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A TWO CRITICAL POINTS THEOREM FOR NON-DIFFERENTIABLE FUNCTIONS AND APPLICATIONS TO HIGHLY DISCONTINUOUS PDE'S

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Abstract. The aim of this paper is to develop an abstract two critical points result for general nonsmooth functions. Based on this, in particular, we are able to show the existence of at least two positive weak solutions for elliptic Dirichlet problems involving the $p-$Laplacian with discontinuous nonlinearities.

1. Introduction

In this paper we are interested in nontrivial weak solutions to discontinuous Dirichlet problems driven by the p-Laplacian. To be more precise, given a bounded domain $\Omega \subseteq \mathbb{R}^N$ with a C^1-boundary, we study the following equation

$$\begin{aligned} -\Delta_p u &= \lambda f(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $1 < p < N$ and the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is only measurable in the first argument, locally essentially bounded in the second argument and $s \to f(x,s)$ may be discontinuous for a.a. $x \in \Omega$.

Since $s \to f(x,s)$ is locally essentially bounded for a.a. $x \in \Omega$, the function

$$F(x,\xi) = \int_0^\xi f(x,t)dt$$

is locally Lipschitz in the second argument and so its generalized directional derivative $F^\circ(x,\cdot)$ as well as its generalized gradient $\partial F(x,\cdot)$ in the sense of Clarke exists, see Section 2 for detailed definitions. This means, that our problem can be written equivalently as a differential inclusion of the form

$$-\Delta_p u \in \lambda \partial F(\cdot,u) \quad \text{in } W^{-1,p'}(\Omega) \tag{1.2}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, where $W^{-1,p'}(\Omega)$ is the dual of the usual Sobolev space $W_0^{1,p}(\Omega)$.

In order to study problem (1.1), in this paper, we obtain some results on existence and multiplicity of critical points for functionals of type $\Phi - \lambda\Psi$, where Φ and Ψ are locally Lipschitz continuous. Our results can be seen as an extension of a paper of Bonanno-D'Aguì [5]. The arguments here are based on a paper of Bonanno-D'Aguì-Winkert [6].

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Let us comment on relevant works in case of discontinuous problems given in (1.1) and (1.2), respectively. In 1996, Bonanno-Marano [7] studied the existence of solutions of the semilinear equation

$$
\begin{aligned}
-\Delta u &= f(u) + h(x) && \text{in } \Omega,\\
u &> 0 && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned} \tag{1.3}
$$

where $\Omega \subseteq \mathbb{R}^n$ with $n \geq 3$, $h \in L^p(\Omega)$ with $p \in]\frac{n}{2}, \infty]$ and $f : \mathbb{R} \to \mathbb{R}$ is a function whose set of discontinuity points has Lebesgue measure zero. Based on arguments of set-valued analysis the authors prove the existence of at least one positive solution of equation (1.3). As seen above, discontinuous problems of the form (1.1) are in fact equivalent to differential inclusion given in (1.2). Carl-Heikkilä [8] considered such differential inclusion with nonmonotone discontinuous multifunctions given in the form

$$Au + f(\cdot, u) \in h(u)\partial j(\cdot, u) \quad \text{in } W^{-1,p'}(\Omega)$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, where A is a second-order quasilinear elliptic differential operator given in divergence form, the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in the first and locally Lipschitz continuous in the second variable and so its Clarke's gradient $\partial j(x, \cdot)$ exists. Furthermore, $h : \mathbb{R} \to \mathbb{R}$ is increasing, bounded, not necessarily continuous and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Since h is allowed to be discontinuous, the multi-valued function $s \to h(s)\partial j(\cdot, s)$ is neither monotone nor continuous. Their existence result is based on a combined use of abstract fixed point results for monotone mappings on partially ordered sets and on the existence and comparison results for multi-valued quasilinear elliptic problems with Clarke's generalized gradient. We also mention a multiplicity result for differential inclusion with nonlinear boundary condition in terms of Clarke's gradient published by Winkert [19] by applying the method of sub- and supersolution and suitable nonlinear methods for nonsmooth functionals. Entire extremal solutions for multivalued quasilinear elliptic problems of hemivariational type in all of $\mathbb{R}^N$ given by

$$-\Delta_p u + \partial j(\cdot, u) \ni 0 \quad \text{in } \mathcal{D}'$$

with $\mathcal{D} = C_0^\infty(\mathbb{R}^N)$ were obtained by Winkert in [18] by applying the method of sub- and supersolution without imposing any condition at infinity.

For $p = 2$, applying a coincidence result based on the Ky Fan's fixed point theorem, an interesting existence result for problem (1.2) is given in Bonanno-Candito-Motreanu [4]. Here, to establish the existence of at least two weak solutions for problem (1.2) we apply an abstract critical point result (Theorem 2.10), see also the papers of Marano-Motreanu [13], [14], Bonanno-Candito [3] and Bonanno [1] for nonsmooth functionals.

The paper is organized as follows. In Section 2, we present new results concerning the existence of critical points of nonsmooth functions of the form

$$I_\lambda = \Phi - \lambda\Psi$$

with locally Lipschitz continuous functionals $\Phi, \Psi : X \to \mathbb{R}$ and a parameter $\lambda > 0$ to be specified. The main theorem is stated as Theorem 2.10 and guarantees the existence of at least two nontrivial critical points of the functional I_λ. Such a result is very strong and of independent interest. In Section 3, we are going to apply Theorem 2.10 to our original problem (1.1) and we obtain the existence of at least two positive weak solutions of (1.1) under further conditions on the nonlinearity. We can give a precise interval for the parameter $\lambda > 0$ for which these solutions exist. Furthermore, we present a useful corollary which gives easy conditions to apply our results.

2. Critical points for non-differentiable functions

Let us start by recalling some basic notions in nonsmooth analysis that are required in the sequel. For a real Banach space $(X, \|\cdot\|_X)$, we denote by X^* its dual space and by $\langle \cdot, \cdot \rangle$ the duality pairing between X and X^*. A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exist a neighborhood U_x of x and a constant $L_x \geq 0$ such that

$$|f(y) - f(z)| \leq L_x \|y - z\|_X \quad \text{for all } y, z \in U_x.$$

For a locally Lipschitz function $f : X \to \mathbb{R}$ on a Banach space X, the generalized directional derivative of f at the point $x \in X$ along the direction $y \in X$ is defined by

$$f^\circ(x; y) := \limsup_{z \to x, t \to 0^+} \frac{f(z + ty) - f(z)}{t},$$

see Clarke [10, Chapter 2]. Note that if $f : X \to \mathbb{R}$ is strictly differentiable, that is, for all $x \in X$, $f'(x) \in X^*$ exists such that

$$\lim_{\substack{z \to x \\ t \to 0^+}} \frac{f(z + ty) - f(z)}{t} = \langle f'(x), y \rangle \quad \text{for all } y \in X,$$

then the usual directional derivative $f'(x; y)$ given by

$$f'(x; y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}$$

exists and coincides with the generalized directional derivative $f^\circ(x; y)$.

If f_1, $f_2 : X \to \mathbb{R}$ are locally Lipschitz functions, then we have

$$(f_1 + f_2)^\circ(x; y) \leq f_1^\circ(x; y) + f_2^\circ(x; y) \quad \text{for all } x, y \in X.$$

The generalized gradient of a locally Lipschitz function $f : X \to \mathbb{R}$ at $x \in X$ is the set

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y \rangle \leq f^\circ(x; y) \quad \text{for all } y \in X\}.$$

Based on the Hahn-Banach theorem we easily verify that $\partial f(x)$ is nonempty. An element $x \in X$ is said to be a critical point of a locally Lipschitz function $f : X \to \mathbb{R}$ if there holds

$$f^\circ(x; y) \geq 0 \quad \text{for all } y \in X$$

or, equivalently, $0 \in \partial f(x)$, see Chang [9].

Let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions. We put

$$I = \Phi - \Psi.$$

We further fix two numbers $r_1, r_2 \in [-\infty, +\infty]$ such that $r_1 < r_2$. The following definition is a special version of the Palais-Smale condition ((PS) for short).

Definition 2.1. We say that the function $I : X \to \mathbb{R}$ fulfills the Palais-Smale condition cut off lower at r_1 and upper at r_2 (${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition for short) if any sequence $(u_n) \subseteq X$ satisfying

(1) $I(u_n)$ is bounded;

(2) there exists a sequence $(\varepsilon_n) \subset \mathbb{R}_+, \varepsilon_n \to 0^+$ such that

$$I^\circ(u_n; v) \geq -\varepsilon_n \|v\|_X \quad \text{for all } v \in X;$$

(3) $r_1 < \Phi(u_n) < r_2$ for all $n \in \mathbb{N}$;

has a convergent subsequence. If $r_1 = -\infty$, $r_2 \in \mathbb{R}$, we write $(\mathrm{PS})^{[r_2]}$ and the case $r_1 \in \mathbb{R}$, $r_2 = +\infty$ will be denoted by ${}^{[r_1]}(\mathrm{PS})$.

It is easy to see that if $r_1 = -\infty$ and $r_2 = +\infty$, the definition above reduces to the well-known (PS)-condition for locally Lipschitz continuous functions, see Motreanu-Rădulescu [15, Definition 1.7]. We should also mention that if I fulfills the ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition, then it satisfies the ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition for all $s_1, s_2 \in [-\infty, +\infty]$ such that $r_1 \leq s_1 < s_2 \leq r_2$. Particularly, if I satisfies the usual (PS)-condition for locally Lipschitz continuous functions, then it fulfills the ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition for all $s_1, s_2 \in [-\infty, +\infty]$ with $s_1 < s_2$.

The following result was proved by the authors in [6, Theorem 2.3].

Theorem 2.2. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions. Put*

$$I = \Phi - \Psi$$

and assume that there exist $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 < \Phi(x_0) < r_2$ such that

$$\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0),$$

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \tag{2.1}$$

Furthermore, suppose that I satisfies the ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition.

Then, there exists a critical point u_0 of I such that $u_0 \in \Phi^{-1}(]r_1, r_2[)$ and $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$.

As a consequence we can state the following result.

Theorem 2.3. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions with Φ being bounded from below. Put*

$$I = \Phi - \Psi$$

and assume that there exist $x_0 \in X$ and $r \in \mathbb{R}$ satisfying $\Phi(x_0) < r$ such that

$$\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) \leq r - \Phi(x_0) + \Psi(x_0).$$

Furthermore, suppose that I satisfies the $(\mathrm{PS})^{[r]}$-condition.

Then, there exists $u_0 \in \Phi^{-1}(]-\infty, r[)$ *such that* $I(u_0) \leq I(u)$ *for all* $u \in \Phi^{-1}(]-\infty, r[)$ *with* u_0 *being a critical point of* I.

Proof. The assertion of the theorem follows directly from Theorem 2.2 by setting $r_1 \in \mathbb{R}$ such that $r_1 < \inf_X \Phi$ and $r_2 = r$. In this case (2.1) is verified (see also the proof of [6, Theorem 2.3]). Here we use the convention $\sup_{\emptyset} \Psi = -\infty$. □

For a real Banach space X and locally Lipschitz continuous functions $\Phi, \Psi : X \to \mathbb{R}$ we define

$$I_\lambda = \Phi - \lambda\Psi$$

with $\lambda > 0$. Moreover, we put

$$\beta(r) = \inf_{v \in \Phi^{-1}(]-\infty, r[)} \frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)}$$

for all $r \in \mathbb{R}$ and

$$\rho(r) = \sup_{v \in \Phi^{-1}(]0, r[)} \frac{\Psi(v)}{\Phi(v)}$$

for all $r \in \mathbb{R}$.

Based on this notation, we can give a direct consequence of Theorem 2.3.

Theorem 2.4. *Let* X *be a real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two locally Lipschitz continuous functionals with* Φ *bounded from below. Fix* $r > \inf_X \Phi$ *such that* $\sup_{\Phi^{-1}(]-\infty, r[)} \Psi(u) < +\infty$ *and assume that, for each*

$$\lambda \in \left]0, \frac{1}{\beta(r)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ *satisfies the* $(\mathrm{PS})^{[r]}$*-condition. Then, for each*

$$\lambda \in \left]0, \frac{1}{\beta(r)}\right[,$$

there exists $u_\lambda \in \Phi^{-1}(]-\infty, r[)$ *such that* $I_\lambda(u_\lambda) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]-\infty, r[)$ *and* u_λ *is a critical point of* I_λ.

Proof. From $\beta(r) < \dfrac{1}{\lambda}$ follows that there is $\bar{v} \in \Phi^{-1}(]-\infty, r[)$ such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(\bar{v})}{r - \Phi(\bar{v})} < \frac{1}{\lambda},$$

that is, $\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} (\lambda\Psi)(u) < r - \Phi(\bar{v}) + (\lambda\Psi)(\bar{v})$. So, the assertion follows from Theorem 2.3 applied to the functional $\Phi - \lambda\Psi$. □

Remark 2.5. If we assume that $\sup_{\Phi^{-1}(]-\infty, r[)} \Psi(u) < +\infty$ for all $r > \inf_X \Phi$ and if $\lambda^* := \sup\limits_{r > \inf_X \Phi} \dfrac{1}{\beta(r)}$, then the conclusion of Theorem 2.4 holds for all $\lambda \in]0, \lambda^*[$.

Remark 2.6. Of course, if $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$, the conclusion of the Theorem 2.4 holds in particular for all

$$\lambda \in \left]0, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$$

with $r > 0$ such that $\sup_{\Phi^{-1}(]-\infty,r[)} \Psi(u) < +\infty$.

Recently, the authors state the following result, see [6, Theorem 2.5], which will be useful in later considerations. Note that the following version is slightly different.

Theorem 2.7. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions. Assume that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$ *and suppose that there exist $r > 0$ such that*

$$\beta(r) < \rho(r) \tag{2.2}$$

and for each $\lambda \in \Lambda^r := \left]\frac{1}{\rho(r)}, \frac{1}{\beta(r)}\right[$ *the function $I_\lambda = \Phi - \lambda\Psi$ fulfills the* $(\mathrm{PS})^{[r]}$*-condition.*

Then, for each $\lambda \in \Lambda^r$ there exists $u_\lambda \in \Phi^{-1}(]0, r[)$ (that is, $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ with u_λ being a critical point of I_λ.

Proof. This follows directly from Theorem 2.4 of [6] by setting $r_1 = 0$ and $r_2 = r$. For completeness, we give a proof that follows from Theorem 2.4. Indeed, fix λ such that $\beta(r) < \frac{1}{\lambda} < \rho(r)$. From Theorem 2.4 there is $u_\lambda \in \Phi^{-1}(]-\infty, r[)$ such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty, r[)$. Arguing by a contradiction, assume $u_\lambda = 0$ so that $0 < \Phi(u) - \lambda\Psi(u)$ for all $u \in \Phi^{-1}(]0, r[)$. It follows $\frac{\Psi(u)}{\Phi(u)} < \frac{1}{\lambda}$ for all $u \in \Phi^{-1}(]0, r[)$ for which $\rho(r) \leq \frac{1}{\lambda}$, that is against our assumption. $\square$

Remark 2.8. If there are $r > 0$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\beta(r) < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},$$

therefore condition (2.2) is satisfied (since $\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \leq \rho(r)$), for which the conclusion of Theorem 2.7 holds, in particular, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{1}{\beta(r)}\right[$. In turn, by assuming that there are $r > 0$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},$$

again the conclusion of Theorem 2.7 holds (indeed, $\beta(r) \leq \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r}$ by choosing $v = 0$). So, I_λ admits a non-zero local minimum, in particular, for each

$$\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[.$$

Now, we point out the following consequence of Theorem 2.7. To this end, put

$$\lambda^* = \frac{1}{\inf_{r>0}\beta(r)}; \qquad \lambda_* = \frac{1}{\sup_{r>0}\rho(r)}.$$

Further, we recall that a functional $I : X \to \mathbb{R}$ fulfills the weak Palais-Smale condition (WPS-condition for short) if any bounded sequence $(u_n) \subseteq X$ satisfying (1) and (2) of Definition 2.1 has a convergent subsequence.

Corollary 2.9. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$, with Φ coercive. Assume that*

$$\lambda_* < \lambda^*$$

and for each $\lambda \in \Lambda^* :=]\lambda_*, \lambda^*[$ the function $I_\lambda = \Phi - \lambda\Psi$ fulfills the (WPS)-condition.

Then, for each $\lambda \in \Lambda^*$ there exist $r > 0$ and $u_\lambda \in \Phi^{-1}(]0, r[)$ (that is, $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ with u_λ being a critical point of I_λ.

Proof. Fix $\lambda \in \Lambda^*$. From $\lambda < \lambda^* = \frac{1}{\inf_{r>0}\beta(r)}$ one has $\inf_{r>0}\beta(r) < \frac{1}{\lambda}$ for which there is $\bar{r} > 0$ such that

$$\beta(\bar{r}) < \frac{1}{\lambda}.$$

Now, since $r \to \rho(r)$ is a nondecreasing function, one has $\lim_{r\to 0^+}\dfrac{1}{\rho(r)} = \dfrac{1}{\sup_{r>0}\rho(r)} = \lambda_* < \lambda$ for which there is $r^* > 0$ such that $\frac{1}{\lambda} < \rho(r)$ for all $r \in]0, r^*[$. Fixed a positive number $\tilde{r} < \min\{\bar{r}; r^*\}$, one has $\frac{1}{\lambda} < \rho(\tilde{r})$ and $\rho(\tilde{r}) \leq \rho(\bar{r})$, for which one has

$$\frac{1}{\lambda} < \rho(\bar{r}).$$

Hence, one has $\beta(\bar{r}) < \frac{1}{\lambda} < \rho(\bar{r})$. So, taking also into account that $(PS)^{[\bar{r}]}$-condition is satisfied since Φ is coercive and I_λ fulfills the (WPS)-condition, Theorem 2.7 ensures the conclusion. □

The following result gives us two nontrivial critical points.

Theorem 2.10. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Suppose that there exist $r \in \mathbb{R}$ and $\hat{u} \in X$ with $0 < \Phi(\hat{u}) < r$ such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\hat{u})}{\Phi(\hat{u})} \tag{2.3}$$

and for each $\lambda \in \Lambda^{r,\hat{u}} := \left]\frac{\Phi(\hat{u})}{\Psi(\hat{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$ the functional $I_\lambda = \Phi - \lambda\Psi$ fulfills the (PS)*-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda^{r,\hat{u}}$, the functional I_λ admits at least two nontrivial critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Proof. We fix λ as in the conclusion of the theorem. First we mention that the (PS)-condition implies the $(\mathrm{PS})^{[r]}$-condition, see Bonanno [1]. Moreover, inequality (2.3) ensures that condition (2.2) holds. From Theorem 2.7 (see also Remark 2.8) follows the existence of $u_{\lambda,1} \in \Phi^{-1}(]0,r[)$ such that $I_\lambda(u_{\lambda,1}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0,r[)$ with $u_{\lambda,1}$ being a critical point of I_λ. In particular, $u_{\lambda,1} \neq 0$.

Claim 1: $I_\lambda(u_{\lambda,1}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty, r])$ and $I_\lambda(u_{\lambda,1}) < 0$.

Since $\lambda > \frac{\Psi(\hat{u})}{\Phi(\hat{u})}$ it follows

$$\Phi(\hat{u}) - \lambda\Psi(\hat{u}) < 0 = \Phi(0) - \lambda\Psi(0),$$

that is

$$I_\lambda(u_{\lambda,1}) \leq I_\lambda(\hat{u}) < I_\lambda(0) = 0.$$

So, $I_\lambda(u_{\lambda,1}) < 0$ and $I_\lambda(u_{\lambda,1}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}([0,r[) = \Phi^{-1}(]-\infty,r[)$. In addition, since

$$\lambda < \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)},$$

we have, for all $\tilde{u} \in X$ satisfying $\Phi(\tilde{u}) = r$, that

$$\Phi(\tilde{u}) - \lambda\Psi(\tilde{u}) \geq \Phi(\tilde{u}) - \lambda \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) > \Phi(\tilde{u}) - r = 0,$$

that is $I_\lambda(\tilde{u}) > I_\lambda(0) > I_\lambda(u_{\lambda,1})$. This proves the Claim 1.

As the functional I_λ is unbounded from below, we find an element $\tilde{u}_{\lambda,2} \in X$ such that $I_\lambda(\tilde{u}_{\lambda,2}) < I_\lambda(u_{\lambda,1})$. Moreover, as $u_{\lambda,1}$ is a global minimum of I_λ on $\Phi^{-1}(]-\infty,r])$ we obviously obtain $\Phi(\tilde{u}_{\lambda,2}) > r$.

Now we are in the position to apply the nonsmooth Mountain Pass Theorem, see Chang [9], which gives us an element $u_{\lambda,2} \in X$ being a critical point of I_λ with corresponding critical value

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = u_{\lambda,1}, \gamma(1) = \tilde{u}_{\lambda,2}\}.$$

Claim 2: $I_\lambda(u_{\lambda,2}) > 0$

First, we put $L = r - \lambda \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)$. Because of

$$\lambda < \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)},$$

we easily see that $L > 0$. Let $\gamma \in \Gamma$. As $\Phi(\gamma(0)) < r$ and $\Phi(\gamma(1)) > r$, we find $\tilde{t} \in]0,1[$ such that $\Phi(\gamma(\tilde{t})) = r$. For $\tilde{u} = \gamma(\tilde{t})$ we derive

$$\Phi(\tilde{u}) - \lambda\Psi(\tilde{u}) \geq L.$$

This gives $I_\lambda(\gamma(\tilde{t})) \geq L$. We conclude that

$$\max_{t\in[0,1]} I_\lambda(\gamma(t)) \geq L \quad \text{for each } \gamma \in \Gamma.$$

This finally yields

$$I_\lambda(u_{\lambda,2}) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \geq L > 0.$$

That proves Claim 2 and the assertion of the theorem follows as well. □

Remark 2.11. Condition (2.3) is equivalent to assume

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}<\rho(r)$$

for some $r>0$, for which the interval of parameters, in this case, becomes

$$\left]\frac{1}{\rho(r)},\frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[.$$

Hence, setting $\bar{\lambda}=\frac{1}{\inf_{r>0}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}}$ and arguing as in the proof of Corollary 2.9, simple computations show that the conclusion of Theorem 2.10 is true for each $\lambda\in]\lambda_*,\bar{\lambda}[$ by assuming

$$\lambda_*<\bar{\lambda}.$$

Remark 2.12. In order to use the mountain pass theorem we have to suppose the much stronger (PS)-condition instead of the $(\mathrm{PS})^{[r]}$-condition, see also Theorem 2.4.

Now we give a version of Bonanno [2, Theorem 2.1] for locally Lipschitz continuous functionals with some slightly different assumptions to obtain two different critical points, one is possibly zero.

Theorem 2.13. *Let X be a real Banach space and let $\Phi,\Psi:X\to\mathbb{R}$ be two locally Lipschitz continuous functionals with Φ bounded from below and $\inf_X\Phi=\Phi(0)=\Psi(0)=0$. Fix $r>0$ such that $\sup_{\Phi^{-1}(]-\infty,r[)}\Psi(u)<+\infty$ and assume that, for each*

$$\lambda\in\left]0,\frac{r}{\sup_{\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[,$$

the functional $I_\lambda=\Phi-\lambda\Psi$ satisfies the (PS)*-condition and is unbounded from below. Then, for each*

$$\lambda\in\left]0,\frac{r}{\sup_{\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[,$$

there exists $u_{\lambda,1}\in\Phi^{-1}(]-\infty,r[)$ such that $I_\lambda(u_{\lambda,1})\le I_\lambda(u)$ for all $u\in\phi^{-1}(]-\infty,r[)$ and $u_{\lambda,1}$ is a critical point of I_λ. Moreover, there exists a second critical point $u_{\lambda,2}$ of I_λ.

Proof. Fix λ as asserted. As before, we mention that the (PS)-condition implies the $(\mathrm{PS})^{[r]}$-condition and so the assumptions of Theorem 2.4 (see also Remark 2.6) are satisfied. This gives us an element $u_{\lambda,1}\in\Phi^{(-1)}(]-\infty,r[)$ such that $I_\lambda(u_{\lambda,1})\le I_\lambda(u)$ for all $u\in\phi^{-1}(]-\infty,r[)$ being a critical point of I_λ.

Since the functional I_λ is unbounded from below, we find an element $\tilde{u}_{\lambda,2}\in X$ such that $I_\lambda(\tilde{u}_{\lambda,2})<I_\lambda(u_{\lambda,1})$.

The nonsmooth mountain pass theorem, see Chang [9], implies the existence of an element $u_{\lambda,2}\in X$ being a critical point of I_λ with corresponding critical value

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\lambda(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_{\lambda,1}, \gamma(1) = \tilde{u}_{\lambda,2}\}.$$

$\square$

Remark 2.14. If we assume that $\sup_{\Phi^{-1}(]-\infty,r[)} \Psi(u) < +\infty$ for all $r > \inf_X \Phi$ and if $\lambda^* := \sup_{r>\inf_X \Phi} \dfrac{1}{\beta(r)}$, then the conclusion of Theorem 2.13 holds for all $\lambda \in]0, \lambda^*[$ (see Remark 2.5).

3. A discontinuous Dirichlet problem

In this section, we are interested in applying the results of Section 2 to problem given in (1.1). We assume the following conditions on the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

H(f): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is nonnegative, belongs to the class $\mathcal{H}$ and there exist $s \in [1, p[, q \in]p, p^*[$ and two positive constants a_s and a_q such that

$$f(x,t) \leq a_s|t|^{s-1} + a_q|t|^{q-1} \tag{3.1}$$

for a.a. $x \in \Omega$ and for all $t \geq 0$, where $p^* = \frac{Np}{N-p}$ denotes the critical exponent of p.

Definition 3.1. We say that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{H}$ if $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$, there exists a set $A \subset \Omega$ with $|A|_N = 0$ such that the set

$$D_f := \bigcup_{x \in \Omega \setminus A} \{t \in \mathbb{R} : f(x,\cdot) \text{ is discontinuous at } t\}$$

has measure zero, $s \to f(x,s)$ is locally essentially bounded for a.a. $x \in \Omega$, and the functions

$$f^-(x,t) := \lim_{\delta \to 0^+} \operatorname*{essinf}_{|t-z|<\delta} f(x,z), \qquad f^+(x,t) := \lim_{\delta \to 0^+} \operatorname*{esssup}_{|t-z|<\delta} f(x,z),$$

are superpositionally measurable, that is, $f^-(x,u(x))$ and $f^+(x,u(x))$ are measurable for all measurable functions $u : \Omega \to \mathbb{R}$. Functions belonging to $\mathcal{H}$ are sometimes called highly discontinuous.

Without loss of generality we can suppose that $f(t) = f(0) = 0$ for all $t \leq 0$. We set $X = W_0^{1,p}(\Omega)$ equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

It is well known that we have the continuous embedding

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\| \quad \text{for all } u \in X, \tag{3.2}$$

where the constant C, given by

$$C = \frac{1}{\sqrt{\pi}} \frac{1}{N^{\frac{1}{p}}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(1+N-\frac{N}{p}\right)}\right)^{\frac{1}{N}},$$

is the best constant, see Talenti [17], and Γ stands for the Gamma function. By Hölder's inequality and (3.2) we obtain

$$\|u\|_{L^{\hat{q}}(\Omega)} = \left(\int_\Omega |u|^{\hat{q}}dx\right)^{\frac{1}{\hat{q}}} \leq |\Omega|_N^{\frac{p^*-\hat{q}}{\hat{q}p^*}} \|u\|_{L^{p^*}(\Omega)} \leq |\Omega|_N^{\frac{p^*-\hat{q}}{\hat{q}p^*}} C\|u\| \tag{3.3}$$

for all $u \in X$ and for all $\hat{q} \in [1, p^*[$, where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. Setting $F(x,\xi) = \int_0^\xi f(x,t)dt$ for a.a. $x \in \Omega$ and for all $\xi \in \mathbb{R}$, we define

$$\Phi(u) = \frac{\|u\|^p}{p}, \quad \Psi(u) = \int_\Omega F(x,u)dx \quad \text{and} \quad I_\lambda = \Phi(u) - \lambda\Psi(u)$$

for all $u \in X$ and for $\lambda > 0$. In addition, we set

$$R(x) = \sup\{\delta : B(x,\delta) \subseteq \Omega\}$$

for all $x \in \Omega$ and $R = \sup_{x\in\Omega} R(x)$ for which there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$. Furthermore, for positive constants γ and δ, we put

$$\begin{aligned} K &= \frac{R^p}{p^{p-1}} \frac{(p-1)^N}{p^N - (p-1)^N} \frac{1}{pC^p|\Omega|_N^{\frac{p}{N}}}, \\ K_\delta &= \frac{1}{K} \frac{1}{pC^p|\Omega|_N^{\frac{p}{N}}} \frac{\delta^p}{\operatorname{essinf}_{x\in\Omega} F(x,\delta)}, \\ K_\gamma &= \frac{1}{pC^p|\Omega|_N^{\frac{p}{N}}} \frac{1}{\frac{a_s}{s}\gamma^{s-p} + \frac{a_q}{q}\gamma^{q-p}}. \end{aligned} \tag{3.4}$$

The main result in this section reads as follows.

Theorem 3.2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying hypothesis H(f), assume that there are two constants γ and δ with $\delta < \gamma$ such that*

$$\frac{a_s}{s}\gamma^{s-p} + \frac{a_q}{q}\gamma^{q-p} < K\frac{\operatorname{essinf}_{x\in\Omega} F(x,\delta)}{\delta^p} \tag{3.5}$$

and suppose there exist two constants $m > p$ and $l > 0$ such that

$$0 < mF(x,\xi) \leq \xi f(x,\xi) \quad \text{for a.a. } x \in \Omega \text{ and for all } \xi \geq l. \tag{3.6}$$

Further, assume that

$$f^-(x,s) = 0 \qquad \text{implies} \qquad f(x,s) = 0 \tag{3.7}$$

for a.a $x \in \Omega$ and for all $s \in D_f$.

Then, for each $\lambda \in]K_\delta, K_\gamma[$, problem (1.1) *admits at least two positive weak solutions.*

Proof. Let $\lambda \in]K_\delta, K_\gamma[$ be fixed. From (3.4) and (3.5) we easily see that the interval $]K_\delta, K_\gamma[$ is nonempty. We want to apply Theorem 2.10. First, we mention that the Ambrosetti-Rabinowitz condition stated in (3.6) implies that the functional I_λ is unbounded from below and satisfies the Palais-Smale condition, see for example

Rabinowitz [16]. So, we only need to show that inequality (2.3) is satisfied. To this end, put

$$r = \frac{|\Omega|_N^{\frac{p}{p^*}}}{pC^p}\gamma^p \tag{3.8}$$

and note that the growth condition in (3.1) implies

$$F(x,t) \leq \frac{a_s}{s}|t|^s + \frac{a_q}{q}|t|^q \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R}. \tag{3.9}$$

Taking into account (3.3), (3.8) and (3.9), we have

$$\begin{aligned}
&\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \\
&\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \left(\frac{a_s}{s}\|u\|^s_{L^s(\Omega)} + \frac{a_q}{q}\|u\|^q_{L^q(\Omega)}\right)}{r} \\
&\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \left(\frac{a_s}{s}C^s|\Omega|_N^{\frac{p^*-s}{p^*}}\|u\|^s + \frac{a_q}{q}C^q|\Omega|_N^{\frac{p^*-q}{p^*}}\|u\|^q\right)}{r} \\
&\leq \frac{\left(\frac{a_s}{s}C^s|\Omega|_N^{\frac{p^*-s}{p^*}}(pr)^{\frac{s}{p}} + \frac{a_q}{q}C^q|\Omega|_N^{\frac{p^*-q}{p^*}}(pr)^{\frac{q}{p}}\right)}{r} \\
&= pC^p|\Omega|^{\frac{p^*-p}{p^*}}\left(\frac{a_s}{s}\left(\frac{pC^pr}{|\Omega|_N^{\frac{p}{p^*}}}\right)^{\frac{s-p}{p}} + \frac{a_q}{q}\left(\frac{pC^pr}{|\Omega|_N^{\frac{p}{p^*}}}\right)^{\frac{q-p}{p}}\right) \\
&= pC^p|\Omega|_N^{\frac{p}{N}}\left(\frac{a_s}{s}\gamma^{s-p} + \frac{a_q}{q}\gamma^{q-p}\right) \\
&= \frac{1}{K_\gamma},
\end{aligned}$$

where

$$\gamma = \left(\frac{pC^pr}{|\Omega|_N^{\frac{p}{p^*}}}\right)^{\frac{1}{p}}.$$

This implies that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{1}{\lambda}. \tag{3.10}$$

In order to prove the other inequality, let

$$v_\delta(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,R), \\ \frac{p\delta}{R}(R - |x-x_0|) & \text{if } x \in B(x_0,R) \setminus B\left(x_0, \frac{p-1}{p}R\right), \\ \delta & \text{if } x \in B\left(x_0, \frac{p-1}{p}R\right). \end{cases}$$

We easily see that $v_\delta \in X$. Moreover, one has

$$\begin{aligned}\Phi(v_\delta) &= \frac{1}{p}\int_{B(x_0,R)\setminus B\left(x_0,\frac{p-1}{p}R\right)} \frac{(p\delta)^p}{R^p}dx \\ &= \frac{1}{p}\frac{(p\delta)^p}{R^p}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(R^N - \left(\frac{p-1}{p}R\right)^N\right) \\ &= \frac{p^{p-1}\left(p^N-(p-1)^N\right)}{p^N}R^{N-p}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\delta^p\end{aligned} \tag{3.11}$$

and

$$\Psi(v_\delta) \geq \int_{B\left(x_0,\frac{p-1}{p}R\right)} F(x,\delta)dx = \operatorname*{essinf}_{x\in\Omega} F(x,\delta)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\frac{(p-1)^N R^N}{p^N}.$$

Combining these estimates yields

$$\begin{aligned}\frac{\Psi(v_\delta)}{\Phi(v_\delta)} &\geq \frac{R^p}{p^{p-1}}\frac{(p-1)^N}{p^N-(p-1)^N}\frac{\operatorname{essinf}_{x\in\Omega} F(x,\delta)}{\delta^p} \\ &= pC^p|\Omega|_N^{\frac{p}{N}} K\frac{\operatorname{essinf}_{x\in\Omega} F(x,\delta)}{\delta^p} = \frac{1}{K_\delta} > \frac{1}{\lambda}.\end{aligned} \tag{3.12}$$

From (3.10) and (3.12) we obtain

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(v_\delta)}{\Phi(v_\delta)}.$$

We only need to show that $\Phi(v_\delta) < r$. Setting

$$\hat{k} = \left(\frac{p^{p-1}\left(p^N-(p-1)^N\right)}{p^N}R^{N-p}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\frac{pC^p}{|\Omega|_N^{\frac{p}{p^*}}}\right)^{\frac{1}{p}},$$

we get from (3.11) that

$$\Phi(v_\delta) = \hat{k}^p\frac{|\Omega|_N^{\frac{p}{p^*}}}{pC^p}\delta^p.$$

Recall that $\delta < \gamma$ we are going to show that $\hat{k}\delta < \gamma$.
First, we observe that

$$\begin{aligned}\frac{1}{\tilde{k}^p} &= \frac{p^N}{p^{p-1}\left(p^N-(p-1)^N\right)}\frac{1}{R^{N-p}}\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\frac{|\Omega|_N^{\frac{p}{p^*}}}{pC^p} \\ &= \frac{R^p}{p^{p-1}}\frac{(p-1)^N}{p^N-(p-1)^N}\frac{1}{pC^p|\Omega|_N^{\frac{p}{N}}}\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\frac{p^N}{(p-1)^N R^N}|\Omega|_N \\ &= \frac{|\Omega|_N}{\left|B\left(x_0,\frac{p-1}{p}R\right)\right|}K \geq K.\end{aligned} \tag{3.13}$$

Now, we apply (3.5) in combination with the growth condition in (3.9) to obtain

$$\frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{\gamma^p} < K \frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^p}. \tag{3.14}$$

Arguing by contradiction and assume that $\hat{k}\delta \geq \gamma$. Then, this fact together with $\delta < \gamma$ and (3.13) gives

$$\frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{\gamma^p} \geq \frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{\hat{k}^p\delta^p} \geq \frac{1}{\hat{k}^p}\frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^p} \geq K \frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^p},$$

which is a contradiction to (3.14). Hence, $\hat{k}\delta < \gamma$ and this implies $\Phi(v_\delta) < r$.

Now, we are in the position to apply Theorem 2.10 which says that I_λ admits two non-zero critical points u_1, u_2. We claim that u_1, u_2 are two weak solutions of our problem. To this end, denote by u^* a critical point of I_λ in X. This means $(\Phi - \lambda\Psi)^\circ(u^*; w) \geq 0$ for all $w \in X$, for which one has $\Phi^{'}(u^*)(w) - \lambda(\Psi)^\circ(u^*; w) \geq 0$ for all $w \in X$, that is $-\int_\Omega |\nabla u^*|^{p-2}\nabla u^* \cdot \nabla w \, dx \leq -\lambda(\Psi)^\circ(u^*; w)$ for all $w \in X$. Defining

$$T^*(w) = -\int_\Omega |\nabla u^*|^{p-2}\nabla u^* \cdot \nabla w \, dx \tag{3.15}$$

for all $w \in X$, one has that T^* is a linear and continuous operator on X such that $T^* \in \lambda\partial(-\Psi)(u^*)$. Taking [9, Theorem 2.2] into account for which $\partial(-\Psi)|_X(u^*) \subseteq \partial(-\Psi)|_{L^p(\Omega)}(u^*)$, one has that T^* is a linear and continuous operator on $L^p(\Omega)$. Thus, there is $\tilde{w} \in L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, such that $T^*(w) = \int_\Omega w(x)\tilde{w}(x) \, dx$ for all $w \in L^p(\Omega)$.

Now, denoting $\tilde{u} \in W^{2,p} \cap X$ (actually, by classical regularity argument, $\tilde{u}$ belongs to $C_0^{1,\beta}$ with $0 < \beta < 1$) the unique solution of the linear problem

$$\begin{aligned} -\Delta_p u &= \tilde{w}(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.16}$$

one has $-\int_\Omega |\nabla \tilde{u}|^{p-2}\nabla\tilde{u} \cdot \nabla w \, dx = \int_\Omega \tilde{w}(x)w(x) \, dx$ for all $w \in X$, that is

$$T^*(w) = -\int_\Omega |\nabla \tilde{u}|^{p-2}\nabla\tilde{u} \cdot \nabla w \, dx \tag{3.17}$$

for all $w \in X$. Since a linear continuous operator on X is uniquely determined (see for instance [12, Theorem 5.9.3]), from (3.15) and (3.17) follows $\tilde{u} = u^*$ which gives $u^* \in W^{2,p} \cap X$ and

$$-\int_\Omega |\nabla u^*|^{p-2}\nabla u^* \cdot \nabla w dx = \int_\Omega \tilde{w} w dx \quad \text{for all } w \in X. \tag{3.18}$$

Moreover, from [9, Theorem 2.2] one has

$$\tilde{w}(x) \in \lambda\left[f^-(x, u^*(x)), f^+(x, u^*(x))\right] \tag{3.19}$$

for a.a. $x \in \Omega$. Clearly, it follows $\tilde{w}(x) = \lambda f(x, u^*(x))$ for a.a. $x \in \Omega \setminus {u^*}^{-1}(D_f)$. Since $|D_f| = 0$, from [11] one has $-\Delta_p u^*(x) = 0$ for a.a. $x \in {u^*}^{-1}(D_f)$ so that, being u^* the unique solution of the above linear problem (3.16), one has $\tilde{w}(x) = 0$ for a.a. $x \in {u^*}^{-1}(D_f)$. Now, we observe that $f^-(x, u^*(x)) = 0$ for a.a. $x \in {u^*}^{-1}(D_f)$

since, on the contrary, from (3.19) one has $\tilde{w}(x) > 0$ for all $x \in \Omega_0 \subseteq u^{*-1}(D_f)$, with $|\Omega_0| \neq 0$, which is a contradiction. Therefore, from (3.7) we obtain $f(x, u^*(x)) = 0$ for a.a. $x \in u^{*-1}(D_f)$. Hence, one has $\tilde{w}(x) = 0 = \lambda f(x, u^*(x))$ for a.a. $x \in u^{*-1}(D_f)$, for which, in conclusion, one has

$$\tilde{w}(x) = \lambda f(x, u^*(x))$$

for a.a. $x \in \Omega$. Therefore, from (3.18) we have

$$-\int_\Omega |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla w dx = \lambda \int_\Omega f(x, u^*(x)) w dx \quad \text{for all } w \in X,$$

that is, u^* is a weak solution of problem (1.1) and our claim is proved.

Finally, u_1, u_2 are weak solutions of problem (1.1) and the maximum principle ensures the conclusion. □

Remark 3.3. If in Theorem 3.2 we do not assume the hypothesis (3.5),then the existence of two weak solutions to problem (1.1) is ensured by Theorem 2.13 for each $\lambda \in]0, K_\gamma[$. Clearly, in this case, one of two solutions may be zero.

A simple and useful corollary can be given next. To do so, let

$$\lambda^\# = \frac{1}{pC^p|\Omega|^{\frac{p}{N}}_N} \left(\frac{s}{a_s}\right)^{\frac{q-p}{q-s}} \left(\frac{q}{a_q}\right)^{\frac{p-s}{q-s}} \left(\frac{p-s}{q-p}\right)^{\frac{p-s}{q-s}} \frac{q-p}{q-s}.$$

Corollary 3.4. *Let hypothesis H(f),* (3.6) *and* (3.7) *be satisfied and assume that*

$$\limsup_{t \to 0^+} \frac{F(x,t)}{t^p} = +\infty \quad \textit{uniformly for a.a.} \, x \in \Omega.$$

Then, for each $\lambda \in]0, \lambda^\#[$, problem (1.1) *admits at least two positive weak solutions.*

Proof. Let $\lambda \in]0, \lambda^\#[$ be fixed. We easily see that $\lambda^\# = \sup_\gamma K_\gamma$ and so there is $\gamma > 0$ such that $\lambda < K_\gamma$. On the other side, since

$$\limsup_{t \to 0^+} \, pC^p K |\Omega|^{\frac{p}{N}}_N \frac{F(x,t)}{t^p} = +\infty \quad \text{uniformly for a.a.} \, x \in \Omega,$$

there exists $\delta < \gamma$ such that

$$pC^p K |\Omega|^{\frac{p}{N}}_N \frac{F(x,t)}{t^p} > \frac{1}{\lambda}.$$

Hence, $\lambda \in]K_\delta, K_\gamma[$ and so condition (3.5) is fulfilled. Therefore, the statement of the corollary follows from Theorem 3.2. □

Let us consider some examples which in our setting of Theorem 3.2 and Corollary 3.4.

Example 3.5. Let $c : \mathbb{R} \to \mathbb{R}$ be given by

$$c(t) = \begin{cases} 0 & \text{if } t \in C, \\ 1 & \text{if } t \notin C, \end{cases}$$

where C is the Cantor set. One easily verifies that c is continuous in every $t \notin C$ and since the Lebesgue measure of C is zero we conclude that c is almost everywhere continuous. Simple computations show that the function $f(t) = c(t) + t^p$, $t \in \mathbb{R}$ satisfies all the assumptions of Corollary 3.4. We note that in this case the set of discontinuity points of f is uncountable.

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ t^{\frac{1}{p}} & \text{if } 0 \le t < 2, \\ t^p & \text{if } t \ge 2. \end{cases}$$

Then we see that this function satisfies the assumptions of Corollary 3.4.

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