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Positivity

Positive solutions for nonlinear nonhomogeneous Dirichlet problems with concave-convex nonlinearities

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Abstract We consider a nonlinear parametric Dirichlet equation driven by a nonhomogeneous differential operator involving a reaction exhibiting the competing effects of concave and convex terms. Using variational methods combined with truncation and comparison techniques we prove a bifurcation near zero theorem describing the dependence of the positive solutions on the parameter $\lambda > 0$.

Keywords Nonhomogeneous differential operator · Nonlinear regularity theory · Nonlinear maximum principle · Bifurcation of positive solutions · Strong comparison · Concave and convex nonlinearities

Mathematics Subject Classification 35J66 · 35J70 · 35J92

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^{1,\alpha}$-boundary $\partial\Omega, \alpha \in (0, 1)$. In this paper, we study the existence, nonexistence, and multiplicity of positive solutions to the following nonhomogeneous parametric Dirichlet problem

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$$\begin{aligned} -\operatorname{div} a(\nabla u) &= f(x,u,\lambda) \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{P_λ}$$

where $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous and strictly monotone mapping satisfying appropriate regularity and growth conditions listed in hypotheses H(a) below. These hypotheses are general enough to incorporate many differential operators of interest in our framework such as the p-Laplacian ($1 < p < \infty$), the (p,q)-differential operator ($1 < q < p < \infty$) and the generalized p-mean curvature differential operator ($1 < p < \infty$). The reaction of the problem depends on a parameter $\lambda > 0$ and is Carathéodory in the variables $(x,s) \in \Omega\times\mathbb{R}$ (that is, $x \to f(x,s,\lambda)$ is measurable for all $s \in \mathbb{R}$, for all $\lambda > 0$ and $s \to f(x,s,\lambda)$ is continuous for a.a. $x \in \Omega$, for all $\lambda > 0$). We assume that $f(x,\cdot,\lambda)$ is $(p-1)$-superlinear near $+\infty$ but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition (AR-condition for short). Near zero, the reaction $f(x,\cdot,\lambda)$ exhibits a concave term (that is, $s \to f(x,s,\lambda)$ is $(p-1)$-superlinear near 0^+). So, we have in problem (P_λ) the competing effects of concave and convex terms. Such problems were studied by Ambrosetti–Brezis–Cerami [2], Li–Wu–Zhou [23] (semilinear equations driven by the Laplace differential operator), and by Filippakis–Kristály–Papageorgiou [10], Gasiński–Papageorgiou [16,17], García Azorero–Peral Alonso–Manfredi [12], Guo–Zhang [18], Hu–Papageorgiou [19], and Marano–Papageorgiou [24] (nonlinear problems driven by the p-Laplace differential operator). In the aforementioned works, the reaction has the form $\lambda s^{q-1}+g(x,s)$ with $g(x,\cdot)$ being $(p-1)$-superlinear. With the exception of Marano–Papageorgiou [24], in all the other works the $(p-1)$-superlinearity of $g(x,\cdot)$ is expressed by employing the AR-condition. Moreover, in the works of García Azorero–Peral Alonso–Manfredi [12] and Guo–Zhang [18], $g(x,s) = g(s) = s^{r-1}$ for all $s \geq 0$ with $p < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$ (see also [2,23]). We mention that the p-Laplacian is a $(p-1)$-homogeneous differential operator and this fact is exploited in the methods used in the aforementioned works. The differential operator here is not homogeneous and this is source of difficulties in the analysis of problem (P_λ). To overcome these difficulties we need a different approach and new techniques. We prove a bifurcation result for $\lambda > 0$ near zero which describes the variation of the set of positive solutions as the parameter $\lambda > 0$ varies. Our theorem contains as special cases the main theorems of [12,16,17,19], and [24]. Recently, a similar bifurcation theorem was proved for Robin problems by Papageorgiou–Rădulescu [26] under stronger conditions on the nonlinearity $f : \Omega \times \mathbb{R} \times (0,\infty) \to \mathbb{R}$.

Our approach is variational based on critical point theory combined with suitable truncation and comparison techniques. In the next section we develop the necessary mathematical background material which will help to follow the arguments in this paper.

2 Mathematical background

Let X be a Banach space and X^* its topological dual while $\langle\cdot,\cdot\rangle$ denotes the duality brackets to the pair (X^*, X). We have the following definition.

Definition 2.1 The functional $\varphi \in C^1(X)$ fulfills the Cerami condition (the C-condition for short) if the following holds: every sequence $(u_n)_{n\geq 1} \subseteq X$ such that $(\varphi(u_n))_{n\geq 1}$ is bounded in $\mathbb{R}$ and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional φ which compensates for the fact that the ambient space X does not need to be locally compact (X is in general infinite dimensional). The C-condition is one of the main tools in proving a deformation theorem which in turn leads to the minimax theory of the critical values of φ. One of the basic results in this theory is the so-called mountain pass theorem due to Ambrosetti–Rabinowitz [3] which we state here in a slightly more general form (see, for example, Gasiński–Papageorgiou [13]).

Theorem 2.2 *Let $\varphi \in C^1(X)$ be a functional satisfying the C-condition and let $u_1, u_2 \in X$, $\|u_2 - u_1\|_X > \rho > 0$,*

$$\max\{\varphi(u_1), \varphi(u_2)\} < \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} =: m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$. Then $c \geq m_\rho$ with c being a critical value of φ.

By $L^p(\Omega)$ $\left(\text{or } L^p\left(\Omega; \mathbb{R}^N\right)\right)$ and $W_0^{1,p}(\Omega)$ we denote the usual Lebesgue and Sobolev spaces with their norms $\|\cdot\|_p$ and $\|\cdot\|_{W_0^{1,p}(\Omega)}$. Thanks to the Poincaré inequality we have

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The norm of $\mathbb{R}^N$ is denoted by $\|\cdot\|_{\mathbb{R}^N}$ and $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. For $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. It is well known that

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

The Lebesgue measure on $\mathbb{R}^N$ is denoted by $|\cdot|_N$ and for a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we define the Nemytskij operator corresponding to the function h by

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently, $x \mapsto N_h(u)(x)$ is measurable.

In addition to the Sobolev space $W_0^{1,p}(\Omega)$ we will also use the ordered Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\right\}$$

and its positive cone

$$C_0^1(\overline{\Omega})_+ = \left\{ u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega} \right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) = \left\{ u \in C_0^1(\overline{\Omega})_+ : u(x) > 0 \text{ for all } x \in \Omega, \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega \right\},$$

where $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$.

Now let $\vartheta \in C^1(0,+\infty)$ be a function satisfying

$$0 < \hat{c} \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2\left(1 + t^{p-1}\right) \tag{2.1}$$

for all $t > 0$ and with some constants $\hat{c}, c_0, c_1, c_2 > 0$. The hypotheses on $a : \mathbb{R}^N \to \mathbb{R}^N$ read as follows.

H(a): $a(\xi) = a_0\left(\|\xi\|_{\mathbb{R}^N}\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \to ta_0(t)$ is strictly increasing in $(0,\infty)$, $\lim_{t\to 0^+} ta_0(t) = 0$, and $\lim_{t\to 0^+} \dfrac{ta_0'(t)}{a_0(t)} = c > -1$;

(ii) $\|\nabla a(\xi)\|_{\mathbb{R}^N} \leq c_3 \dfrac{\vartheta\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}}$ for all $\xi \in \mathbb{R}^N\setminus\{0\}$ and for some $c_3 > 0$;

(iii) $(\nabla a(\xi)y, y)_{\mathbb{R}^N} \geq \dfrac{\vartheta\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}} \|y\|^2_{\mathbb{R}^N}$ for all $\xi \in \mathbb{R}^N\setminus\{0\}$ and all $y \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t sa_0(s)ds$ for all $t > 0$, then there exists $d, \nu \in (1,p)$, $1 < \mu < \min\{d,\nu\}$, and $\xi > 0$ such that

(1) $t \mapsto G_0\left(t^{\frac{1}{d}}\right)$ is convex in $(0,+\infty)$;

(2) $\limsup_{t\to 0^+} \dfrac{G_0(t)}{t^\nu} < +\infty$;

(3) $t^2 a_0(t) - \mu G_0(t) \geq \tilde{c} t^p$ for all $t > 0$ and for some $\tilde{c} > 0$;

(4) $pG_0(t) - t^2 a_0(t) \geq -\hat{\xi}$ for all $t > 0$ and for some $\hat{\xi} > 0$.

Remark 2.3 We point out that the assumption H(a)(iii) is equivalent to $\|\nabla a(\xi)\|_{\mathbb{R}^{N^2}} \geq \frac{\vartheta(\|\xi\|_{\mathbb{R}^N})}{\|\xi\|_{\mathbb{R}^N}}$ since $a(\xi) = a_0(\|\xi\|_{\mathbb{R}^N})\xi$ which gives that $\nabla a(\xi)$ is symmetric. Therefore, one also could write conditions H(a)(ii),(iii) together in the form

$$\frac{\vartheta\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}} \leq \|\nabla a(\xi)\|_{\mathbb{R}^{N^2}} \leq c_3 \frac{\vartheta\left(\|\xi\|_{\mathbb{R}^N}\right)}{\|\xi\|_{\mathbb{R}^N}}.$$

Hypotheses H(a)(i), (ii), (iii) allow the usage of the nonlinear global regularity results of Lieberman [22]. Hypothesis H(a)(iv) is dictated by the needs of our problem. However, as we will see in the examples that follow, it is satisfied in many cases of interest.

Note that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing. Let $G(\xi) = G_0\left(\|\xi\|_{\mathbb{R}^N}\right)$ for all $\xi \in \mathbb{R}^N$. Then $G(\cdot)$ is convex and differentiable. We have

$$\nabla G(\xi) = G_0'\left(\|\xi\|_{\mathbb{R}^N}\right)\frac{\xi}{\|\xi\|_{\mathbb{R}^N}} = a_0\left(\|\xi\|_{\mathbb{R}^N}\right)\xi = a(\xi) \quad \text{for all } \xi \in \mathbb{R}^N.$$

Hence, $G(\cdot)$ is the primitive of $a(\cdot)$ and the convexity of $G(\cdot)$ along with $G(0) = 0$ imply

$$G(\xi) \leq (a(\xi), \xi)_{\mathbb{R}^N} \quad \text{for all } \xi \in \mathbb{R}^N. \tag{2.2}$$

Using hypotheses H(a) as well as (2.1) and (2.2) we have the following lemma summarizing the main properties of the map $a(\cdot)$.

Lemma 2.4 *Under the hypotheses H(a)(i)–(iii) there holds*

(i) $\xi \to a(\xi)$ *is maximal monotone and strictly monotone;*
(ii) $\|a(\xi)\|_{\mathbb{R}^N} \leq c_4\left(1 + \|\xi\|_{\mathbb{R}^N}^{p-1}\right)$ *for all* $\xi \in \mathbb{R}^N$ *and for some* $c_4 > 0$*;*
(iii) $(a(\xi), \xi)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}\|\xi\|_{\mathbb{R}^N}^p$ *for all* $\xi \in \mathbb{R}^N$.

From this lemma we easily deduce the following growth restrictions for the primitive $G(\cdot)$.

Corollary 2.5 *If hypotheses H(a)(i)–(iii) hold, then*

$$\frac{c_1}{p(p-1)}\|\xi\|_{\mathbb{R}^N}^p \leq G(\xi) \leq c_5\left(1 + \|\xi\|_{\mathbb{R}^N}^p\right) \quad \textit{for all } \xi \in \mathbb{R}^N \textit{ and for some } c_5 > 0.$$

Example 2.6 The following maps satisfy hypotheses H(a).

(i) Let $1 < p < \infty$ and let $a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\xi$. Then $a(\cdot)$ represents the well-known p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(\|\nabla u\|_{\mathbb{R}^N}^{p-2}\nabla u\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

(ii) Let $1 < q < p < \infty$ and let $a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\xi + \|\xi\|_{\mathbb{R}^N}^{q-2}\xi$. Then $a(\cdot)$ becomes the (p, q)-differential operator defined by

$$\Delta_p u + \Delta_q u = \operatorname{div}\left(\|\nabla u\|_{\mathbb{R}^N}^{p-2}\nabla u\right) + \operatorname{div}\left(\|\nabla u\|_{\mathbb{R}^N}^{q-2}\nabla u\right)$$

for all $u \in W_0^{1,p}(\Omega)$. Such differential operators arise in many physical applications (see Cherfils–Il′yasov [5] and the references therein).

(iii) Let $1 < p < \infty$ and let $a(\xi) = \left(1 + \|\xi\|_{\mathbb{R}^N}^2\right)^{\frac{p-2}{2}}\xi$. In this case $a(\cdot)$ represents the generalized p-mean curvature differential operator which is defined by

$$\operatorname{div}\left[\left(1 + \|\nabla u\|_{\mathbb{R}^N}^2\right)^{\frac{p-2}{2}}\nabla u\right] \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

(iv) Let $1 < p < \infty$ and let $a(\xi) = \|\xi\|^{p-2}\xi + \frac{\|\xi\|^{p-2}\xi}{1+\|\xi\|^p}$. In this case the corresponding differential operator is

$$\Delta_p u + \operatorname{div}\left(\frac{\|\nabla u\|_{\mathbb{R}^N}^{p-2}\nabla u}{1+\|\nabla u\|_{\mathbb{R}^N}^p}\right) \quad \text{for all } u \in W^{1,p}(\Omega),$$

which arises in plasticity theory (see Fuchs–Gongbao [11]).

Now, let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^* \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega (a(\nabla u), \nabla v)_{\mathbb{R}^N}\, dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega). \tag{2.3}$$

The next proposition gives the main properties of A (see, for example, Gasiński–Papageorgiou [14]).

Proposition 2.7 *Let hypotheses H(a)(i)–(iii) be satisfied. Then $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by (2.3) is bounded (that is, it maps bounded sets to bounded sets), demicontinuous, strictly monotone (hence maximal monotone), and of type $(S)_+$, that is, if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

Now, let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the subsequent growth condition

$$|f_0(x,s)| \leq a_0(x)\left(1 + |s|^{r-1}\right) \qquad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^*$. Setting $F_0(x,s) = \int_0^s f_0(x,t)dt$ we define the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ through

$$\varphi_0(u) = \int_\Omega G(\nabla u)dx - \int_\Omega F_0(x,u)dx.$$

From Gasiński–Papageorgiou [15] we have the following result.

Proposition 2.8 *Let the assumptions in H(a)(i)–(iii) be satisfied. If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in C_0^1(\overline{\Omega}) \textit{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \textit{for all } h \in W_0^{1,p}(\Omega) \textit{ with } \|h\|_{W_0^{1,p}(\Omega)} \leq \rho_1.$$

Let $g, h \in L^{\infty}(\Omega)$. We write $g \prec h$ if for every compact set $K \subseteq \Omega$ there exists $\varepsilon = \varepsilon(K) > 0$ such that $g(x) + \varepsilon \leq h(x)$ for a.a. $x \in K$. Clearly, if $g, h \in C(\Omega)$ and $g(x) < h(x)$ for all $x \in \Omega$, then $g \prec h$.

Using this order $\prec$ we can have the following strong comparison result which extends Proposition 2.6 of Arcoya–Ruiz [4] where the case of the p-Laplacian is considered.

Proposition 2.9 *Let hypotheses H(a)(i)—(iii) be satisfied,* $\xi \geq 0, g, h \in L^{\infty}(\Omega)$, $g \prec h$, *and let* $u, v \in W_0^{1,p}(\Omega)$ *be solutions of the following Dirichlet problems*

$$-\operatorname{div}(\nabla u) + \xi |u|^{p-2}u = g \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0$$
$$-\operatorname{div}(\nabla v) + \xi |v|^{p-1} = h \quad \text{in } \Omega, \quad v\big|_{\partial\Omega} = 0$$

with $v \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. *Then* $v - u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Proof From Ladyzhenskaya–Ural′tseva [20, p. 286] we know that $u \in L^{\infty}(\Omega)$. Invoking the regularity results of Lieberman [22, p. 320] we have that $u \in C_0^1(\overline{\Omega})$. Note that

$$A(u) + \xi |u|^{p-2}u = g \leq h = A(v) + \xi v^{p-1} \quad \text{in } W^{-1,p'}(\Omega).$$

Acting with $(u - v)^+ \in W_0^{1,p}(\Omega)$ we obtain

$$\left\langle A(u) - A(v), (u - v)^+ \right\rangle + \xi \int_{\Omega} \left(|u|^{p-2}u - v^{p-1} \right) (u - v)^+ dx \leq 0,$$

which gives

$$\int_{\{u>v\}} (a(\nabla u) - a(\nabla v), \nabla u - \nabla v)_{\mathbb{R}^N}\, dx + \xi \int_{\{u>v\}} \left(u^{p-1} - v^{p-1} \right) (u - v) dx \leq 0.$$

Therefore, $|\{u > v\}|_N = 0$ and consequently, $u \leq v$.

First, we are going to show that $u(x) < v(x)$ for all $x \in \Omega$. For this purpose, we introduce the following two sets

$$E_0 = \{x \in \Omega : u(x) = v(x)\}, \quad E_1 = \{x \in \Omega : \nabla u(x) = \nabla v(x) = 0\}.$$

Claim $E_0 \subseteq E_1$

Letting $x_0 \in E_0$, the function $x \mapsto y(x) = (u - v)(x)$ attains its maximum at x_0. Hence, $\nabla u(x_0) = \nabla v(x_0)$. If $\nabla u(x_0) \neq 0$, then we can find $\overline{B}_{\rho}(x_0) \subseteq \Omega$ such that

$$|\nabla u(x)| > 0, \quad |\nabla v(x)| > 0, \quad (\nabla u(x), \nabla v(x))_{\mathbb{R}^N} > 0 \quad \text{for all } x \in \overline{B}_{\rho}(x_0),$$

where $\overline{B}_{\rho}(x_0)$ is the closed ball with center x_0 and radius $\rho > 0$. Setting $w = v - u \in C_0^1(\overline{\Omega}) \setminus \{0\}$, we point out that this function satisfies the following linear elliptic equation

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\beta_{ij}(x)\frac{\partial w}{\partial x_j}\right) = -\xi\left(v^{p-1} - |u|^{p-2}u\right) + h - g \tag{2.4}$$

whereby the coefficients $\beta_{ij}(\cdot)$ of the differential operator are given by

$$\beta_{ij}(x) = \int_0^1 \frac{\partial a_i}{\partial y_j}\left((1-t)\nabla u(x) + t\nabla v(x)\right) dx$$

(see Arcoya–Ruiz [4], Cuesta–Takáč [6]). Therefore, $\beta_{ij} \in C^{\beta}\left(\overline{B}_\rho(x_0)\right)$ for some $\beta \in (0,1)$ and they form a uniformly elliptic differential operator in (2.4). Moreover, by taking $\rho > 0$ even smaller if necessary we can show, using $g \prec h$, that the right-hand side in (2.4) is positive on $\overline{B}_\rho(x_0)$. Invoking the strong maximum principle (see, for example, Pucci–Serrin [27, p. 111]) there holds

$$w(x) > 0 \quad \text{for all } x \in B_\rho(x_0),$$

or equivalently

$$u(x) < v(x) \quad \text{for all } x \in B_\rho(x_0),$$

which contradicts the fact that $x_0 \in E_0$. This proves the claim.

Owing to $v \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, we have $E_1 \subseteq \Omega$ and E_1 is closed, that is, $E_1 \subset\subset \Omega$. Now, because of $E_0 \subseteq E_1$ and the closedness of E_1, it follows that E_0 is compact as well. Hence, we can find a smooth open set Ω_1 such that

$$E_0 \subseteq \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega.$$

Then, we can find a number $\varepsilon \in (0,1)$ such that

$$u(x) + \varepsilon \le v(x) \quad \text{for all } x \in \partial\Omega_1 \quad \text{and} \quad g(x) + \varepsilon \le h(x) \quad \text{for a.a. } x \in \Omega_1.$$

Now, let $\delta \in (0, \varepsilon)$ such that

$$\xi\left||s|^{p-2}s - |\tau|^{p-2}\tau\right| < \varepsilon \quad \text{for all } s, \tau \in [-\eta, \eta], |s-\tau| < 2\delta,$$

where $\eta = \max\{\|u\|_\infty, \|v\|_\infty\}$. We get

$$\begin{aligned}
-\operatorname{div} a(\nabla(u+\delta)) + \xi|u+\delta|^{p-2}(u+\delta) &= -\operatorname{div} a(\nabla u) + \xi|u+\delta|^{p-2}(u+\delta)\\
&= \xi\left[|u+\delta|^{p-2}(u+\delta) - |u|^{p-2}u\right] + g\\
&\le g + \varepsilon\\
&\le h\\
&= -\operatorname{div} a(\nabla v) + \xi v^{p-1} \quad \text{for a.a. } x \in \Omega.
\end{aligned}$$

Then, due to Damascelli [7, p. 495] it follows that $u + \delta \le v$ in Ω_1. Since $E_0 \subseteq \Omega_1$ we infer that $E_0 = \emptyset$ and

$$u(x) < v(x) \quad \text{for all } x \in \Omega.$$

Moreover, by virtue of Proposition 2.4 of Cuesta–Takáč [6], we obtain

$$\frac{\partial v}{\partial n} < \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega,$$

which implies $v - u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

From Filippakis–Kristály–Papageorgiou [10, Lemma 3.3] we borrow the following lemma.

Lemma 2.10 *Let X be an ordered Banach space, K_+ is an order cone of X, $\operatorname{int} K_+ \neq \emptyset$, and $e \in \operatorname{int} K_+$. Then, for every $u \in K_+$, there exists $t = t(u) > 0$ such that*

$$te - u \in \operatorname{int} K_+.$$

3 Bifurcation theorem

Our hypotheses on the nonlinearity $f : \Omega \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ are the following.

H: $f : \Omega \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is a function such that $(x, s) \to f(x, s, \lambda)$ is a Carathéodory mapping for all $\lambda > 0$, $\lambda \to f(x, s, \lambda)$ is nondecreasing, $f(x, 0, \lambda) = 0$ for a.a. $x \in \Omega$, for all $\lambda > 0$, and

- (i) for every $\rho > 0$ and every $\lambda > 0$, there exists $a_\rho(\lambda) \in L^\infty(\Omega)_+$ such that
  - (1) $\lambda \mapsto \left\|a_\rho(\lambda)\right\|_\infty$ is bounded on bounded sets;
  - (2) $|f(x, s, \lambda)| \le a_\rho(\lambda)(x)$ for a.a. $x \in \Omega$ and for all $s \in [0, \rho]$;
- (ii) if $F(x, s, \lambda) = \int_0^s f(x, t, \lambda)dt$, then, for all $\lambda > 0$,

$$\lim_{s \to +\infty} \frac{F(x, s, \lambda)}{s^p} = +\infty \quad \text{uniformly for a.a. } x \in \Omega$$

  and there exist $r(\lambda) \in (p, p^*)$ with $\lambda \to r(\lambda)$ nondecreasing, $r(\lambda) \to r_0 \in (p, p^*)$ as $\lambda \to 0^+$, and functions $\hat{\eta}_\infty(\lambda), \eta_\infty(\lambda) \in L^\infty(\Omega)$ such that
  - (1) $\lambda \to \left\|\hat{\eta}_\infty(\lambda)\right\|_\infty$ and $\lambda \to \|\eta_\infty(\lambda)\|_\infty$ are bounded on bounded sets;
  - (2) $\hat{\eta}_\infty(\lambda)(x) \le \liminf_{s \to +\infty} \dfrac{f(x, s, \lambda)}{s^{r(\lambda)-1}} \le \limsup_{s \to +\infty} \dfrac{f(x, s, \lambda)}{s^{r(\lambda)-1}} \le \eta_\infty(\lambda)(x)$ uniformly for a.a. $x \in \Omega$;
- (iii) for every $\lambda > 0$, there exist $\tau(\lambda) \in \left(\max\left\{(r(\lambda) - p)\frac{N}{p}, 1\right\}, p^*\right)$ and $\beta_0(\lambda) > 0$ such that
  - (1) $\lambda \to \tau(\lambda)$ and $\lambda \to \beta_0(\lambda)$ are nondecreasing;
  - (2) $\beta_0(\lambda) \le \liminf_{s \to +\infty} \dfrac{f(x, s, \lambda)s - pF(x, s, \lambda)}{s^{\tau(\lambda)}}$ uniformly for a.a. $x \in \Omega$;

(iv) for every $\lambda > 0$ there exist $q(\lambda), \theta \in (1, \mu)$ [see hypothesis H(a)(iv)] with $q(\lambda) \le \theta$ and $\delta_0(\lambda) \in (0,1)$, $\hat{c}_0(\lambda) > 0$ such that
  (1) $q(\lambda) \to q_0 \in (1, p)$ as $\lambda \to 0^+$;
  (2) $\lambda \to \hat{c}_0(\lambda)$ is strictly increasing and $\hat{c}_0(\lambda) \to +\infty$ as $\lambda \to +\infty$;
  (3) $\hat{c}_0(\lambda)s^\theta \le f(x,s,\lambda)s \le q(\lambda)F(x,s,\lambda)$ for a.a. $x \in \Omega$ and for all $s \in [0, \delta_0(\lambda)]$;
  and there exists a function $\eta_0(\cdot,\lambda) \in L^\infty(\Omega)_+$ such that
  (4) $\|\eta_0(\cdot,\lambda)\|_\infty \to 0$ as $\lambda \to 0^+$;
  (5) $\limsup_{s\to 0^+} \dfrac{F(x,s,\lambda)}{s^{q(\lambda)}} \le \eta_0(x,\lambda)$ uniformly for a.a. $x \in \Omega$;

(v) there exist $r^* \in (p, p^*]$ and $c_0^* > 0$ such that $f(x,s,\lambda) \ge -c_0^* s^{r^*-1}$ for a.a. $x \in \Omega$, for all $s \ge 0$, and for all $\lambda > 0$.

Remark 3.1 Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume, without loss of generality, that $f(x,s,\lambda) = 0$ for a.a. $x \in \Omega$, for all $s \le 0$, and for all $\lambda > 0$. Note that hypotheses H(ii),(iii) imply that, for all $\lambda > 0$,

$$\lim_{s\to+\infty} \frac{f(x,s,\lambda)}{s^{p-1}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$

This means that $f(x,\cdot,\lambda)$ is $(p-1)$-superlinear near $+\infty$. Such problems are usually treated using the AR-condition (unilateral version) which says that there exist $\tau = \tau(\lambda) > 0$ and $M = M(\lambda) > 0$ such that

$$0 < \tau F(x,s,\lambda) \le f(x,s,\lambda)s \quad \text{for a.a. } x \in \Omega \text{ and for all } s \ge M; \tag{3.1}$$

$$0 < \operatorname{ess\,inf}_\Omega F(\cdot, M, \lambda), \tag{3.2}$$

(see Ambrosetti–Rabinowitz [3] and Mugnai [25]). Integrating (3.1) and using (3.2) we reach a weaker condition, namely that

$$c_6 s^\tau \le F(x,s,\lambda) \quad \text{for a.a. } x \in \Omega, \ \text{ for all } s \ge M, \text{ with } c_6 = c_6(\lambda) > 0. \tag{3.3}$$

From (3.3) follows the much weaker condition (recall that $\tau > p$)

$$\lim_{s\to+\infty} \frac{F(x,s,\lambda)}{s^p} = +\infty \quad \text{uniformly for a.a. } x \in \Omega. \tag{3.4}$$

In the present work we employ (3.4) together with condition H(iii) which are weaker than the AR-condition and permit the consideration of superlinear reactions with slower growth near $+\infty$ which fail to satisfy the AR-condition. If the AR-condition is satisfied, then we may assume that $\tau = \tau(\lambda) > \max\left\{(r(\lambda) - p)\frac{N}{p}, 1\right\}$. Hence, (3.1) and (3.3) imply

$$\begin{aligned}
&\frac{f(x,s,\lambda)s - pF(x,s,\lambda)}{s^\tau} \\
&= \frac{f(x,s,\lambda)s - \tau F(x,s,\lambda)}{s^\tau} + (\tau - p)\frac{F(x,s,\lambda)}{s^\tau} \\
&\geq (\tau - p)c_6 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq M.
\end{aligned}$$

In consequence, hypotheses H(iii)(2) is fulfilled.

Example 3.2 The following functions satisfy hypotheses H (for the sake of simplicity we drop the x-dependence).

(i) $f_1(s) = \lambda s^{q-1} + s^{r-1}$ for all $s \geq 0$ and with $1 < q < p < r < p^*$.
This is the nonlinearity considered in Ambrosetti–Brezis–Cerami [2] where $p = 2$ (semilinear equations driven by the Laplacian) and in García Azorero–Peral Alonso–Manfredi [12], Guo-Zhang [18] where $1 < p < \infty$ (nonlinear equations driven by the p-Laplacian).

(ii) A reaction which does not satisfy the AR-condition can be given by
$f_2(s) = \lambda s^{q-1} + s^{p-1}\left[\ln(s) + \frac{1}{p}\right]$ for all $s \geq 0$ with $1 < q < p$.

(iii) Other admissible reactions are the following.

(1) $f_3(s) = \xi(\lambda)\left(s^{q-1} + s^{r-1}\right)$ for all $s \geq 0$ with $1 < q < p < r < p^*$, $\xi(\lambda) > 0$, $\lambda \to \xi(\lambda)$ is increasing, $\xi(\lambda) \to 0^+$ as $\lambda \to 0^+$, and $\xi(\lambda) \to +\infty$ as $\lambda \to +\infty$.

(2) $f_4(s) = \begin{cases} \lambda s^{q-1} & \text{if } s \in [0, \rho(\lambda)], \\ s^{r-1} + \lambda\rho(\lambda)^{q-1} - \rho(\lambda)^{r-1} & \text{if } \rho(\lambda) < s \end{cases}$
with $1 < q < p < r < p^*$, $\rho(\lambda) \in [0, 1]$, $\lambda \to \rho(\lambda)$ is nondecreasing, $\rho(\lambda) \to 0^+$ as $\lambda \to 0^+$, and $\rho(\lambda) \to 1^-$ as $\lambda \to +\infty$.

First, we introduce the following sets

$$\begin{aligned}
\mathcal{L} &= \{\lambda > 0 : \text{ problem } (\mathrm{P}_\lambda) \text{ admits a positive solution}\}, \\
\mathcal{S}(\lambda) &= \text{ the set of positive solutions of problem } (\mathrm{P}_\lambda).
\end{aligned}$$

We define, for every $\lambda > 0$, the corresponding C^1-energy functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ to problem (P_λ) by

$$\varphi_\lambda(u) = \int_\Omega G(\nabla u)dx - \int_\Omega F(x,u,\lambda)dx.$$

We start with an observation concerning the solution set $\mathcal{S}(\lambda)$.

Proposition 3.3 *If hypotheses H(a)(i)–(iii) and H(i),(iv) hold, then* $\mathcal{S}(\lambda) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ *for every* $\lambda > 0$.

Proof We may assume that $\lambda \in \mathcal{L}$, otherwise $\mathcal{S}(\lambda) = \emptyset$. Therefore, there exists $u \in W_0^{1,p}(\Omega)$, $u \geq 0$, $u \neq 0$ such that

$$-\operatorname{div} a(\nabla u) = f(x,u,\lambda) \quad \text{for a.a. } x \in \Omega. \tag{3.5}$$

From Ladyzhenskaya–Ural′tseva [20, p. 286] it follows that $u \in L^\infty(\Omega)$ and the regularity results of Lieberman [22, p. 320] imply $u \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$.

Owing to hypotheses H(i),(iv), for a given $\rho > 0$, we can find $\xi_\rho^\lambda > 0$ such that

$$f(x, s, \lambda) + \xi_\rho^\lambda s^{p-1} \geq 0 \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leq s \leq \rho. \tag{3.6}$$

Let $\rho = \|u\|_\infty > 0$ and let $\xi_\rho^\lambda > 0$ be as in (3.6). Combining (3.5) and (3.6) gives

$$-\operatorname{div} a(\nabla u) + \xi_\rho^\lambda u^{p-1} \geq 0 \quad \text{for a.a. } x \in \Omega,$$

equivalently

$$\operatorname{div} a(\nabla u) \leq \xi_\rho^\lambda u^{p-1} \quad \text{for a.a. } x \in \Omega. \tag{3.7}$$

Letting $\chi(t) = t a_0(t)$ for all $t > 0$, hypothesis H(a)(iii) and (2.1) ensure that

$$t\chi'(t) = t^2 a_0'(t) + t a_0(t) \geq c_1 t^{p-1}.$$

Integrating by parts leads to

$$\int_0^t s\chi'(s)ds = t\chi(t) - \int_0^t \chi(s)ds = t^2 a_0(t) - G_0(t) \geq \frac{c_1}{p} t^p. \tag{3.8}$$

We set $H(t) = t^2 a_0(t) - G_0(t)$ and $H_0(t) = \frac{c_1}{p} t^p$ for all $t \geq 0$. Let $\delta \in (0, 1)$ and $s > 0$. We introduce the sets

$$C_1 = \{t \in (0, 1) : H(t) \geq s\} \quad \text{and} \quad C_2 = \{t \in (0, 1) : H_0(t) \geq s\}.$$

It is easy to see that $C_2 \subseteq C_1$ [see (3.8)] and so $\inf C_1 \leq \inf C_2$. Therefore, due to Leoni [21, p. 6],

$$H^{-1}(s) \leq H_0^{-1}(s).$$

Hence

$$\int_0^\delta \frac{1}{H^{-1}\left(\frac{\xi_\rho^\lambda}{p} s^p\right)} ds \geq \int_0^\delta \frac{1}{H_0^{-1}\left(\frac{\xi_\rho^\lambda}{p} s^p\right)} ds = \frac{\xi_\rho^\lambda}{c_1} \int_0^\delta \frac{ds}{s} = +\infty.$$

Then, because of (3.7), we may apply the strong maximum principle of Pucci–Serrin [27, p. 111] which ensures that $u(x) > 0$ for all $x \in \Omega$. The boundary point lemma of Pucci–Serrin [27, p. 120] implies then $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. We conclude that $\mathcal{S}(\lambda) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

Proposition 3.4 *If hypotheses H(a) and H(i)–(iv) hold, then the energy functional* φ_λ *satisfies the C-condition for every* $\lambda > 0$.

Proof Let $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_\lambda(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ for all } n \geq 1, \tag{3.9}$$

$$\left(1 + \|u_n\|_{W_0^{1,p}(\Omega)}\right) \varphi_\lambda'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.10}$$

Thanks to (3.10) there holds

$$\left|\left\langle \varphi_\lambda'(u_n), h\right\rangle\right| \leq \frac{\varepsilon_n \|h\|_{W_0^{1,p}(\Omega)}}{1 + \|u_n\|_{W_0^{1,p}(\Omega)}} \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+,$$

that is

$$\left|\langle A(u_n), h\rangle - \int_\Omega f(x, u_n, \lambda) h dx\right| \leq \frac{\varepsilon_n \|h\|_{W_0^{1,p}(\Omega)}}{1 + \|u_n\|_{W_0^{1,p}(\Omega)}} \quad \text{for all } n \geq 1. \tag{3.11}$$

Taking $h = -u_n^- \in W_0^{1,p}(\Omega)$ in (3.11) gives

$$\int_\Omega \left(a\left(-\nabla u_n^-\right), -\nabla u_n^-\right)_{\mathbb{R}^N} dx \leq \varepsilon_n \quad \text{for all } n \geq 1,$$

which results in, due to Lemma 2.4(iii),

$$\frac{c_1}{p-1} \left\|\nabla u_n^-\right\|_p^p \leq \varepsilon_n \quad \text{for all } n \geq 1.$$

Hence,

$$u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{3.12}$$

Moreover, combining (3.9) and (3.12), yields

$$\int_\Omega pG\left(\nabla u_n^+\right) dx - \int_\Omega pF\left(x, u_n^+, \lambda\right) dx \leq M_2 \quad \text{for all } n \geq 1, \tag{3.13}$$

for some $M_2 > 0$. In (3.11) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ to obtain

$$-\int_\Omega \left(a\left(\nabla u_n^+\right), \nabla u_n^+\right)_{\mathbb{R}^N} dx + \int_\Omega f\left(x, u_n^+, \lambda\right) u_n^+ dx \leq \varepsilon_n \quad \text{for all } n \geq 1. \tag{3.14}$$

Adding (3.13) and (3.14) gives

$$\int_\Omega \left[pG\left(\nabla u_n^+\right) - \left(a\left(\nabla u_n^+\right), \nabla u_n^+\right)_{\mathbb{R}^N}\right] dx \\ + \int_\Omega \left[f\left(x, u_n^+, \lambda\right) u_n^+ - pF\left(x, u_n^+, \lambda\right)\right] dx \leq M_3 \quad \text{for all } n \geq 1,$$

for some $M_3 > 0$. Taking into account hypothesis H(a)(iv)(4) we get

$$\int_\Omega \left[f\left(x, u_n^+, \lambda\right) u_n^+ - pF\left(x, u_n^+, \lambda\right)\right] dx \leq M_4 \quad \text{for all } n \geq 1, \tag{3.15}$$

for some $M_4 > 0$. By virtue of hypotheses H(i)–(iii) we can find $\beta_1 \in (0, \beta_0(\lambda))$ and $c_7 = c_7(\lambda) > 0$ such that

$$f(x, s, \lambda)s - pF(x, s, \lambda) \geq \beta_1 s^{\tau(\lambda)} - c_7 \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq 0. \tag{3.16}$$

Using (3.16) in (3.15) we infer that

$$(u_n^+)_{n\geq 1} \subseteq L^{\tau(\lambda)}(\Omega) \text{ is bounded.} \tag{3.17}$$

First we assume that $N \neq p$. Having regard to hypothesis H(iii), without loss of generality, we may assume that $\tau(\lambda) < r(\lambda) < p^*$. Therefore, there exists $t \in (0, 1)$ such that

$$\frac{1}{r(\lambda)} = \frac{1-t}{\tau(\lambda)} + \frac{t}{p^*}. \tag{3.18}$$

Invoking the interpolation theory (see, for example, Gasiński–Papageorgiou [13, p. 905]) in combination with (3.17) and the Sobolev embedding theorem we have

$$\left\|u_n^+\right\|_{r(\lambda)} \leq \left\|u_n^+\right\|_{\tau(\lambda)}^{1-t} \left\|u_n^+\right\|_{p^*}^{t} \leq c_8 \left\|u_n^+\right\|_{W_0^{1,p}(\Omega)}^{t} \quad \text{for all } n \geq 1 \tag{3.19}$$

and for some $c_8 > 0$.

Hypotheses H(i),(ii) imply that

$$f(x, s, \lambda) \leq c_9\left(1 + s^{r(\lambda)}\right) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \geq 0, \tag{3.20}$$

and for some $c_9 > 0$. Now we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ in (3.11) to get

$$\int_\Omega \left(a\left(\nabla u_n^+\right), \nabla u_n^+\right)_{\mathbb{R}^N} dx - \int_\Omega f\left(x, u_n^+, \lambda\right) u_n^+ dx \leq \varepsilon_n \quad \text{for all } n \geq 1.$$

From this, by applying Lemma 2.4(iii), (3.20), and (3.19) we conclude that

$$\frac{c_1}{p-1}\left\|\nabla u_n^+\right\|_p^p \leq c_{10}\left(1+\left\|u_n^+\right\|_{r(\lambda)}^{r(\lambda)}\right) \leq c_{11}\left(1+\left\|u_n^+\right\|_{W_0^{1,p}(\Omega)}^{r(\lambda)t}\right) \tag{3.21}$$

for all $n \geq 1$ and for some $c_{10}, c_{11} > 0$.

The hypotheses on $\tau(\lambda)$ [see H(iii)] and (3.18) imply that $tr(\lambda) < p$. Hence, from (3.21) it follows that

$$(u_n^+)_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.22}$$

If $N = p$, then by definition $p^* = \infty$ while from the Sobolev embedding theorem we know that $W_0^{1,p}(\Omega)$ is compactly embedded in $L^\eta(\Omega)$ for all $\eta \in [1, \infty)$. So, for the previous argument to work, we need to replace p^* by $\eta > r(\lambda)$ large enough such that

$$tr(\lambda) = \frac{\eta(r(\lambda) - \tau(\lambda))}{\eta - \tau(\lambda)} < p.$$

Then we reach again (3.22).

From (3.12) and (3.22) we know that $(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded and so by passing to a suitable subsequence if necessary we may assume that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^{r(\lambda)}(\Omega). \tag{3.23}$$

In (3.11) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$, and apply (3.23). This gives

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

which by the $(S)_+$-property of A (see Proposition 2.7) results in

$$u_n \to u \text{ in } W_0^{1,p}(\Omega).$$

This proves that the functional φ_λ satisfies the C-condition for every $\lambda > 0$. □

Next we prove the nonemptiness and a structural property of $\mathcal{L}$.

Proposition 3.5 *If hypotheses H(a) and H hold, then $\mathcal{L} \neq \emptyset$ and for every $\lambda \in \mathcal{L}$ we have $(0, \lambda] \subseteq \mathcal{L}$.*

Proof We are going to show that the functional φ_λ satisfies the mountain pass geometry (see Theorem 2.2) for $\lambda > 0$ small enough. This fact in conjunction with Proposition 3.4 will permit the usage of the mountain pass theorem (see Theorem 2.2) which will show that, for $\lambda > 0$ small enough, the solution set $\mathcal{S}(\lambda)$ is nonempty and so $\mathcal{L} \neq \emptyset$.

Claim *There exists* $\hat{\lambda} > 0$ *such that, for all* $\lambda \in (0, \hat{\lambda})$, *we can find* $\varrho_\lambda > 0$ *such that*

$$\inf \left[\varphi_\lambda(u) : \|u\|_{W_0^{1,p}(\Omega)} = \varrho_\lambda \right] = m_\lambda > 0 = \varphi_\lambda(0).$$

For every $\lambda > 0$, by virtue of hypotheses H(i), (ii), and (iv), we can find $c_{12}(\lambda) > 0$, $c_{13}(\lambda) > 0$ such that

$$c_{12}(\lambda) \to 0^+ \text{ as } \lambda \to 0^+, \quad \lambda \to c_{13}(\lambda) \text{ is bounded on bounded sets,}$$

and

$$F(x, s, \lambda) \le c_{12}(\lambda) s^{q(\lambda)} + c_{13}(\lambda) s^{r(\lambda)} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \ge 0. \tag{3.24}$$

Taking into account Corollary 2.5, (3.24), and the Sobolev embedding theorem we derive

$$\begin{aligned} \varphi_\lambda(u) &= \int_\Omega G(\nabla u) dx - \int_\Omega F(x, u, \lambda) dx \\ &\ge \frac{c_1}{p(p-1)} \|\nabla u\|_p^p - \int_\Omega F(x, u, \lambda) dx \\ &\ge c_{14} \|u\|^p_{W_0^{1,p}(\Omega)} - c_{15}(\lambda) \|u\|^{q(\lambda)}_{W_0^{1,p}(\Omega)} - c_{16}(\lambda) \|u\|^{r(\lambda)}_{W_0^{1,p}(\Omega)} \end{aligned}$$

with $c_{14} = \frac{c_1}{p(p-1)}$, $c_{15}(\lambda) > 0$ satisfying $c_{15}(\lambda) \to 0^+$ as $\lambda \to 0^+$, and $c_{16}(\lambda) > 0$ with $\lambda \to c_{16}(\lambda)$ being bounded on bounded sets. Therefore,

$$\varphi_\lambda(u) \ge \left[c_{14} - c_{15}(\lambda) \|u\|^{q(\lambda)-p}_{W_0^{1,p}(\Omega)} - c_{16}(\lambda) \|u\|^{r(\lambda)-p}_{W_0^{1,p}(\Omega)} \right] \|u\|^p_{W_0^{1,p}(\Omega)}. \tag{3.25}$$

Now, let $\xi_\lambda(t) = c_{15}(\lambda) t^{q(\lambda)-p} + c_{16}(\lambda) t^{r(\lambda)-p}$ for all $t > 0$. Clearly, $\xi_\lambda \in C^1(0, \infty)$ and since $q(\lambda) < p < r(\lambda)$ for all $\lambda > 0$, we see that

$$\xi_\lambda(t) \to +\infty \text{ as } t \to 0^+ \text{ and as } t \to +\infty.$$

Thus, we can find a number $t_0 \in (0, +\infty)$ such that $\xi_\lambda(t_0) = \inf_{t>0} \xi_\lambda(t)$, that is, $\xi'_\lambda(t_0) = 0$. This gives

$$(p - q(\lambda)) c_{15}(\lambda) t_0^{q(\lambda)-p-1} = (r(\lambda) - p) c_{16}(\lambda) t_0^{r(\lambda)-p},$$

respectively

$$t_0 = t_0(\lambda) = \left[\frac{(p - q(\lambda)) c_{15}(\lambda)}{(r(\lambda) - p) c_{16}(\lambda)} \right]^{\frac{1}{r(\lambda)-q(\lambda)}}.$$

The hypotheses on $\lambda \to q(\lambda)$ and on $\lambda \to r(\lambda)$ [see H(iii), (iv)] and the properties of $\lambda \to c_{15}(\lambda)$ as well as $\lambda \to c_{16}(\lambda)$ imply that

$$\xi_\lambda(t_0) \to 0^+ \text{ as } \lambda \to 0^+.$$

So, we can find a number $\hat{\lambda} > 0$ small enough such that

$$\xi_\lambda(t_0) < c_{14} \quad \text{for all } \lambda \in (0, \hat{\lambda}).$$

Then, from (3.25) we see that

$$\varphi_\lambda(u) \geq m_\lambda > 0 = \varphi_\lambda(0) \quad \text{for all } \|u\|_{W_0^{1,p}(\Omega)} = t_0(\lambda) = \varrho_\lambda.$$

This proves the Claim.

Hypothesis H(ii) implies that, for all $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$, there holds

$$\varphi_\lambda(tu) \to -\infty \text{ as } t \to +\infty \text{ and for all } \lambda > 0. \tag{3.26}$$

Then, the Claim, (3.26), and Proposition 3.4 permit the usage of the mountain pass theorem (see Theorem 2.2) to find an element $u_\lambda \in W_0^{1,p}(\Omega)$ (for $\lambda \in (0, \hat{\lambda})$) such that

$$\varphi_\lambda'(u_\lambda) = 0 \quad \text{and} \quad \varphi_\lambda(0) = 0 < m_\lambda \leq \varphi_\lambda(u_\lambda). \tag{3.27}$$

The second assertion in (3.27) gives $u_\lambda \neq 0$ and the first one reads as

$$A(u_\lambda) = N_{f_\lambda}(u_\lambda), \tag{3.28}$$

where $f_\lambda(x, s) = f(x, s, \lambda)$. Acting on (3.28) with $-u_\lambda^- \in W_0^{1,p}(\Omega)$ we directly obtain, using Lemma 2.4(iii), that

$$\frac{c_1}{p-1} \|\nabla u_\lambda^-\|_p^p \leq 0$$

implying $u_\lambda \geq 0$, $u_\lambda \neq 0$. Therefore, $u_\lambda \in \mathcal{S}(\lambda) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ (see Proposition 3.3) and so $\left(0, \hat{\lambda}\right) \subseteq \mathcal{L}$, hence $\mathcal{L} \neq \emptyset$. This proves the first assertion of the proposition.

Next, let $\lambda \in \mathcal{L}$ and take $\gamma \in (0, \lambda)$. Since $\lambda \in \mathcal{L}$ there exists $u_\lambda \in \mathcal{S}(\lambda) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Thus,

$$-\operatorname{div} a(\nabla u_\lambda) = f(x, u_\lambda, \lambda) \geq f(x, u_\lambda, \gamma) \quad \text{for a.a. } x \in \Omega, \tag{3.29}$$

because $\gamma < \lambda$ and the fact that $\lambda \to f(x, s, \lambda)$ is nondecreasing (see H).

We introduce the following Carathéodory function

$$\hat{f}_\gamma(x,s) = \begin{cases} f(x,s,\gamma) & \text{if } s \leq u_\lambda(x), \\ f(x,u_\lambda(x),\gamma) & \text{if } u_\lambda(x) < s. \end{cases} \tag{3.30}$$

Setting $\hat{F}_\gamma(x,s) = \int_0^s \hat{f}_\gamma(x,t)dt$ we define the C^1-functional $\hat{\psi}_\gamma : W_0^{1,p}(\Omega) \to \mathbb{R}$ through

$$\hat{\psi}_\gamma(u) = \int_\Omega G(\nabla u)dx - \int_\Omega \hat{F}_\gamma(x,u)dx.$$

From Corollary 2.5 and the truncation defined in (3.30) it is clear that $\hat{\psi}_\gamma$ is coercive. Moreover, the convex integral $u \to \int_\Omega G(\nabla u)dx$ is sequentially weakly lower semicontinuous (follows from Mazur's lemma) while, by applying the Sobolev embedding theorem, the same property can be shown for the functional $u \to \int_\Omega \hat{F}_\gamma(x,u)dx$. It follows that the functional $u \to \hat{\psi}_\gamma(u)$ is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. Then, by the Weierstrass theorem, we find $u_\gamma \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_\gamma\left(u_\gamma\right) = \inf\left[\hat{\psi}_\gamma(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.31}$$

Owing to hypothesis H(a)(iv)(2) we find numbers $\tilde{\eta} > 0$ and $\delta_1 \in (0, \delta_0(\gamma)\big]$ such that

$$G_0(t) \leq \tilde{\eta} t^\nu \quad \text{for all } t \in (0, \delta_1]. \tag{3.32}$$

Let $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and recall that $u_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. By Lemma 2.10 there exists a number $\tilde{t} \in (0,1)$ small enough such that

$$\tilde{t}u(x), \tilde{t}\,|\nabla u(x)| \in [0, \delta_1] \text{ for all } x \in \overline{\Omega} \text{ and } \tilde{t}u \leq u_\lambda. \tag{3.33}$$

Applying (3.32) and (3.33) as well as hypothesis H(iv)(3) yields

$$\begin{aligned} \hat{\psi}_\gamma\left(\tilde{t}u\right) &= \int_\Omega G\left(\tilde{t}\nabla u\right)dx - \int_\Omega \hat{F}_\gamma\left(x, \tilde{t}u\right)dx \\ &\leq \tilde{\eta}\left(\tilde{t}\right)^\nu \|\nabla u\|_\nu^\nu - \hat{c}_0(\lambda)\left(\tilde{t}\right)^\theta \|u\|_\theta^\theta. \end{aligned} \tag{3.34}$$

Since $\theta < \nu$ [see hypotheses H(a)(iv) and H(iv)] we see that by taking $\tilde{t} \in (0,1)$ even smaller if necessary we will have from (3.34)

$$\hat{\psi}_\gamma\left(\tilde{t}u\right) < 0$$

which gives, due to (3.31),

$$\hat{\psi}_\gamma\left(u_\gamma\right) < 0 = \hat{\psi}_\gamma(0).$$

Hence, $u_\gamma \neq 0$. As u_γ is a critical point of $\hat{\psi}_\gamma$ there holds $\left(\hat{\psi}_\gamma\right)'\left(u_\gamma\right) = 0$, that is

$$A\left(u_\gamma\right) = N_{\hat{f}_\gamma}\left(u_\gamma\right). \tag{3.35}$$

Acting on (3.35) with $-u_\gamma^- \in W_0^{1,p}(\Omega)$ gives

$$\frac{c_1}{p-1}\left\|\nabla u_\gamma^-\right\|_p^p \leq 0,$$

thanks to the truncation in (3.30) and Lemma 2.4(iii). Hence, $u_\gamma \geq 0$, $u_\gamma \neq 0$.

Now, taking $\left(u_\gamma - u_\lambda\right)^+ \in W_0^{1,p}(\Omega)$ as test function in (3.35) results in, due to (3.29) and (3.30),

$$\begin{aligned}\left\langle A\left(u_\gamma\right), \left(u_\gamma - u_\lambda\right)^+\right\rangle &= \int_\Omega \hat{f}_\gamma\left(x, u_\gamma\right)\left(u_\gamma - u_\lambda\right)^+ dx \\ &= \int_\Omega f\left(x, u_\lambda, \gamma\right)\left(u_\gamma - u_\lambda\right)^+ dx \\ &\leq \left\langle A\left(u_\lambda\right), \left(u_\gamma - u_\lambda\right)^+\right\rangle.\end{aligned}$$

Therefore

$$\int_{\{u_\gamma > u_\lambda\}} \left(a\left(\nabla u_\gamma\right) - a\left(\nabla u_\lambda\right), \nabla u_\gamma - \nabla u_\lambda\right)_{\mathbb{R}^N} dx \leq 0,$$

which means that $\left|\left\{u_\gamma > u_\lambda\right\}\right|_N = 0$ and so $u_\gamma \leq u_\lambda$.

To sum up we have proved that

$$u_\gamma \in [0, u_\lambda] = \left\{u \in W_0^{1,p}(\Omega) : 0 \leq u(x) \leq u_\lambda(x) \text{ for a.a. } x \in \Omega\right\}.$$

Then according to (3.30), Eq. (3.35) becomes

$$A\left(u_\gamma\right) = N_{f_\gamma}\left(u_\gamma\right) \quad \text{with} \quad f_\gamma(x, s) = f(x, s, \gamma).$$

Hence, $u_\gamma \in \mathcal{S}(\gamma) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and so $\gamma \in \mathcal{L}$.

Therefore, we can say that if $\lambda \in \mathcal{L}$, then $(0, \lambda] \subseteq \mathcal{L}$. □

Remark 3.6 The above structural property of the admissible set $\mathcal{L}$ means that $\mathcal{L}$ is in fact an interval in $(0, +\infty)$.

Hypotheses H(iv), (v) imply that, for all $\lambda > 0$,

$$f(x, s, \lambda) \geq \hat{c}_0(\lambda)s^{\theta-1} - c_0^* s^{r^*-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geq 0. \tag{3.36}$$

This unilateral growth estimate on $f(x, \cdot, \lambda)$ leads to the following auxiliary Dirichlet problem

$$\begin{aligned} -\operatorname{div} a(\nabla u) &= \hat{c}_0(\lambda)u^{\theta-1} - c_0^* u^{r^*-1} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{Au$_\lambda$}$$

We have the following existence and uniqueness result for (Au$_\lambda$).

Proposition 3.7 *Let hypotheses H(a) be satisfied and let $\theta < \mu < d < p < r^* < p^*$ as well as $\lambda > 0$. Then, problem (Au$_\lambda$) has a unique positive solution $\tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and the map $\lambda \to \tilde{u}_\lambda$ is increasing, that is, if $\lambda < \gamma$, then $\tilde{u}_\gamma - \tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

Proof First, we establish the existence of a positive solution to (Au$_\lambda$) for all $\lambda > 0$. To this end, let $\xi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\xi_\lambda(u) = \int_\Omega G(\nabla u)dx - \frac{\hat{c}_0(\lambda)}{\theta} \left\|u^+\right\|_\theta^\theta + \frac{c_0^*}{r^*} \left\|u^+\right\|_{r^*}^{r^*}.$$

Since $r^* > p$ and because of Corollary 2.5 we easily verify that ξ_λ is coercive. Similar to the arguments in the proof of Proposition 3.5 we can conclude that ξ_λ is sequentially weakly lower semicontinuous. Hence, we find $\tilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\xi_\lambda\left(\tilde{u}_\lambda\right) = \inf\left[\xi_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.37}$$

As in the proof of Proposition 3.5 and since $\theta < \mu < p < r^* < p^*$ we infer that if $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $t \in (0, 1)$ sufficiently small, then $\xi_\lambda(tu) < 0$, which implies, because $\tilde{u}_\lambda$ is the global minimizer of ξ_λ [see (3.37)], that

$$\xi_\lambda\left(\tilde{u}_\lambda\right) < 0 = \xi_\lambda(0).$$

Thus, $\tilde{u}_\lambda \neq 0$. Furthermore, (3.37) gives $\xi_\lambda'\left(\tilde{u}_\lambda\right) = 0$, that is

$$A\left(\tilde{u}_\lambda\right) = \hat{c}_0\left(\left(\tilde{u}_\lambda\right)^+\right)^{\theta-1} - c_0^*\left(\left(\tilde{u}_\lambda\right)^+\right)^{r^*-1}. \tag{3.38}$$

Taking $-\left(\tilde{u}_\lambda\right)^- \in W_0^{1,p}(\Omega)$ as test function in (3.38) yields, owing to Lemma 2.4(iii),

$$\frac{c_1}{p-1}\left\|\nabla\left(\tilde{u}_\lambda\right)^-\right\|_p^p \leq 0.$$

So, $\tilde{u}_\lambda \geq 0$, $\tilde{u}_\lambda \neq 0$. Then, (3.38) becomes

$$A\left(\tilde{u}_\lambda\right) = \hat{c}_0\left(\tilde{u}_\lambda\right)^{\theta-1} - c_0^*\left(\tilde{u}_\lambda\right)^{r^*-1}$$

meaning that $\tilde{u}_\lambda$ is a positive solution of (Au$_\lambda$). As before (see the proof of Proposition 3.3), the nonlinear regularity theory (see Ladyzhenskaya–Ural′tseva [20] and Lieberman [22]) and the nonlinear maximum principle (see Pucci–Serrin [27, pp. 111, 120]) imply that $\tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.

Now, we are going to prove the uniqueness of $\tilde{u}_\lambda$. To this end, let $T : L^1(\Omega) \to \mathbb{R} \cup \{\infty\}$ be the integral functional defined by

$$T(u) = \begin{cases} \int_\Omega G\left(\nabla u^{\frac{1}{d}}\right) dx & \text{if } u \geq 0,\, u^{\frac{1}{d}} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let u_1, u_2 be in the domain of T, i.e. $u_1, u_2 \in \operatorname{dom}(T) = \left\{u \in L^1(\Omega) : T(u) < +\infty\right\}$ and let further $y = ((1-t)u_1 + tu_2)^{\frac{1}{d}}$ with $t \in [0, 1]$. Let $y_1 = u_1^{\frac{1}{d}}$, $y_2 = u_2^{\frac{1}{d}}$, then $y_1, y_2 \in W_0^{1,p}(\Omega)$. Now, we apply Lemma 1 in Díaz–Saá [8] to obtain

$$\|\nabla y(x)\|_{\mathbb{R}^N} \leq \left((1-t)\,\|\nabla y_1(x)\|_{\mathbb{R}^N}^d + t\,\|\nabla y_2(x)\|_{\mathbb{R}^N}^d\right)^{\frac{1}{d}} \quad \text{a.e. in } \Omega.$$

Since G_0 is increasing and thanks to hypotheses H(a)(iv)(1) we obtain

$$\begin{aligned} &G_0\left(\|\nabla u(x)\|_{\mathbb{R}^N}\right) \\ &\quad \leq G_0\left(\left((1-t)\,\|\nabla y_1(x)\|_{\mathbb{R}^N}^d + t\,\|\nabla y_2(x)\|_{\mathbb{R}^N}^d\right)^{\frac{1}{d}}\right) \\ &\quad \leq (1-t)G_0\left(\|\nabla y_1(x)\|_{\mathbb{R}^N}\right) + tG_0\left(\|\nabla y_2(x)\|_{\mathbb{R}^N}\right) \quad \text{a.e. in } \Omega. \end{aligned}$$

In view of $G(\xi) = G_0(\|\xi\|)$ for all $\xi \in \mathbb{R}^N$ it follows

$$G(\nabla u(x)) \leq (1-t)G\left(\nabla y_1(x)\right) + tG\left(\nabla y_2(x)\right) \quad \text{a.e. in } \Omega.$$

Therefore, T is convex. In addition, via Fatou's lemma, we see that T is lower semicontinuous.

Suppose that $\overline{u}_\lambda$ is another positive solution of (Au$_\lambda$). As done for $\tilde{u}_\lambda$, via the nonlinear regularity theory and the nonlinear maximum principle, we have $\overline{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Therefore, if $h \in C_0^1(\overline{\Omega})$ and $t \in (-1, 1)$ is small enough in its absolute value, then

$$(\tilde{u}_\lambda)^d + th \in \operatorname{dom}(T) \quad \text{and} \quad (\overline{u}_\lambda)^d + th \in \operatorname{dom}(T).$$

So, the Gateaux derivative of T at $(\tilde{u}_\lambda)^d$ and $(\overline{u}_\lambda)^d$ in the direction h exists and using the chain rule it follows

$$\begin{aligned} T'\left((\tilde{u}_\lambda)^d\right)(h) &= \frac{1}{d}\int_\Omega \frac{-\operatorname{div} a(\nabla \tilde{u}_\lambda)}{(\tilde{u}_\lambda)^{d-1}} h\,dx, \\ T'\left((\overline{u}_\lambda)^d\right)(h) &= \frac{1}{d}\int_\Omega \frac{-\operatorname{div} a(\nabla \overline{u}_\lambda)}{(\overline{u}_\lambda)^{d-1}} h\,dx. \end{aligned}$$

The convexity of T implies the monotonicity of T'. This leads to

$$
\begin{aligned}
0 &\le \left\langle T'\left((\tilde{u}_\lambda)^d\right) - T'\left((\overline{u}_\lambda)^d\right), (\tilde{u}_\lambda)^d - (\overline{u}_\lambda)^d \right\rangle_{L^1(\Omega)} \\
&= \frac{1}{d}\int_\Omega \left(\frac{-\operatorname{div} a(\nabla \tilde{u}_\lambda)}{(\tilde{u}_\lambda)^{d-1}} - \frac{-\operatorname{div} a(\nabla \overline{u}_\lambda)}{(\overline{u}_\lambda)^{d-1}} \right) \left((\tilde{u}_\lambda)^d - (\overline{u}_\lambda)^d\right) dx \\
&= \frac{1}{d}\int_\Omega \left(\frac{\hat{c}_0(\lambda)\, (\tilde{u}_\lambda)^{\theta-1} - c_0^* \,(\tilde{u}_\lambda)^{r^*-1}}{(\tilde{u}_\lambda)^{d-1}} - \frac{\hat{c}_0(\lambda)\overline{u}_\lambda^{\theta-1} - c_0^* \overline{u}_\lambda^{r^*-1}}{(\overline{u}_\lambda)^{d-1}} \right) \\
&\quad \times \left((\tilde{u}_\lambda)^d - (\overline{u}_\lambda)^d\right) dx \\
&= \frac{1}{d}\int_\Omega \left(\hat{c}_0(\lambda) \left[\frac{1}{(\tilde{u}_\lambda)^{d-\theta}} - \frac{1}{(\overline{u}_\lambda)^{d-\theta}} \right] + c_0^* \left[(\overline{u}_\lambda)^{r^*-d} - (\tilde{u}_\lambda)^{r^*-d} \right] \right) \\
&\quad \times \left((\tilde{u}_\lambda)^d - (\overline{u}_\lambda)^d\right) dx.
\end{aligned}
$$

Since $\theta < \mu < d < p < r^* < p^*$, the last inequality implies $\tilde{u}_\lambda = \overline{u}_\lambda$. This proves the uniqueness of the positive solution of (Au$_\lambda$) for all $\lambda > 0$.

Next, we examine the monotonicity of the map $\lambda \to \tilde{u}_\lambda$ from $(0, \infty)$ into $C_0^1(\overline{\Omega})_+ \setminus \{0\}$. Letting $0 < \lambda < \gamma$, we first observe, due to hypothesis H(iv)(2), that

$$
\begin{aligned}
-\operatorname{div} a\left(\nabla \tilde{u}_\gamma\right) &= \hat{c}_0(\gamma)\tilde{u}_\gamma^{\theta-1} - c_0^*\tilde{u}_\gamma^{r^*-1} \\
&\ge \hat{c}_0(\lambda)\tilde{u}_\gamma^{\theta-1} - c_0^*\tilde{u}_\gamma^{r^*-1} \quad \text{for a.a. } x \in \Omega.
\end{aligned} \tag{3.39}
$$

Introducing the Carathéodory function

$$
v_\lambda(x, s) = \begin{cases} 0 & \text{if } s < 0, \\ \hat{c}_0(\lambda)s^{\theta-1} - c_0^* s^{r^*-1} & \text{if } 0 \le s \le \tilde{u}_\gamma(x), \\ \hat{c}_0(\lambda)\left(\tilde{u}_\gamma(x)\right)^{\theta-1} - c_0^*\left(\tilde{u}_\gamma(x)\right)^{r^*-1} & \text{if } \tilde{u}_\gamma(x) < s, \end{cases} \tag{3.40}
$$

and setting $V_\lambda(x, s) = \int_0^s v_\lambda(x, t)dt$, we consider the C^1-functional $\sigma_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$
\sigma_\lambda(u) = \int_\Omega G(\nabla u)dx - \int_\Omega V_\lambda(x, u)dx.
$$

Applying Corollary 2.5 and the truncation defined in (3.40) we conclude that σ_λ is coercive. In addition, it is sequentially weakly lower semicontinuous. Therefore, we find an element $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$
\sigma_\lambda\left(\hat{u}_\lambda\right) = \inf\left[\sigma_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.41}
$$

As in the proof of Proposition 3.5 and since $\theta < \mu < r^*$, for $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $t \in (0, 1)$ small enough (at least such that $tu \le \tilde{u}_\gamma$, see Lemma 2.10), we have

$\sigma_\lambda(tu) < 0$ implying

$$\sigma_\lambda\left(\hat{u}_\lambda\right) < 0 = \sigma_\lambda(0).$$

Thus, $\hat{u}_\lambda \neq 0$. The assertion in (3.41) gives $\sigma_\lambda'\left(\hat{u}_\lambda\right) = 0$ and so

$$A\left(\hat{u}_\lambda\right) = N_{v_\lambda}\left(\hat{u}_\lambda\right). \tag{3.42}$$

Acting on (3.42) with $-\left(\hat{u}_\lambda\right)^- \in W_0^{1,p}(\Omega)$ and applying Lemma 2.4(iii) as well as (3.40) gives

$$\frac{c_1}{p-1}\left\|\nabla\left(\hat{u}_\lambda\right)^-\right\|_p^p \leq 0.$$

Hence, $\hat{u}_\lambda \geq 0, \hat{u}_\lambda \neq 0$. Now, we choose $\left(\hat{u}_\lambda - \tilde{u}_\gamma\right)^+ \in W_0^{1,p}(\Omega)$ in (3.42). By means of (3.39) and (3.40) we obtain

$$\begin{aligned}\left\langle A\left(\hat{u}_\lambda\right), \left(\hat{u}_\lambda - \tilde{u}_\gamma\right)^+\right\rangle &= \int_\Omega v_\lambda\left(x, \hat{u}_\lambda\right)\left(\hat{u}_\lambda - \tilde{u}_\gamma\right)^+ dx\\ &= \int_\Omega \left[\hat{c}_0(\lambda)\left(\hat{u}_\lambda\right)^{\theta-1} - c_0^*\left(\hat{u}_\lambda\right)^{r^*-1}\right]\left(\hat{u}_\lambda - \tilde{u}_\gamma\right)^+ dx\\ &\leq \left\langle A\left(\tilde{u}_\gamma\right), \left(\hat{u}_\lambda - \tilde{u}_\gamma\right)^+\right\rangle,\end{aligned}$$

which implies

$$\int_{\{\hat{u}_\lambda > \tilde{u}_\gamma\}} \left(a\left(\nabla\hat{u}_\lambda\right) - a\left(\nabla\tilde{u}_\gamma\right), \nabla\hat{u}_\lambda - \nabla\tilde{u}_\gamma\right)_{\mathbb{R}^N} \leq 0.$$

Taking into account Lemma 2.4(i) we conclude that $\left|\left\{\hat{u}_\lambda > \tilde{u}_\gamma\right\}\right|_N = 0$ and hence, $\hat{u}_\lambda \leq \tilde{u}_\gamma$. So, we have proved

$$\hat{u}_\lambda \in \left[u, \tilde{u}_\gamma\right] = \left\{u \in W_0^{1,p}(\Omega) : 0 \leq u(x) \leq \tilde{u}_\gamma \text{ for a.a. } x \in \Omega\right\}. \tag{3.43}$$

Then, Eq. (3.42) becomes

$$A\left(\hat{u}_\lambda\right) = \hat{c}_0(\lambda)\left(\hat{u}_\lambda\right)^{\theta-1} - c_0^*\left(\hat{u}_\lambda\right)^{r^*-1},$$

due to the truncation function defined in (3.40). Therefore, $\hat{u}_\lambda$ is a positive solution of (Au_λ) and because of the uniqueness of the positive solutions of (Au_λ) we infer that $\hat{u}_\lambda = \tilde{u}_\lambda$. In particular, we conclude that

$$\tilde{u}_\lambda \leq \tilde{u}_\gamma \tag{3.44}$$

[see (3.43)].

Note that, for a given $\rho > 0$, we can find $\xi_\rho > 0$ such that

$$s \to \xi_\rho s^{p-1} - c_0^* s^{r-1} \text{ is nondecreasing on } [0, \rho]. \tag{3.45}$$

Let $\rho = \left\|\tilde{u}_\gamma\right\|_\infty$ and let ξ_ρ be as in (3.45). Then, by applying (3.44), (3.45), and hypothesis H(iv)(2), we obtain

$$\begin{aligned}
-\operatorname{div} a\,(\nabla \tilde{u}_\lambda) + \xi_\rho\,(\tilde{u}_\lambda)^{p-1} &= \hat{c}_0(\lambda)\,(\tilde{u}_\lambda)^{\theta-1} - c_0^*\,(\tilde{u}_\lambda)^{r^*-1} + \xi_\rho\,(\tilde{u}_\lambda)^{p-1} \\
&\leq \hat{c}_0(\lambda)\,(\tilde{u}_\lambda)^{\theta-1} - c_0^*\left(\tilde{u}_\gamma\right)^{r^*-1} + \xi_\rho\left(\tilde{u}_\gamma\right)^{p-1} \\
&\leq \hat{c}_0(\gamma)\left(\tilde{u}_\gamma\right)^{\theta-1} - c_0^*\left(\tilde{u}_\gamma\right)^{r^*-1} + \xi_\rho\left(\tilde{u}_\gamma\right)^{p-1} \\
&= -\operatorname{div} a\left(\nabla \tilde{u}_\gamma\right) + \xi_\rho\left(\tilde{u}_\gamma\right)^{p-1} \quad \text{for a.a. } x \in \Omega.
\end{aligned}$$

Now, let

$$\begin{aligned}
g(x) &= \hat{c}_0(\lambda)\,(\tilde{u}_\lambda(x))^{\theta-1} - c_0^*\,(\tilde{u}_\lambda(x))^{r^*-1} + \xi_\rho\,(\tilde{u}_\lambda(x))^{p-1}, \\
\hat{h}(x) &= \hat{c}_0(\lambda)\,(\tilde{u}_\lambda(x))^{\theta-1} - c_0^*\left(\tilde{u}_\gamma(x)\right)^{r^*-1} + \xi_\rho\left(\tilde{u}_\gamma(x)\right)^{p-1}, \\
h(x) &= \hat{c}_0(\gamma)\left(\tilde{u}_\gamma(x)\right)^{\theta-1} - c_0^*\left(\tilde{u}_\gamma(x)\right)^{r^*-1} + \xi_\rho\left(\tilde{u}_\gamma(x)\right)^{p-1}.
\end{aligned}$$

Evidently, $g(x) \leq \hat{h}(x) \leq h(x)$ for a.a. $x \in \Omega$. Note that, by means of (3.44),

$$\begin{aligned}
h(x) - \hat{h}(x) &= \left(\hat{c}_0(\gamma) - \hat{c}_0(\lambda)\right)\left(\tilde{u}_\gamma(x)\right)^{\theta-1} + \hat{c}_0(\gamma)\left(\left(\tilde{u}_\gamma(x)\right)^{\theta-1} - (\tilde{u}_\lambda(x))^{\theta-1}\right) \\
&\geq \left(\hat{c}_0(\gamma) - \hat{c}_0(\lambda)\right)\left(\tilde{u}_\gamma(x)\right)^{\theta-1}.
\end{aligned}$$

Since $\tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $\hat{c}_0(\gamma) > \hat{c}_0(\lambda)$ [see H(iv)(2)], it follows that $\hat{h} \prec h$ which implies $g \prec h$. Then, Proposition 2.9 gives $\tilde{u}_\gamma - \tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Therefore, $\lambda \to \tilde{u}_\lambda$ is increasing. □

Proposition 3.8 *Let hypotheses H(a) and H be satisfied and let $\lambda \in \mathcal{L}$. Then, $\tilde{u}_\lambda \leq u$ for all $u \in \mathcal{S}(\lambda)$, where $\tilde{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ is the unique positive solution of (Au_λ) obtained in Proposition 3.7.*

Proof Let $u \in \mathcal{S}(\lambda) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ (see Proposition 3.3) and consider the following Carathéodory function

$$k(x,s) = \begin{cases} 0 & \text{if } s < 0, \\ \hat{c}_0(\lambda) s^{\theta-1} - c_0^* s^{r^*-1} & \text{if } 0 \leq s \leq u(x), \\ \hat{c}_0(\lambda) u(x)^{\theta-1} - c_0^* u(x)^{r^*-1} & \text{if } u(x) < s. \end{cases} \tag{3.46}$$

Let $K(x,s) = \int_0^s k(x,t)dt$ and consider the C^1-functional $\hat{\sigma} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\sigma}(u) = \int_\Omega G(\nabla u)dx - \int_\Omega K(x, u)dx.$$

It is clear that $\hat{\sigma}$ is coercive and sequentially weakly lower semicontinuous which implies the existence of $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma}\left(\hat{u}\right) = \inf\left[\hat{\sigma}(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{3.47}$$

As before, exploiting the fact that $\theta < \mu < p < r^*$, for $u \in \text{int}\left(C_0^1(\overline{\Omega})_+\right)$ and $t \in (0, 1)$ small enough, we can show that $\hat{\sigma}(tu) < 0$, which implies $\hat{\sigma}\left(\hat{u}\right) < 0 = \hat{\sigma}(0)$. Hence, $\hat{u} \neq 0$.

From (3.47) we have $\left(\hat{\sigma}\right)'\left(\hat{u}\right) = 0$, that is

$$A\left(\hat{u}\right) = N_k\left(\hat{u}\right). \tag{3.48}$$

As before, acting on (3.48) with $-\left(\hat{u}\right)^- \in W_0^{1,p}(\Omega)$ and using (3.46) as well as Lemma 2.4(iii) we have $\hat{u} \geq 0$, $\hat{u} \neq 0$. Next, we choose $\left(\hat{u} - u\right)^+ \in W_0^{1,p}(\Omega)$ as test function in (3.48). Based on (3.36), (3.46) and since $u \in \mathcal{S}(\lambda)$, we obtain

$$\begin{aligned}
\left\langle A\left(\hat{u}\right), \left(\hat{u} - u\right)^+\right\rangle &= \int_\Omega k\left(x, \hat{u}\right)\left(\hat{u} - u\right)^+ dx \\
&= \int_\Omega \left[\hat{c}_0(\lambda)u^{\theta-1} - c_0^* u^{r^*-1}\right]\left(\hat{u} - u\right)^+ dx \\
&\leq \int_\Omega f(x, u, \lambda)\left(\hat{u} - u\right)^+ dx \\
&= \left\langle A(u), \left(\hat{u} - u\right)^+\right\rangle.
\end{aligned}$$

Consequently,

$$\int_{\{\hat{u}>u\}} \left(a\left(\nabla\hat{u}\right) - a(\nabla u), \nabla\hat{u} - \nabla u\right)_{\mathbb{R}^N} dx \leq 0.$$

Therefore, $\left|\left\{\hat{u} > u\right\}\right|_N = 0$ [see Lemma 2.4(i)] and so, $\hat{u} \leq u$. We have proved that

$$\hat{u} \in [0, u] = \left\{v \in W_0^{1,p}(\Omega) : 0 \leq v(x) \leq u(x) \text{ for a.a. } x \in \Omega\right\}.$$

Having regard to (3.46) and (3.48) we see that $\hat{u}$ is a positive solution of (Au_λ). Taking into account Proposition 3.7 we easily verify that $\hat{u} = \tilde{u}_\lambda$ which implies $\tilde{u}_\lambda \leq u$ for all $u \in \mathcal{S}(\lambda)$. □

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 3.9 *If hypotheses H(a) and H hold, then* $\lambda^* < \infty$.

Proof Arguing by contradiction, suppose we can find a sequence $(\lambda_n)_{n\geq 1} \subseteq \mathcal{L}$ such that $\lambda_n \nearrow +\infty$ as $n \to \infty$. For every $n \geq 1$ we find $u_n \in \mathcal{S}(\lambda_n) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ satisfying

$$\varphi_{\lambda_n}(u_n) < 0 \tag{3.49}$$

(see the proof of Proposition 3.5). Inequality (3.49) reads as

$$\int_\Omega pG(\nabla u_n)dx - \int_\Omega pF(x, u_n, \lambda_n)\, dx < 0 \quad \text{for all } n \geq 1. \tag{3.50}$$

Moreover, there holds

$$A(u_n) = N_{f_{\lambda_n}}(u_n) \quad \text{for all } n \geq 1.$$

Taking $u_n \in W_0^{1,p}(\Omega)$ as test function gives

$$-\int_\Omega (a(\nabla u_n), \nabla u_n)_{\mathbb{R}^N} dx + \int_\Omega f(x, u_n, \lambda_n)\, u_n dx = 0 \quad \text{for all } n \geq 1. \tag{3.51}$$

Adding both (3.50) and (3.51) and making use of hypothesis H(a)(iv)(3) results in

$$\int_\Omega [f(x, u_n, \lambda_n)\, u_n - pF(x, u_n, \lambda_n)]\, dx \leq M_5 \quad \text{for all } n \geq 1, \tag{3.52}$$

and for some $M_5 > 0$.

By virtue of hypotheses H(i), (iv) there exist $\hat{\beta} \in (0, \beta(\lambda_1))$ and $c_{17} > 0$ such that

$$\hat{\beta} s^{\tau(\lambda_1)} - c_{17} \leq f(x, s, \lambda_n) - pF(x, s, \lambda_n) \quad \text{for a.a. } x \in \Omega, \text{ for all } s \geq 0, \tag{3.53}$$

and for all $n \geq 1$. Applying (3.53) in (3.52) shows that

$$(u_n)_{n\geq 1} \subseteq L^{\tau(\lambda_1)}(\Omega) \text{ is bounded.} \tag{3.54}$$

Now, applying (3.54) and reasoning as in the proof of Proposition 3.4 [see the part of the proof after (3.17)], we obtain that

$$(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.55}$$

From (3.51), (3.55), and Lemma 2.4(ii), we see that there exists $M_6 > 0$ such that

$$\int_\Omega f(x, u_n, \lambda_n) u_n dx \leq M_6 \quad \text{for all } n \geq 1.$$

This gives, due to (3.36),

$$\hat{c}_0(\lambda_n)\, \|u_n\|_\theta^\theta - c_0^*\, \|u_n\|_{r^*}^{r^*} \leq M_6 \quad \text{for all } n \geq 1.$$

Recall that $r^* \in (p, p^*]$ [see hypothesis H(v)]. Then, from the last inequality and the Sobolev embedding theorem combined with (3.55) it follows

$$\hat{c}_0(\lambda_n) \|u_n\|_\theta^\theta \leq M_7 \quad \text{for all } n \geq 1 \text{ and with some } M_7 > 0.$$

Now, we may apply Propositions 3.8 and 3.7 to obtain

$$\hat{c}_0(\lambda_n) \left\|\tilde{u}_{\lambda_1}\right\|_\theta^\theta \leq M_7 \quad \text{for all } n \geq 1,$$

which contradicts the fact that $\hat{c}_0(\lambda_n) \to +\infty$ as $n \to \infty$ [see hypothesis H(iv)(2)]. This proves that $\lambda^* < \infty$. □

Proposition 3.5 implies that $(0, \lambda^*) \subseteq \mathcal{L}$.

Next, we establish a multiplicity result if $\lambda \in (0, \lambda^*)$. To do this, we need to strengthen the conditions on $f(x, \cdot, \lambda)$.

H': $f : \Omega \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is a function such that $(x, s, \lambda) \to f(x, s, \lambda)$ is a Carathéodory mapping on $\Omega \times [\mathbb{R} \times (0, \infty)]$, $\lambda \to f(x, s, \lambda)$ is nondecreasing, $f(x, 0, \lambda) = 0$ for a.a. $x \in \Omega$, for all $\lambda > 0$, hypotheses H'(i)–(v) are the same as the corresponding hypotheses H(i)–(v) and

(vi) for every $\rho > 0$ and every $\lambda > 0$, there exists $\xi_\rho^\lambda > 0$ such that

$$s \to f(x, s, \lambda) + \xi_\rho^\lambda s^{p-1} \text{ is nondecreasing on } [0, \rho]$$

for a.a. $x \in \Omega$ and for $\lambda > \mu > 0$ there holds

$$\text{ess inf}_\Omega \, [f(x, s, \lambda) - f(x, s, \mu) : s \geq \rho] \geq m_\rho > 0.$$

Remark 3.10 The examples of functions f presented after hypotheses H still satisfy the new conditions stated in H'.

Proposition 3.11 *Let hypotheses H(a) and H' be satisfied and let* $\lambda \in (0, \lambda^*)$. *Then, problem* ($P_\lambda$) *admits at least two positive solutions*

$$u_0, \hat{u} \in \text{int}\left(C_0^1(\overline{\Omega})_+\right) \text{ with } u_0 \leq \hat{u} \text{ and } u_0 \neq \hat{u}.$$

Proof Let $\gamma \in (\lambda, \lambda^*)$ and let $u_\gamma \in \mathcal{S}(\gamma) \subseteq \text{int}\left(C_0^1(\overline{\Omega})_+\right)$. We have

$$-\operatorname{div} a\left(\nabla u_\gamma\right) = f(x, u_\gamma, \gamma) \geq f(x, u_\gamma, \lambda) \quad \text{for a.a. } x \in \Omega. \tag{3.56}$$

We introduce the following Carathéodory function

$$\hat{f}_\lambda(x, s) = \begin{cases} f(x, s, \lambda) & \text{if } s \leq u_\gamma(x), \\ f(x, u_\gamma(x), \lambda) & \text{if } u_\gamma(x) < s. \end{cases}$$

Setting $\hat{F}_\lambda(x,s) = \int_0^s \hat{f}_\lambda(x,t)dt$, we define the C^1-functional $\hat{\psi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ through

$$\hat{\psi}_\lambda(u) = \int_\Omega G(\nabla u)dx - \int_\Omega \hat{F}_\lambda(x,u)dx.$$

Reasoning as in the proof of Proposition 3.5 [see the part of the proof after (3.30)] and using (3.56), we can show the existence of a solution $u_0 \in \mathcal{S}(\lambda)$ such that

$$u_0 \in \left[0, u_\gamma\right] = \left\{u \in W_0^{1,p}(\Omega) : 0 \le u(x) \le u_\gamma(x) \text{ for a.a. } x \in \Omega\right\}.$$

In fact we can say more. Let $\rho = \|u_\gamma\|_\infty$ and let $\xi_\rho^\lambda, \xi_\rho^\gamma$ be as postulated by hypothesis H'(vi). Choosing $\hat{\xi}_\rho > \max\left\{\xi_\rho^\lambda, \xi_\rho^\gamma\right\}$ and using H'(vi), $u_0 \le u_\gamma$, and the fact that $u_\gamma \in \mathcal{S}(\gamma)$ we derive

$$\begin{aligned} -\operatorname{div} a\left(\nabla u_0\right) + \hat{\xi}_\rho u_0^{p-1} &= f\left(x, u_0, \lambda\right) + \hat{\xi}_\rho u_0^{p-1} \\ &= f\left(x, u_0, \gamma\right) + \hat{\xi}_\rho u_0^{p-1} - \left[f\left(x, u_0, \gamma\right) - f\left(x, u_0, \lambda\right)\right] \\ &\le f\left(x, u_\gamma, \gamma\right) + \hat{\xi}_\rho u_\gamma^{p-1} \\ &= -\operatorname{div} a\left(\nabla u_\gamma\right) + \hat{\xi}_\rho u_\gamma^{p-1} \quad \text{for a.a. } x \in \Omega. \end{aligned}$$

Note that, if $\sigma(x) = f\left(x, u_0(x), \gamma\right) - f\left(x, u_0(x), \lambda\right)$, then since $u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and owing to hypotheses H'(vi) we have $0 \prec \sigma$ and so we may apply Proposition 2.9 to conclude that $u_\gamma - u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. Therefore, we have

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0, u_\gamma]. \tag{3.57}$$

Applying u_0 we introduce the following truncation of the mapping $s \to f(x,s,\lambda)$

$$e_\lambda(x,s) = \begin{cases} f(x, u_0(x), \lambda) & \text{if } s \le u_0(x), \\ f(x,s,\lambda) & \text{if } u_0(x) < s, \end{cases} \tag{3.58}$$

which is known to be a Carathéodory function. We set $E_\lambda(x,s) = \int_0^s e_\lambda(x,t)dt$ and consider the C^1-functional $w_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$w_\lambda(u) = \int_\Omega G(\nabla u)dx - \int_\Omega E_\lambda(x,u)dx.$$

Claim $K_{w_\lambda} = \left\{u \in W_0^{1,p}(\Omega) : w_\lambda'(u) = 0\right\} \subseteq [u_0)$
with $[u_0) = \left\{u \in W_0^{1,p}(\Omega) : u_0(x) \le u(x) \text{ for a.a. } x \in \Omega\right\}$
Let $u \in K_{w_\lambda}$, that is, $w_\lambda'(u) = 0$ and so

$$A(u) = N_{e_\lambda}(u). \tag{3.59}$$

Acting on (3.59) with $(u_0 - u)^+ \in W_0^{1,p}(\Omega)$ yields

$$\begin{aligned}\left\langle A(u), (u_0 - u)^+\right\rangle &= \int_\Omega e_\lambda(x, u)\,(u_0 - u)^+\,dx\\ &= \int_\Omega f\,(x, u_0, \lambda)\,(u_0 - u)^+\,dx\\ &= \left\langle A\,(u_0)\,,\,(u_0 - u)^+\right\rangle\end{aligned}$$

due to the truncation defined in (3.58) and the fact that $u_0 \in \mathcal{S}(\lambda)$. Therefore

$$\int_{\{u_0>u\}} (a\,(\nabla u_0) - a\,(\nabla u)\,,\,\nabla u_0 - \nabla u)_{\mathbb{R}^N}\,dx = 0$$

implying $|\{u_0 > u\}|_N = 0$ [see Lemma 2.4(i)] and thus, $u_0 \le u$. This proves the Claim.

By virtue of the Claim and (3.57) we see that the critical points of w_λ are positive solutions of problem (P_λ). So, we may assume that

$$K_{w_\lambda} \cap \left[\,[u_0, u_\gamma]\setminus\{u_0\}\right] = \emptyset \tag{3.60}$$

[see (3.57)], otherwise we would already have a second solution $\hat{u} \ge u_0$, $\hat{u} \ne u_0$.

Now, we introduce the following truncation of $e_\lambda(x, \cdot)$

$$\hat{e}_\lambda(x, s) = \begin{cases} e_\lambda(x, s) & \text{if } s \le u_\gamma(x),\\ e_\lambda(x, u_\gamma(x)) & \text{if } u_\gamma(x) < s,\end{cases} \tag{3.61}$$

being again a Carathéodory function. We set $\hat{E}_\lambda(x, s) = \int_0^s \hat{e}_\lambda(x, t)dt$ and consider the C^1-functional $\hat{w}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{w}_\lambda(u) = \int_\Omega G(\nabla u)dx - \int_\Omega \hat{E}_\lambda(x, u)dx.$$

By means of (3.61) and Corollary 2.5 we see that $\hat{w}_\lambda$ is coercive. As before, it is also sequentially weakly lower semicontinuous. Then, the Weierstrass theorem implies the existence of $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{w}_\lambda\,(\tilde{u}_0) = \inf\left[\hat{w}_\lambda(u) : u \in W_0^{1,p}(\Omega)\right],$$

that is, $\left(\hat{w}_\lambda\right)'(\tilde{u}_0) = 0$, hence

$$A\,(\tilde{u}_0) = N_{\hat{e}_\lambda}\,(\tilde{u}_0)\,. \tag{3.62}$$

As before, acting on (3.62) with $\left(\tilde{u}_0 - u_\gamma\right)^+ \in W_0^{1,p}(\Omega)$ and using the Claim, we derive that

$$\tilde{u}_0 \in \left[u_0, u_\gamma\right] = \left\{u \in W_0^{1,p}(\Omega) : u_0(x) \leq u(x) \leq u_\gamma(x) \text{ for a.a. } x \in \Omega\right\}.$$

Then, from (3.60) (3.61) we see that $\tilde{u}_0 = u_0$.

Note that $\hat{w}_\lambda\big|_{[0,u_\gamma]} = w_\lambda\big|_{[0,u_\gamma]}$ which follows from the definition of the truncations in (3.58) and (3.61). Recall that $u_\gamma - u_0 \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ [see (3.57)]. Therefore, we know that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of w_λ and taking into account Proposition 2.8 we have that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of w_λ as well.

Let us assume that K_{w_λ} is finite, otherwise we would have infinity distinct positive solutions u of (P_λ) with $u \geq u_0$ (see the Claim). Hence, there exists $\rho \in (0,1)$ small enough such that

$$w_\lambda(u_0) < \inf\left[w_\lambda(u) : \|u - u_0\|_{W_0^{1,p}(\Omega)} = \rho\right] = m_\rho \tag{3.63}$$

(see Aizicovici–Papageorgiou–Staicu [1, Proof of Proposition 29]). Note that, due to (3.58),

$$w_\lambda = \varphi_\lambda + \xi_\lambda \quad \text{with } \xi_\lambda \in \mathbb{R}. \tag{3.64}$$

From (3.26) and (3.64) it follows, for $u \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$,

$$w_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.65}$$

Furthermore, owing to (3.64) and Proposition 3.4, we have that

$$w_\lambda \text{ satisfies the C-condition.} \tag{3.66}$$

Now, based on (3.63), (3.65), and (3.66), we may apply the mountain pass theorem stated in Theorem 2.2. Hence, there exists $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{w_\lambda} \quad \text{and} \quad w_\lambda(u_0) < m_\rho \leq w_\lambda\left(\hat{u}\right). \tag{3.67}$$

The first assertion in (3.67) in combination with the Claim and Proposition 3.3 says that

$$\hat{u} \in \mathcal{S}(\lambda) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right) \quad \text{and} \quad u_0 \leq \hat{u}.$$

The second assertion gives $u_0 \neq \hat{u}$. □

Next, we examine what happens at the critical case $\lambda = \lambda^*$ (bifurcation point).

Proposition 3.12 *If hypotheses H(a) and H' hold, then* $\lambda^* \in \mathcal{L}$ *and so* $\mathcal{L} = (0, \lambda^*]$.

Proof Let $(\lambda_n)_{n\geq 1} \subseteq \mathcal{L}$ be a sequence such that $\lambda_n \nearrow \lambda^*$ as $n \to \infty$. Then we can find $u_n \in \mathcal{S}(\lambda_n)$ such that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \geq 1. \tag{3.68}$$

Since $u_n \in \mathcal{S}(\lambda_n)$, there holds

$$A(u_n) = N_{f_{\lambda_n}}(u_n) \quad \text{for all } n \geq 1. \tag{3.69}$$

From (3.68) and (3.69), as in the proof of Proposition 3.9, we obtain that

$$(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \rightharpoonup u_* \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^{r(\lambda^*)}(\Omega). \tag{3.70}$$

Acting on (3.69) with $u_n - u_* \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to \infty$, and using (3.70) (recall that $r(\lambda^*) \geq r(\lambda_n)$ for all $n \geq 1$, see H'(ii)), we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0,$$

which by the $(S)_+$-property of the operator A (see Proposition 2.7) results in

$$u_n \to u_* \text{ in } W_0^{1,p}(\Omega). \tag{3.71}$$

So, if we pass in (3.69) to the limit as $n \to \infty$ and apply (3.71), we get

$$A(u_*) = N_{f_{\lambda^*}}(u_*).$$

Additionally, Propositions 3.7 and 3.8 imply that

$$\tilde{u}_{\lambda_1} \leq \tilde{u}_{\lambda_n} \leq u_n \quad \text{for all } n \geq 1.$$

Therefore, $\tilde{u}_{\lambda_1} \leq u_*$. From this we see that $u_* \in \mathcal{S}(\lambda^*)$ and so $\lambda^* \in \mathcal{L}$, that is $\mathcal{L} = (0, \lambda^*]$. □

Next, we show the existence of a smallest positive solution to problem (P_λ) for every $\lambda \in \mathcal{L} = (0, \lambda^*]$

Proposition 3.13 *Let hypotheses H(a) and H' be satisfied and let $\lambda \in \mathcal{L} = (0, \lambda^*]$. Then, problem (P_λ) admits a smallest positive solution $\overline{u}_\lambda \in \mathcal{S}(\lambda) \subseteq \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$ and the map $\lambda \to \overline{u}_\lambda$ from $(0, \infty)$ into $C_0^1(\overline{\Omega})_+ \setminus \{0\}$ is increasing, that is, if $\lambda < \gamma$, then $\overline{u}_\gamma - \overline{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$.*

Proof As done in Filippakis–Kristály–Papageorgiou [10], due to the monotonicity of the operator A (see Proposition 2.7), we can check that $\mathcal{S}(\lambda)$ is downward directed, that is, if $u, \hat{u} \in \mathcal{S}(\lambda)$, then there exists $\tilde{u} \in \mathcal{S}(\lambda)$ such that $\tilde{u} \leq u$ and $\tilde{u} \leq \hat{u}$. Since we are looking for the smallest positive solution of problem (P_λ), we may assume, without loss of generality, that there exists $M_8 > 0$ such that

$$\|u\|_\infty \leq M_8 \quad \text{for all } u \in \mathcal{S}(\lambda). \tag{3.72}$$

From Dunford–Schwartz [9, p. 336] we know that there exists a sequence $(u_n)_{n\geq 1} \subseteq \mathcal{S}(\lambda)$ such that

$$\inf \mathcal{S}(\lambda) = \inf_{n\geq 1} u_n.$$

Moreover, since $u_n \in \mathcal{S}(\lambda)$, we have

$$A\left(u_n\right) = N_{f_\lambda}\left(u_n\right) \quad \text{for all } n \geq 1. \tag{3.73}$$

From (3.72) and (3.73) it follows that

$$(u_n)_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Then, as in the proof of Proposition 3.12, by applying Proposition 2.7, we have (for a subsequence if necessary) that

$$u_n \to \overline{u}_\lambda \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty.$$

Hence, (3.73) implies

$$A\left(\overline{u}_\lambda\right) = N_{f_\lambda}\left(\overline{u}_\lambda\right) \quad \text{for all } n \geq 1.$$

Moreover, due to Proposition 3.8, $\tilde{u}_\lambda \leq u_n$ for all $n \geq 1$, hence $\tilde{u}_\lambda \leq \overline{u}_\lambda$ and so $\overline{u}_\lambda \in \mathcal{S}(\lambda)$. Evidently, $\overline{u}_\lambda = \inf \mathcal{S}(\lambda)$.

Finally, if $\gamma \in (\lambda, \lambda^*]$, then, as in the proof of Proposition 3.11, we can prove the existence of

$$\overline{u}_\lambda \in \mathcal{S}(\lambda) \quad \text{such that} \quad \overline{u}_\lambda \in \operatorname{int}_{C_0^1(\overline{\Omega})}\left[0, \overline{u}_\gamma\right].$$

Thus, $\overline{u}_\gamma - \overline{u}_\lambda \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right)$. □

We can also prove a continuity property of the map $\lambda \to \overline{u}_\lambda$ from $(0, \lambda^*]$ into $C_0^1(\overline{\Omega})$.

Proposition 3.14 *If hypotheses H(a) and H' hold, then* $\lambda \to \overline{u}_\lambda$ *from* $(0, \lambda^*]$ *into* $C_0^1(\overline{\Omega})$ *is left continuous.*

Proof Let $(\lambda_n)_{n\geq 1} \subseteq \mathcal{L}$ be a sequence such that $\lambda_n \nearrow \lambda$ as $n \to \infty$. By means of Proposition 3.13 we know that $\left(\overline{u}_{\lambda_n}\right)_{n\geq 1}$ is increasing and $\overline{u}_{\lambda_n} \leq \overline{u}_\lambda$ for all $n \geq 1$. We have

$$A\left(\overline{u}_{\lambda_n}\right) = N_{f_{\lambda_n}}\left(\overline{u}_{\lambda_n}\right) \quad \text{for all } n \geq 1,$$

that is

$$\begin{aligned} -\operatorname{div} a\left(\nabla \overline{u}_{\lambda_n}\right) &= f\left(x, \overline{u}_{\lambda_n}, \lambda_n\right) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The regularity results of Lieberman [22] imply the existence of $\alpha \in (0,1)$ and $M_9 > 0$ such that

$$\overline{u}_{\lambda_n} \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \left\|\overline{u}_{\lambda_n}\right\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq M_9 \quad \text{for all } n \geq 1.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ gives, due to the monotonicity of the sequence $\left(\overline{u}_{\lambda_n}\right)_{n\geq 1}$,

$$\overline{u}_{\lambda_n} \nearrow \tilde{u}^* \text{ in } C_0^1(\overline{\Omega}), \quad \tilde{u}^* \in \mathcal{S}\left(\lambda^*\right). \tag{3.74}$$

Suppose that $\tilde{u}^*$ is not the minimal positive solution of problem (P_λ). Then we can find $x_0 \in \Omega$ such that

$$\overline{u}_\lambda(x_0) < \tilde{u}^*(x_0).$$

Moreover, taking into account (3.74), we find a number $n_0 \geq 1$ such that

$$\overline{u}_\lambda(x_0) < \overline{u}_{\lambda_n}(x_0) \quad \text{for all } n \geq n_0,$$

which is a contradiction to Proposition 3.13. Hence, $\tilde{u}^* = \overline{u}_\lambda$ and we have proved the desired continuity of $\lambda \to \overline{u}_\lambda$. □

Summarizing the situation for problem (P_λ), we can state the following bifurcation-type theorem.

Theorem 3.15 *If hypotheses H(a) and H' hold, then there exists $\lambda^* > 0$ such that*

(i) *for all $\lambda \in (0, \lambda^*)$, problem (P_λ) admits at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right), \quad u_0 \leq \hat{u}, \quad u_0 \neq \hat{u};$$

(ii) *for $\lambda = \lambda^*$, problem (P_λ) has at least one positive solution*

$$u_* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right);$$

(iii) *for all* $\lambda > \lambda^*$, *problem* (P_λ) *has no positive solution.*

Furthermore, for every $\lambda \in (0, \lambda^*]$, *problem* (P_λ) *has a smallest positive solution* $\overline{u}_\lambda \in \operatorname{int}\left(C^1_0(\overline{\Omega})_+\right)$ *and the map* $\lambda \to \overline{u}_\lambda$ *from* $(0, \lambda^*]$ *into* $C^1_0(\overline{\Omega})$ *is*

- *increasing, that is, if* $\lambda < \gamma$, *then* $\overline{u}_\gamma - \overline{u}_\lambda \in \operatorname{int}\left(C^1_0(\overline{\Omega})_+\right)$;
- $\lambda \to \overline{u}_\lambda$ *is left continuous, that is, if* $\lambda_n \nearrow \lambda$, *then* $\overline{u}_{\lambda_n} \to \overline{u}_\lambda$ *in* $C^1_0(\overline{\Omega})$.

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