

where $\tilde{y}_n$ is given by Lemma 5.10.

Remark 6.2 The results of Lemma 6.1 holds true for negative solutions of the auxiliary problem (2.4) since $\tilde{E}_{\varepsilon_n}(\cdot)$ is even under our hypotheses.

Remark 6.3 The results of Lemma 6.1 holds true for positive solution of the auxiliary problem (2.5) under the hypothesis (H0), (H1)(i)–(iii) and (H5). The proof is similar.

Proof *(Proof of Lemma 6.1)* By an argument similar to that of (5.14), we can show that $\tilde{E}_{\varepsilon_n}(u_n) \to c_0^r$. Let $R > 1$, $\eta_R \in C^\infty(\mathbb{R}^N)$ such that $0 \le \eta_R \le 1$, $\eta_R \equiv 0$ in $B_{R/2}(0)$, $\eta_R \equiv 1$ in $B_R(0)^c$ and $|\nabla \eta_R| \le C/R$. We set $\eta_n(x) = \eta_R(x - \tilde{y}_n)$. Let $h > 0$ and define $u_{n,h} := \min\{u_n, h\}$. Choose $v_{n,h} = \eta_n^p u_n u_{n,h}^{\kappa p}$ as test function in

(2.4) with $\kappa > 0$ to be determined later. A direct calculation yields

$$
\begin{aligned}
&\int_{\mathbb{R}^N} a(\varepsilon_n x)\left(|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla v_{n,h}+|u_n|^{q-2}u_n v_{n,h}\right)\mathrm{d}x\\
&=\int_{\mathbb{R}^N} a(\varepsilon_n x)\Big(|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\eta_n p\eta_n^{p-1}u_n u_{n,h}^{\kappa p}+|\nabla u_n|^q\eta_n^p u_{n,h}^{\kappa p}\\
&\quad+\kappa p|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla u_{n,h}\eta_n^p u_n u_{n,h}^{\kappa p-1}+|u_n|^q\eta_n^p u_{n,h}^{\kappa p}\Big)\mathrm{d}x\\
&=\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} a(\varepsilon_n x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\eta_n p\eta_n^{p-1}u_n u_{n,h}^{\kappa p}\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N} a(\varepsilon_n x)\left(|\nabla u_n|^q\eta_n^p u_{n,h}^{\kappa p}+|u_n|^q\eta_n^p u_{n,h}^{\kappa p}\right)\mathrm{d}x\\
&\quad+\kappa p\int_{\{u_n\le h\}} a(\varepsilon_n x)|\nabla u_n|^q\eta_n^p u_n u_{n,h}^{\kappa p-1}\,\mathrm{d}x.
\end{aligned}
\tag{6.1}
$$

Applying Young's inequality, we have

$$
\begin{aligned}
&p\left|\frac{1}{\xi}|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\eta_n\eta_n^{p-1}u_n\xi\right|\\
&\le p\cdot\left(\frac{1}{q}\frac{1}{\xi^q}|u_n|^q|\nabla\eta_n|^q+\frac{q-1}{q}\xi^{\frac{q}{q-1}}\eta_n^{q\cdot\frac{p-1}{q-1}}|\nabla u_n|^q\right)\\
&=\frac{p}{q}\frac{1}{\xi^q}|u_n|^q|\nabla\eta_n|^q+\frac{p(q-1)}{q}\xi^{\frac{q}{q-1}}\eta_n^{q\cdot\frac{p-1}{q-1}}|\nabla u_n|^q
\end{aligned}
$$

and so

$$
\begin{aligned}
&\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} a(\varepsilon_n x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\eta_n p\eta_n^{p-1}u_n u_{n,h}^{\kappa p}\,\mathrm{d}x\\
&\le\frac{p}{q}\frac{1}{\xi^q}\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} a(\varepsilon_n x)|u_n|^q|\nabla\eta_n|^q u_{n,h}^{\kappa p}\,\mathrm{d}x\\
&\quad+\frac{p(q-1)}{q}\xi^{\frac{q}{q-1}}C\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} a(\varepsilon_n x)\eta_n^p|\nabla u_n|^q u_{n,h}^{\kappa p}\,\mathrm{d}x.
\end{aligned}
$$

Substituting this expression into the formula (6.1) yields

$$
\begin{aligned}
&\int_{\mathbb{R}^N} a(\varepsilon_n x)\left(|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla v_{n,h}+|u_n|^{q-2}u_n v_{n,h}\right)\mathrm{d}x\\
&\ge C\int_{\mathbb{R}^N} a(\varepsilon_n x)|\nabla u_n|^q\eta_n^p u_{n,h}^{\kappa p}\,\mathrm{d}x+\int_{\mathbb{R}^N} a(\varepsilon_n x)|u_n|^q\eta_n^p u_{n,h}^{\kappa p}\,\mathrm{d}x\\
&\quad-C\int_{\mathbb{R}^N} a(\varepsilon_n x)|u_n|^q|\nabla\eta_n|^q u_{n,h}^{\kappa p}\,\mathrm{d}x\\
&\ge C\int_{\mathbb{R}^N} a(\varepsilon_n x)|\nabla u_n|^q\eta_n^p u_{n,h}^{\kappa p}\,\mathrm{d}x-C\int_{\mathbb{R}^N} a(\varepsilon_n x)|u_n|^q|\nabla\eta_n|^q u_{n,h}^{\kappa p}\,\mathrm{d}x,
\end{aligned}
\tag{6.2}
$$

since

$$\kappa p \int_{\{u_n \le h\}} a(\varepsilon_n x) |\nabla u_n|^q \eta_n^p u_n u_{n,h}^{\kappa p - 1} \, \mathrm{d}x \ge 0$$

and

$$\int_{\mathbb{R}^N} a(\varepsilon_n x) |u_n|^q \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \ge 0.$$

Next, we calculate

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v_{n,h} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \Big(|\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \eta_n p \eta_n^{p-1} u_n u_{n,h}^{\kappa p} + |\nabla u_n|^p \eta_n^p u_{n,h}^{\kappa p} \\
&\quad + \kappa p |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_{n,h} \eta_n^p u_n u_{n,h}^{\kappa p - 1} \Big) \, \mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} \Big(|\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \eta_n p \eta_n^{p-1} u_n u_{n,h}^{\kappa p} + |\nabla u_n|^p \eta_n^p u_{n,h}^{\kappa p} \Big) \, \mathrm{d}x,
\end{aligned}$$

since

$$\begin{aligned}
&\int_{\mathbb{R}^N} \kappa p |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_{n,h} \eta_n^p u_n u_{n,h}^{\kappa p - 1} \, \mathrm{d}x \\
&= \kappa p \int_{\{u_n \le h\}} \eta_n^p u_n^{\kappa p} |\nabla u_n|^p \, \mathrm{d}x \ge 0.
\end{aligned}$$

Hölder's and Young's inequalities yield

$$\begin{aligned}
&p \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \eta_n \eta_n^{p-1} u_n u_{n,h}^{\kappa p} \, \mathrm{d}x \\
&\le p \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} \eta_n^{p-1} u_{n,h}^{(p-1)\kappa} u_n |\nabla \eta_n| u_{n,h}^{\kappa} \, \mathrm{d}x \\
&\le \left(p\xi \int_{\mathbb{R}^N} |\nabla u_n|^p \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \right)^{\frac{p-1}{p}} \left(p\xi^{\frac{1-p}{p}} \int_{\mathbb{R}^N} |\nabla \eta_n|^p u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\le \xi(p-1) \int_{\mathbb{R}^N} |\nabla u_n|^p \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x + \xi^{\frac{1-p}{p}} \int_{\mathbb{R}^N} |\nabla \eta_n|^p u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x
\end{aligned}$$

and so

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v_{n,h} \, \mathrm{d}x \\
&\ge C \int_{\mathbb{R}^N} |\nabla u_n|^p \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x - C \int_{\mathbb{R}^N} |\nabla \eta_n|^p u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x.
\end{aligned} \tag{6.3}$$

We have $\left\langle \tilde{E}'_{\varepsilon_n}(u_n), v_{n,h} \right\rangle = 0$, that is,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2} \nabla u_n \nabla v_{n,h} + |u_n|^{p-2} u_n v_{n,h} \right) \mathrm{d}x \\
&+ \int_{\mathbb{R}^N} a(\varepsilon_n x) \left(|\nabla u_n|^{q-2} \nabla u_n \nabla v_{n,h} + |u_n|^{q-2} u_n v_{n,h} \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \tilde{g}(\varepsilon_n x, u_n) v_{n,h} \, \mathrm{d}x.
\end{aligned}$$

This together with (2.3) yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla v_{n,h} \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} a(\varepsilon_n x) \left(|\nabla u_n|^{q-2} \nabla u_n \nabla v_{n,h} + |u_n|^{q-2} u_n v_{n,h} \right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n) v_{n,h} \, \mathrm{d}x - \int_{\mathbb{R}^N} \eta_n^p |u_n|^p u_{n,h}^{\kappa p} \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} \left(\xi |u_n|^{p-1} + C_\xi |u_n|^{r-1} \right) \eta_n^p u_n u_{n,h}^{\kappa p} \, \mathrm{d}x - \int_{\mathbb{R}^N} \eta_n^p |u_n|^p u_{n,h}^{\kappa p} \, \mathrm{d}x \\
&\leq C \int_{\mathbb{R}^N} \eta_n^p u_n^r u_{n,h}^{\kappa p} \, \mathrm{d}x.
\end{aligned}$$

Then, from (6.2) and (6.3), we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^p \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x + \int_{\mathbb{R}^N} a(\varepsilon_n x) |\nabla u_n|^q \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \\
&\leq C \left(\int_{\mathbb{R}^N} \eta_n^p u_n^r u_{n,h}^{\kappa p} \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla \eta_n|^p u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \right. \\
&\qquad \left. + \int_{\mathbb{R}^N} a(\varepsilon_n x) |u_n|^q |\nabla \eta_n|^q u_{n,h}^{\kappa p} \, \mathrm{d}x \right).
\end{aligned} \tag{6.4}$$

On the other hand, denoting by $w_{n,h} = \eta_n u_n u_{n,h}^{\kappa}$, we have

$$\begin{aligned}
&\|w_{n,h}\|_{p^*}^p \leq C \int_{\mathbb{R}^N} |\nabla w_{n,h}|^p \, \mathrm{d}x = C \int_{\mathbb{R}^N} \left| \nabla \left(\eta_n u_n u_{n,h}^{\kappa} \right) \right|^p \mathrm{d}x \\
&= C \int_{\mathbb{R}^N} \left| \left(\nabla \eta_n u_n u_{n,h}^{\kappa} + \eta_n \nabla u_n u_{n,h}^{\kappa} + \kappa \eta_n u_n u_{n,h}^{\kappa-1} \nabla u_{n,h} \right) \right|^p \mathrm{d}x \\
&\leq C \int_{\mathbb{R}^N} |\nabla \eta_n|^p u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x + C \int_{\mathbb{R}^N} \eta_n^p |\nabla u_n|^p u_{n,h}^{\kappa p} \, \mathrm{d}x \\
&\quad + C\kappa^p \int_{\mathbb{R}^N} \eta_n^p u_n^p u_{n,h}^{p(\kappa-1)} \left| \nabla u_{n,h} \right|^p \mathrm{d}x \\
&= C \int_{\mathbb{R}^N} |\nabla \eta_n|^p u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x + C \int_{\mathbb{R}^N} \eta_n^p |\nabla u_n|^p u_{n,h}^{\kappa p} \, \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+ C\kappa^p \int_{\{u_n \le h\}} \eta_n^p u_{n,h}^{\kappa p} |\nabla u_n|^p \, \mathrm{d}x \\
&\le C(\kappa+1)^p \left(\int_{\mathbb{R}^N} |\nabla \eta_n|^p \, u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^p \, \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \right) \\
&\le C(\kappa+1)^p \left(\int_{\mathbb{R}^N} |\nabla \eta_n|^p \, u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^p \, \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \right. \\
&\qquad \left. + \int_{\mathbb{R}^N} a(\varepsilon_n x) \, |\nabla u_n|^q \, \eta_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \right) \\
&\le C(\kappa+1)^p \left(\int_{\mathbb{R}^N} |\nabla \eta_n|^p \, u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x + \int_{\mathbb{R}^N} a(\varepsilon_n x) \, |u_n|^q \, |\nabla \eta_n|^q \, u_{n,h}^{\kappa p} \, \mathrm{d}x \right. \\
&\qquad \left. + \int_{\mathbb{R}^N} \eta_n^p u_n^r u_{n,h}^{\kappa p} \, \mathrm{d}x \right),
\end{aligned}
\tag{6.5}
$$

where we have used (6.4). Then we estimate

$$
\begin{aligned}
\int_{\mathbb{R}^N} \eta_n^p u_n^r u_{n,h}^{\kappa p} \, \mathrm{d}x &= \int_{\mathbb{R}^N} u_n^{r-p} \left(\eta_n u_n u_{n,h}^{\kappa} \right)^p \, \mathrm{d}x \\
&\le \left(\int_{\mathbb{R}^N} u_n^{p^*} \, \mathrm{d}x \right)^{\frac{r-p}{p^*}} \left(\int_{\mathbb{R}^N} \left(\eta_n u_n u_{n,h}^{\kappa} \right)^{\frac{pp^*}{p^*-(r-p)}} \, \mathrm{d}x \right)^{\frac{p^*-(r-p)}{p^*}} \\
&\le C \left\| w_{n,h} \right\|_{\alpha^*}^p \\
&= C \left(\int_{B_{R/2}(\tilde{y}_n)^c} \left(u_n u_{n,h}^{\kappa} \right)^{\alpha^*} \, \mathrm{d}x \right)^{\frac{p}{\alpha^*}},
\end{aligned}
$$

where $p < \alpha^* = pp^*/(p^* - (r-p)) < p^*$. Further, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \eta_n|^p \, u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \\
&= \int_{B_R(\tilde{y}_n) \setminus B_{R/2}(\tilde{y}_n)} |\nabla \eta_n|^p \, u_n^p u_{n,h}^{\kappa p} \, \mathrm{d}x \\
&\le \left(\int_{B_R(\tilde{y}_n) \setminus B_{R/2}(\tilde{y}_n)} |\nabla \eta_n|^{\frac{pp^*}{r-p}} \, \mathrm{d}x \right)^{\frac{r-p}{p^*}} \left(\int_{B_R(\tilde{y}_n) \setminus B_{R/2}(\tilde{y}_n)} \left(u_n u_{n,h}^{\kappa} \right)^{\alpha^*} \, \mathrm{d}x \right)^{\frac{p}{\alpha^*}}.
\end{aligned}
$$

Since $r < p^*$, we have $\frac{pp^*}{r-p} > N$. Therefore,

$$
\int_{B_R(\tilde{y}_n) \setminus B_{R/2}(\tilde{y}_n)} |\nabla \eta_n|^{\frac{pp^*}{r-p}} \, \mathrm{d}x \le \frac{C}{R^{\frac{pp^*}{r-p} - N}} \le C
$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(\varepsilon_n x)\,|u_n|^q\,|\nabla\eta_n|^q\,u_{n,h}^{\kappa p}\,\mathrm{d}x\\
&\quad=\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} a(\varepsilon_n x)\,|u_n|^q\,|\nabla\eta_n|^q\,u_{n,h}^{\kappa p}\,\mathrm{d}x\\
&\quad\le \|a\|_{L^\infty}\left(\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} |\nabla\eta_n|^{\frac{qp^*}{r-q}}\,\mathrm{d}x\right)^{\frac{r-q}{p^*}}\left(\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} u_n^{p^*}\,\mathrm{d}x\right)^{\frac{q-p}{p^*}}\\
&\qquad\times\left(\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} \left(u_n u_{n,h}^{\kappa}\right)^{\alpha^*}\,\mathrm{d}x\right)^{\frac{p}{\alpha^*}}.
\end{aligned}$$

Moreover, as $q<r<p^*$, it holds $\frac{qp^*}{r-q}>\frac{pp^*}{r-p}>N$ and so

$$\int_{B_R(\tilde{y}_n)\setminus B_{R/2}(\tilde{y}_n)} |\nabla\eta_n|^{\frac{qp^*}{r-q}}\,\mathrm{d}x\le\frac{C}{R^{\frac{qp^*}{r-q}-N}}\le C.$$

Substituting the above estimations into (6.5) yields

$$\begin{aligned}
\left(\int_{B_R(\tilde{y}_n)^c}\left(u_n u_{n,h}^{\kappa}\right)^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} &\le \left\|w_{n,h}\right\|_{p^*}^{p}\\
&\le C(\kappa+1)^p\left(\int_{B_{R/2}(\tilde{y}_n)^c}\left(u_n u_{n,h}^{\kappa}\right)^{\alpha^*}\,\mathrm{d}x\right)^{\frac{p}{\alpha^*}}\\
&\le C(\kappa+1)^p\left(\int_{B_{R/2}(\tilde{y}_n)^c} u_n^{(\kappa+1)\alpha^*}\,\mathrm{d}x\right)^{\frac{p}{\alpha^*}}.
\end{aligned}$$

Using Fatou's lemma in the variable h gives

$$\left(\int_{B_R(\tilde{y}_n)^c} u_n^{(\kappa+1)p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}\le C(\kappa+1)^p\left(\int_{B_{R/2}(\tilde{y}_n)^c} u_n^{(\kappa+1)\alpha^*}\,\mathrm{d}x\right)^{\frac{p}{\alpha^*}}$$

or

$$\|u_n\|_{L^{(\kappa+1)p^*}(B_R(\tilde{y}_n)^c)}\le C_1^{\frac{1}{\kappa+1}}(\kappa+1)^{\frac{1}{\kappa+1}}\,\|u_n\|_{L^{(\kappa+1)\alpha^*}(B_{R/2}(\tilde{y}_n)^c)}.$$

Set $\gamma:=\kappa+1=p^*/\alpha^*>1$. We rewrite

$$\|u_n\|_{L^{\gamma p^*}(B_R(\tilde{y}_n)^c)}\le C_1^{\frac{1}{\gamma}}\gamma^{\frac{1}{\gamma}}\,\|u_n\|_{L^{\gamma\alpha^*}(B_{R/2}(\tilde{y}_n)^c)}.$$

Then we iterate, beginning with $\gamma, \gamma^2, \gamma^3, \ldots, \gamma^m$, to obtain

$$\|u_n\|_{L^{\gamma^m p^*}(B_R(\tilde{y}_n)^c)} \leq C_1^{\sum_{i=1}^m \gamma^{-i}} \gamma^{\sum_{i=1}^m i\gamma^{-i}} \|u_n\|_{L^{p^*}(B_{R/2}(\tilde{y}_n)^c)}.$$

Letting $m \to \infty$, we get

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n)^c)} \leq C_2 \|u_n\|_{L^{p^*}(B_{R/2}(\tilde{y}_n)^c)}.$$

By the change of variables $z = x - \tilde{y}_n$, we obtain

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n)^c)} \leq C_2 \|u_n\|_{L^{p^*}(B_{R/2}(\tilde{y}_n)^c)} = C_2 \left(\int_{B_{R/2}(\tilde{y}_n)^c} |u_n(z + \tilde{y}_n)|^{p^*} \, \mathrm{d}z \right)^{\frac{1}{p^*}}.$$

It follows from Lemma 5.10 that $v_n(z) = u_n(z + \tilde{y}_n)$ strongly converges in $L^{p^*}(\mathbb{R}^N)$. Thus, for $R > 0$ and n large enough, we have

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n)^c)} \leq \sigma.$$

□

Now, we are able to give the proofs of Theorems 1.2–1.3.

Proof *(Proof of Theorems 1.2 and 1.3)* We choose $\delta > 0$ small enough such that $A_\delta \subset \Omega$ and the sets A_δ^-, A_δ are homotopically equivalent to A. We claim that there exists $\tilde{\varepsilon} > 0$ such that, for any $0 < \varepsilon < \tilde{\varepsilon}$ and any solution $u \in \tilde{\mathcal{N}}_\varepsilon$ of the problem (2.4), there holds

$$\|u\|_{L^\infty(\Omega_\varepsilon^c)} \leq \tau. \tag{6.6}$$

Indeed, suppose by contradiction that for $\varepsilon_n \to 0$ as $n \to +\infty$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ such that $\tilde{E}'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|_{L^\infty(\Omega_{\varepsilon_n}^c)} > \tau. \tag{6.7}$$

From Lemma 6.1 it follows that $\tilde{E}_{\varepsilon_n}(u_n) \to c_0^r$. Then we can use Lemma 5.10 to get a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\varepsilon_n \tilde{y}_n \to y \in A$ as $n \to +\infty$. We choose $R_0 > 0$ such that $B_{R_0}(y) \subset B_{2R_0}(y) \subset \Omega$. Then we have

$$B_{R_0/\varepsilon_n}(y/\varepsilon_n) = \frac{1}{\varepsilon_n} B_{R_0}(y) \subset \Omega_{\varepsilon_n}.$$

Furthermore, for any $x \in B_{R_0/\varepsilon_n}(\tilde{y}_n)$, when n is large enough, we have

$$\left| x - \frac{y}{\varepsilon_n} \right| \leq |x - \tilde{y}_n| + \left| \tilde{y}_n - \frac{y}{\varepsilon_n} \right| \leq \frac{R_0}{\varepsilon_n} + \frac{1}{\varepsilon_n} o_n(1) < \frac{2R_0}{\varepsilon_n},$$

which implies that $B_{R_0/\varepsilon_n}(\tilde{y}_n) \subset \Omega_{\varepsilon_n}$. Consequently $B_{R_0/\varepsilon_n}(\tilde{y}_n)^c \supset \Omega_{\varepsilon_n}^c$. Then by Lemma 6.1 with $\sigma = \tau$ and $n \geq n_0$ large enough such that $R_0/\varepsilon_n > R$, we have

$$\|u_n\|_{L^\infty(\Omega_{\varepsilon_n}^c)} \leq \|u_n\|_{L^\infty(B_{R_0/\varepsilon_n}(\tilde{y}_n)^c)} \leq \|u_n\|_{L^\infty(B_R(\tilde{y}_n)^c)} < \tau,$$

which contradicts (6.7) and our claim is true. The same holds for solutions of (2.5), see Remark 6.3.

By (6.6) and the definition of $\tilde{g}$ (resp. $\hat{g}$) we conclude that $\tilde{g}(\varepsilon x, u) = f(u)$ (resp. $\hat{g}(\varepsilon x, u) = f(u)$). Thus solutions of the auxiliary problems (2.4) and (2.5) are also solutions of (1.1). Hence the existence results in Theorems 1.2 and 1.3 follow from Theorems 5.6–5.8 and 5.13.

In the last part, we want to study the concentration behavior of the solutions of the equation (1.1). Let $\varepsilon_n \to 0$ as $n \to +\infty$ and $u_n \in W^{1,\mathcal{H}_\varepsilon}(\mathbb{R}^N)$ be a solution of equation (2.4). As in the beginning of this proof, we can see that $u_n(x + \tilde{y}_n) \to 0$ as $n \to +\infty$ and $|x| \to +\infty$. Thus, for any $\tau > 0$ and some large fixed $R > 0$, there exists N_τ such that

$$\|u_n\|_{L^\infty(B_R^c(\tilde{y}_n))} < \tau \quad \text{for all } n > N_\tau. \tag{6.8}$$

We claim that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geqslant \sigma' \quad \text{for some } \sigma' > 0, \tag{6.9}$$

where R is given in (6.8). Indeed, suppose not, for any $\tau > 0$, by (6.8) we have that

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} < \tau \text{ for } n \text{ large enough.}$$

From $\tilde{E}'_{\varepsilon_n}(u_n) \to 0$ (resp. $\hat{E}'_{\varepsilon_n}(u_n) \to 0$) as $n \to +\infty$ and (H4) (ii) (resp. (H5) (ii)), we have

$$\begin{aligned}
\|u_n\|_{1,p}^p &\leq \|u_n\|_{1,p}^p + \int_{\mathbb{R}^N} a(\varepsilon_n x)\,|\nabla u_n|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} a(\varepsilon_n x)\,|u_n|^q \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \tilde{g}(x, u_n)u_n \,\mathrm{d}x \quad \left(\text{resp. } \int_{\mathbb{R}^N} \hat{g}(x, u_n)u_n \,\mathrm{d}x\right) \\
&\leq \frac{1}{k}\int_{\mathbb{R}^N} u_n^p \,\mathrm{d}x,
\end{aligned}$$

which implies $u_n = 0$, but this does not occur.

From (6.8) and (6.9) we conclude that the maximum point $\tilde{p}_n \in \mathbb{R}^N$ of u_n belongs to $B_R(\tilde{y}_n)$. Write $\tilde{p}_n = \tilde{y}_n + q_n$ for some $q_n \in B_R(0)$. We now apply Lemma 5.10 again to obtain $\varepsilon_n \tilde{y}_n \to y \in A$ as $n \to +\infty$. We note that q_n is bounded. Hence we conclude

$$\lim_{n\to+\infty} a(\varepsilon_n \tilde{p}_n) = a(y) = 0.$$

The same holds for solutions of (2.5) by Lemma 5.11 and Remark 6.3. □

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Declarations

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