Nonlinear Anal. TMA 251 (2025) 113677

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

The asymptotic behavior of constant sign and nodal solutions of (p,q)-Laplacian problems as p goes to 1

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ARTICLE INFO

Communicated by Jan Kristensen

MSC:
26B30
35B40
35J60
35J62

Keywords:
1-laplacian
Asymptotic behavior
Functions of bounded variation
p goes to 1
Sign-changing solutions

ABSTRACT

In this paper we study the asymptotic behavior of solutions to the (p,q)-equation

$$-\Delta_p u - \Delta_q u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

as $p \to 1^+$, where $N \geq 2$, $1 < p < q < 1^* := N/(N-1)$ and f is a Carathéodory function that grows superlinearly and subcritically. Based on a Nehari manifold treatment, we are able to prove that the $(1,q)$-Laplace problem given by

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \Delta_q u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

has at least two constant sign solutions and one sign-changing solution, whereby the sign-changing solution has least energy among all sign-changing solutions. Furthermore, the solutions belong to the usual Sobolev space $W_0^{1,q}(\Omega)$ which is in contrast with the case of 1-Laplacian problems, where the solutions just belong to the space $\mathrm{BV}(\Omega)$ of all functions of bounded variation. As far as we know this is the first work dealing with $(1,q)$-Laplace problems even in the direction of constant sign solutions.

1. Introduction

In the last three decades, problems involving the 1-Laplacian have gained great interest and were subject of several research activities using different techniques. This 1-Laplacian is formally defined as

$$\Delta_1 u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \tag{1.1}$$

and was studied intensively in the groundbreaking works of Andreu–Ballester–Caselles–Mazón [5–7], see also the monograph by Andreu–Vaillo–Caselles–Mazón [8]. Among the very first works on this topic were the papers of Kawohl [24] and Demengel [15]. However, the operator is not only of great interest from a mathematical point of view, it also appears in several applications, see, for example, the papers of Chen–Levine–Rao [14] and Rudin–Osher–Fatemi [34] in the field of image restoration.

The natural function space in order to make (1.1) well-defined is the space $\mathrm{BV}(\Omega)$ of all functions of bounded variation. The main disadvantage of this space is the lack of reflexivity which makes the proof of compactness conditions as the Cerami or the Palais Smale conditions quite challenging. One possible approach can be done by dealing with a solution in the space $\mathrm{BV}(\Omega)$ in which the

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https://doi.org/10.1016/j.na.2024.113677
Received 1 March 2024; Accepted 24 September 2024
Available online 1 October 2024

subdifferential of the energy functional vanishes. The advantage of this treatment is the fact that one can apply variational methods in order to get solutions of corresponding equations driven by the 1-Laplacian. We refer to the papers of Alves–Figueiredo–Pimenta [2] and Figueiredo–Pimenta [17–19].

Another method to deal problem with the 1-Laplacian is the pairing theory developed by Anzellotti [9] in order to give a meaning to the quotient $Du/|Du|$ for $u \in \mathrm{BV}(\Omega)$. This approach has been applied, for example, by Mercaldo–Segura de León–Trombetti [29] who studied the problem

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

with a function $f \in L^1(\Omega)$ independent of u. The authors prove the existence of a renormalized solution u_p of the corresponding p-Laplace problem which turns out to be a solution of the limit problem (1.2). In 2018, Latorre–Segura de León [25] have been considered the Dirichlet problem

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |Du| = f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

and proved existence results and a comparison principle when $f \in L^1(\Omega)$, see also Mazón–Segura de León [26] for the case $f \in L^q(\Omega)$ with $q > N$. Recently, Figueiredo–Pimenta [21] treated the 1-Laplacian problem in $\mathbb{R}^N$ defined by

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \frac{u}{|u|} = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

with a continuous function $f : \mathbb{R} \to \mathbb{R}$ that has subcritical growth and satisfies the Ambrosetti–Rabinowitz condition. By considering the associated Nehari manifold to (1.3) the existence of a sign-changing solution with least energy has been proved. In particular the authors apply ideas from the paper of Molina Salas–Segura de León [30]. Finally, we mention further existence results in the direction of 1-Laplacian problems which have been published within the last decades. We refer to the works of Abdellaoui–Dall'Aglio–Segura de León [1] involving critical gradient terms, Alves–Pimenta [3] for unbounded domains using the concentration–compactness principle, Chang [12] for the spectrum of the 1-Laplacian, Demengel [16] for variational problems, Figueiredo–Pimenta [20] involving gradient terms, Mercaldo–Rossi–Segura de León–Trombetti [27,28] for anisotropic and Neumann boundary values problems, respectively, Parini [31] for the second eigenvalue of the p-Laplacian as p goes to 1 and Pimenta–dos Santos–Santos Júnior [32] for discontinuous problems, see also the references therein.

In this work, we study the asymptotic behavior of the solutions of the following (p,q)-Laplacian problem

$$-\Delta_p u - \Delta_q u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

as $p \to 1^+$, where $N \geq 2$, $1 < p < q < 1^*$ and $1^* = N/(N-1)$. The nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is assumed to satisfy the following conditions:

(f_1) There exists $c_1 > 0$ such that

$$|f(x,s)| \leq c_1 \left(1 + |s|^{r-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R},$$

where $q < r < 1^*$.

(f_2)

$$\lim_{s \to \pm\infty} \frac{f(x,s)}{|s|^{q-2}s} = +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$

(f_3) There exists $\alpha > 0$ such that

$$\limsup_{s \to 0} \frac{|f(x,s)|}{|s|^{\alpha}} < +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$

(f_4) $s \mapsto f(x,s)s - qF(x,s)$ is nondecreasing in $\mathbb{R}_+$ and nonincreasing in $\mathbb{R}_-$, for a.a. $x \in \Omega$, where

$$F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t.$$

(f_5) $s \mapsto f(x,s)$ is increasing for a.a. $x \in \Omega$.

As $p \to 1^+$, the solutions u_p of (1.4) are expected to converge to a function $u_0 \in W_0^{1,q}(\Omega)$, which satisfies

$$\begin{aligned} -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \Delta_q u &= f(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

in the weak sense, that is,

$$\int_\Omega \frac{\nabla u_0 \cdot \nabla v}{|\nabla u_0|}\,\mathrm{d}x + \int_\Omega |\nabla u_0|^{q-2}\nabla u_0 \cdot \nabla v\,\mathrm{d}x = \int_\Omega f(x,u_0)v\,\mathrm{d}x \tag{1.6}$$

holds for all $v \in W_0^{1,q}(\Omega)$.

Our first main result is the following one.

Theorem 1.1. *Suppose $N \geq 2$, $1 < q < 1^*$ and that f satisfies (f_1)–(f_5). Then there exist non-trivial constant sign solutions $u_0, v_0 \in W_0^{1,q}(\Omega)$ of (1.5) in the sense of (1.6) such that u_0 is nonnegative and v_0 is nonpositive, respectively.*

In the second part of this paper we prove the following result, which states the existence of a sign-changing solution of (1.5) in the sense of (1.6).

Theorem 1.2. *Suppose $N \geq 2$, $1 < q < 1^*$ and that f satisfies $(f_1) - (f_5)$. Then there exists a sign-changing solution $w_0 \in W_0^{1,q}(\Omega)$ of (1.5), which turns out to be a least energy sign-changing solution of (1.5).*

The proofs of our results are mainly based on the usage of the Nehari manifold to get a sign-changing solution to problem (1.4). To be more precise we consider the so called nodal Nehari set defined by

$$\mathcal{M}_p = \left\{ u \in W_0^{1,q}(\Omega) : u^{\pm} \neq 0 \text{ and } \langle \Phi_p'(u), u^{\pm} \rangle = 0 \right\},$$

where

$$\Phi_p(u) := I_p(u) + \frac{p-1}{p}|\Omega|$$

with I_p being the energy functional to (1.4). This method is very powerful and does not need any regularity on the solutions. As far as we know, the set $\mathcal{M}_p$ was first used by Bartsch–Weth [11] in order to get nodal solutions for the semilinear equation

$$-\Delta u + u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with differentiable f growing superlinearly and subcritically provided Ω contains a large ball.

We point out that, up to our knowledge, our work is the first one dealing with a $(1,q)$-Laplacian instead of a 1-Laplacian. This fact gives us, in addition to the existence results in Theorems 1.1 and 1.2, more regularity on the solutions. Indeed, in our results we obtain that the solutions belong to the usual Sobolev space $W_0^{1,q}(\Omega)$ which implies that their weak derivative exist. Such property is in general not true for functions on $\mathrm{BV}(\Omega)$ for which the distributional derivative is just a vectorial Radon measure. As a result of this, our solutions satisfy (1.5) in the weak sense given in (1.6), what is unusual for problems involving this operator.

This paper is organized as follows. In Section 2 we introduce our function space $\mathrm{BV}(\Omega)$ of all functions of bounded variation including its properties and we present the pairing theory introduced by Anzellotti [9]. Section 3 is devoted to the proof of Theorem 1.1 while in Section 4 we give the proof of Theorem 1.2.

2. Preliminaries

In this section we present the main function space and tools that will be needed in the sequel. First of all, let us introduce the space of functions of bounded variation, denoted by $\mathrm{BV}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, see the monograph of Attouch–Buttazzo–Michaille [10]. We say that $u \in \mathrm{BV}(\Omega)$, or is a function of bounded variation, if $u \in L^1(\Omega)$ and its distributional derivative Du is a vectorial Radon measure, *i.e.*,

$$\mathrm{BV}(\Omega) = \left\{ u \in L^1(\Omega) : \ Du \in \mathcal{M}(\Omega, \mathbb{R}^N) \right\}.$$

It can be proved that $u \in \mathrm{BV}(\Omega)$ if and only if $u \in L^1(\Omega)$ and

$$\int_\Omega |Du| := \sup \left\{ \int_\Omega u \operatorname{div} \phi \, \mathrm{d}x : \phi \in C_c^1(\Omega, \mathbb{R}^N), \ \|\phi\|_\infty \leq 1 \right\} < +\infty.$$

The space $\mathrm{BV}(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathrm{BV}} := \int_\Omega |Du| + \int_\Omega |u| \, \mathrm{d}x,$$

which is continuously embedded into $L^r(\Omega)$ for all $r \in [1, 1^*]$, where $1^* = N/(N-1)$. Since the domain Ω is bounded, it also holds the compactness of the embedding of $\mathrm{BV}(\Omega)$ into $L^r(\Omega)$ for all $r \in [1, 1^*)$.

The space $C^\infty(\overline{\Omega})$ is not dense in $\mathrm{BV}(\Omega)$ with respect to the strong topology. However, with respect to the strict convergence, it does. We say that $(u_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ converges to $u \in \mathrm{BV}(\Omega)$ in the sense of the strict convergence, if

$$u_n \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \int_\Omega |Du_n| \to \int_\Omega |Du|$$

as $n \to \infty$. In Ambrosio–Fusco–Pallara [4] it has been shown that the trace operator $\mathrm{BV}(\Omega) \hookrightarrow L^1(\partial\Omega)$ is well defined in such a way that

$$\|u\| := \int_\Omega |Du| + \int_{\partial\Omega} |u| \, \mathrm{d}\mathcal{H}^{N-1}$$

is an equivalent norm to $\|\cdot\|_{\mathrm{BV}}$.

Given $u \in \mathrm{BV}(\Omega)$, we can decompose its distributional derivative as

$$Du = D^a u + D^s u,$$

where $D^a u$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$ while $D^s u$ is singular with respect to the same measure. Moreover, we denote the total variation of Du as $|Du|$.

In several arguments used in this work, it is mandatory to have a sort of Green's formula to expressions like $w\,\mathrm{div}(\mathbf{z})$, where $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$, $\mathrm{div}(\mathbf{z}) \in L^N(\Omega)$ and $w \in \mathrm{BV}(\Omega)$. For this we have to somehow deal with the product between $\mathbf{z}$ and Dw, which we denote by $(\mathbf{z}, Dw)$. This can be done through the pairings theory developed by Anzellotti in [9], see also Frid–Chen in [13]. Below, we describe the main results of this theory.

Let us denote

$$X_N(\Omega) = \left\{ \mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) : \ \mathrm{div}(\mathbf{z}) \in L^N(\Omega) \right\}.$$

For $\mathbf{z} \in X_N(\Omega)$ and $w \in \mathrm{BV}(\Omega)$, we define the distribution $(\mathbf{z}, Dw) \in \mathcal{D}'(\Omega)$ as

$$\langle (\mathbf{z}, Dw), \varphi \rangle := -\int_\Omega w\varphi \ \mathrm{div}(\mathbf{z})\,\mathrm{d}x - \int_\Omega w\mathbf{z} \cdot \nabla\varphi\,\mathrm{d}x$$

for every $\varphi \in \mathcal{D}(\Omega)$. With this definition, it can be proved that $(\mathbf{z}, Dw)$ is in fact a Radon measure such that

$$\left| \int_B (\mathbf{z}, Dw) \right| \le \|\mathbf{z}\|_\infty \int_B |Dw| \tag{2.1}$$

for every Borel set $B \subset \Omega$.

In order to define an analogue of Green's formula, it is also necessary to describe a weak trace theory for $\mathbf{z}$. In fact, there exists a trace operator $[\,\cdot\,, \nu] : X_N(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\| [\mathbf{z}, \nu] \|_{L^\infty(\partial\Omega)} \le \|\mathbf{z}\|_\infty$$

and, if $\mathbf{z} \in C^1(\overline{\Omega}_\delta, \mathbb{R}^N)$,

$$[\mathbf{z}, \nu](x) = \mathbf{z}(x) \cdot \nu(x) \quad \text{on } \Omega_\delta,$$

where Ω_δ is a δ-neighborhood of $\partial\Omega$. Thanks to these definitions, it can be proved that the following Green's formula holds for every $\mathbf{z} \in X_N(\Omega)$ and $w \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$:

$$\int_\Omega w\,\mathrm{div}(\mathbf{z})\,\mathrm{d}x + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w \,\mathrm{d}\mathcal{H}^{N-1}. \tag{2.2}$$

3. Existence of constant sign solutions

In this section we are going to prove Theorem 1.1. In order to get the solutions we are interested in, the first step is to consider the following problem for $q > p > 1$,

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

It is well known that weak solutions of (3.1) are critical points of the energy functional $I_p : W_0^{1,q}(\Omega) \to \mathbb{R}$ given by

$$I_p(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega |\nabla u|^q\,\mathrm{d}x - \int_\Omega F(x, u)\,\mathrm{d}x,$$

which is well defined as by (f_1) it holds

$$\int_\Omega F(x, u)\,\mathrm{d}x \le C_1\|u\|_1 + C_2\|u\|_r^r.$$

Since we are looking for nonnegative (or nonpositive) solutions, let us consider the truncated version of I_p (denoted by the sake of simplicity also by I_p), as

$$I_p(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega |\nabla u|^q\,\mathrm{d}x - \int_\Omega F^+(x, u)\,\mathrm{d}x,$$

where $F^+(x, s) = \int_0^s f^+(x, t)\,\mathrm{d}t$ and $f^+(x, s) = 0$ for $s \le 0$ and $f^+(x, s) = f(x, s)$ for $s > 0$.

We consider the functional $\Phi_p : W_0^{1,q}(\Omega) \to \mathbb{R}$ given by

$$\Phi_p(u) := I_p(u) + \frac{p-1}{p}|\Omega|,$$

which is well defined and $(\Phi_p(u))_{p>1}$ is non-decreasing in p, for all $u \in W_0^{1,q}(\Omega)$. Indeed, if $1 < p_1 \le p_2 < N$, then by Young's inequality with exponents p_2/p_1 and $p_2/(p_2 - p_1)$, it follows that

$$\int_\Omega |\nabla u|^{p_1}\,\mathrm{d}x \le \frac{p_1}{p_2}\int_\Omega |\nabla u|^{p_2}\,\mathrm{d}x + \frac{p_2 - p_1}{p_2}|\Omega|.$$

Hence, it follows that, for all $u \in W_0^{1,q}(\Omega)$,

$$\begin{aligned}\Phi_{p_1}(u) &= \frac{1}{p_1}\int_\Omega |\nabla u|^{p_1}\,dx + \frac{p_1-1}{p_1}|\Omega| + \frac{1}{q}\int_\Omega |\nabla u|^q\,dx - \int_\Omega F(u)\,dx \\ &\le \frac{1}{p_1}\left(\frac{p_1}{p_2}\int_\Omega |\nabla u|^{p_2}\,dx + \frac{p_2-p_1}{p_2}|\Omega|\right) + \frac{p_1-1}{p_1}|\Omega| + \frac{1}{q}\int_\Omega |\nabla u|^q\,dx \\ &\quad - \int_\Omega F(u)\,dx \\ &= \Phi_{p_2}(u).\end{aligned} \tag{3.2}$$

Since Φ_p and I_p differs just by a constant, we can study (3.1) from a variational point of view by dealing with either I_p or Φ_p.

By (f_3), for $p > 1$ sufficiently close to 1, the function f satisfies

$$\limsup_{s\to 0} \frac{|f(x,s)|}{|s|^{p-1}} = 0 \quad \text{uniformly for a.a. } x \in \Omega.$$

Then, for each such p, standard arguments imply that Φ_p satisfies the geometric conditions of the mountain-pass theorem. Moreover, as in Molina Salas–Segura de León [30], one can prove that there exists $e \in W_0^{1,q}(\Omega)$, such that

$$\Phi_p(e) < 0 \quad \text{for all } 1 < p < q.$$

Then, we can find a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,q}(\Omega)$ such that, as $n \to +\infty$,

$$\Phi_p(u_n) \to c_p,$$

and

$$\left(1 + \|u_n\|_{W^{1,q}}\right)\Phi_p'(u_n) \to 0 \quad \text{in } W^{-1,q'}(\Omega),$$

where

$$c_p = \inf_{\gamma\in\Gamma_p}\max_{t\in[0,1]} \Phi_p(\gamma(t))$$

and

$$\Gamma_p = \{\gamma \in C([0,1], W_0^{1,q}(\Omega)) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Moreover, as in Gasiński–Winkert [23], one can prove that I_p satisfy the Cerami compactness condition. Then, there exists an element $u_p \in W_0^{1,q}(\Omega)$ such that

$$u_n \to u_p \quad \text{in } W_0^{1,q}(\Omega) \text{ as } n \to +\infty.$$

Hence, u_p is a weak solution of (3.1), that is,

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla v\,dx + \int_\Omega |\nabla u_p|^{q-2}\nabla u_p \cdot \nabla v\,dx = \int_\Omega f(x,u_p)v\,dx \tag{3.3}$$

with

$$\Phi_p(u_p) = c_p.$$

In addition, by taking u_p^- as test function in (3.3), we have that

$$\int_\Omega |\nabla u_p^-|^p\,dx + \int_\Omega |\nabla u_p^-|^q\,dx = 0,$$

which implies that u_p is a nonnegative solution.

It is also possible to show that the mountain pass solution u_p is the minimum of Φ_p over the Nehari manifold associated to (3.1) (for instance, see [33][Proposition 3.11), defined by

$$\mathcal{N}_p = \left\{u \in W_0^{1,q}(\Omega)\setminus\{0\} : \langle \Phi_p'(u), u\rangle = 0\right\},$$

i.e.,

$$\Phi_p(u_p) = \min_{u\in\mathcal{N}_p} \Phi_p(u). \tag{3.4}$$

Hence, since $\mathcal{N}_p$ contains all nontrivial nonnegative solutions of (3.1), it follows that u_p is a nonnegative ground state solution of (3.1).

In Gasiński–Winkert [23] (as well as in Figueiredo–Ramos Quoirin [22]), the authors deal with (3.1) by using the Nehari manifold and prove that for each $u \in W_0^{1,q}(\Omega)$, $u \ge 0$ and $u \ne 0$, there exists a unique $t_u > 0$ such that $\Phi_p'(t_u u) = 0$ and

$$\Phi_p(t_u u) = \max_{t>0} \Phi_p(tu). \tag{3.5}$$

Moreover, $t_{u_p} = 1$, *i.e.*

$$\Phi_p(u_p) = \max_{t>0} \Phi_p(tu_p).$$

Now we prove some technical lemmas that will imply that the family $(u_p)_{1<p<q}$ is bounded in $W_0^{1,q}(\Omega)$.

Lemma 3.1. *The family* $(\Phi_p(u_p))_p$ *is nondecreasing for* $p \in (1,q)$.

Proof. Let $1 < p_1 \le p_2 < q$ and $u_{p_1}, u_{p_2} \in W_0^{1,q}(\Omega)$ satisfying (3.4). Since $u_{p_2} \neq 0$, there exists $t > 0$ such that

$$tu_{p_2} \in \mathcal{N}_{p_1}. \tag{3.6}$$

Then, from (3.2), (3.4), (3.5) and (3.6), it follows that

$$\Phi_{p_2}(u_{p_2}) \ge \Phi_{p_2}(tu_{p_2}) \ge \Phi_{p_1}(tu_{p_2}) \ge \Phi_{p_1}(u_{p_1}). \quad \square$$

Lemma 3.2. *There exists* $C > 0$ *such that*

$$\|u_p\|_{1,q} \le C \quad \text{for all } p \in (1,q). \tag{3.7}$$

The family $(u_p)_{1<p<q}$ *is bounded in* $W_0^{1,q}(\Omega)$.

Proof. Let us assume by contradiction that

$$\|u_p\|_{1,q} \to +\infty \quad \text{as } p \to 1^+. \tag{3.8}$$

Let $w_p := u_p/\|u_p\|_{1,q}$. Since $(w_p)_{1<p<q}$ is bounded in $W_0^{1,q}(\Omega)$, there exists $w \in W_0^{1,q}(\Omega)$ such that $w \ge 0$ and

$$w_p \rightharpoonup w \quad \text{in } W_0^{1,q}(\Omega)$$
$$w_p \to w \quad \text{in } L^r(\Omega) \text{ for all } 1 \le r < \frac{Nq}{N-q}.$$

From Lemma 3.1, there exists $C > 0$ such that $\Phi_p(u_p) \le C$ for $1 < p < q$. Hence,

$$\frac{\Phi_p(u_p)}{\|u_p\|_{1,q}^q} = o_p(1). \tag{3.9}$$

Moreover, note that by (f_2), for a given $\eta > 0$, there exists $\delta > 0$, such that

$$\frac{F(x,s)}{|s|^q} \ge \eta \quad \text{for } |s| \ge \delta.$$

Then, from (3.9), we have that

$$\begin{aligned}
\frac{1}{p}\frac{\|u_p\|_{1,p}^p}{\|u_p\|_{1,q}^q} + \frac{1}{q} &= \int_\Omega \frac{F(x,u_p)}{\|u_p\|_{1,q}^q}\,\mathrm{d}x + o_p(1) \\
&\ge \int_{\Omega\cap\{u_p\ge\delta\}} \frac{F(x,u_p)}{\|u_p\|_{1,q}^q}\,\mathrm{d}x + o_p(1) \\
&\ge \int_{\Omega\cap\{u_p\ge\delta\}} \frac{F(x,u_p)}{u_p^q} w_p^q\,\mathrm{d}x + o_p(1) \\
&\ge \eta \int_{\Omega\cap\{u_p\ge\delta\}} w_p^q\,\mathrm{d}x + o_p(1).
\end{aligned} \tag{3.10}$$

On the other hand, since $W_0^{1,q}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$, it follows that

$$\frac{\|u_p\|_{1,p}^p}{\|u_p\|_{1,q}^q} = \left(\frac{\|u_p\|_{1,p}}{\|u_p\|_{1,q}}\right)^p \|u_p\|_{1,q}^{p-q} = o_p(1). \tag{3.11}$$

Hence, from (3.10), (3.11), Fatou's Lemma and the fact that $\chi_{\{u_p\ge\delta\}} \to \chi_{\{w>0\}}$ a.e., we get

$$\frac{1}{q} \ge \eta \int_\Gamma w^p\,\mathrm{d}x \quad \text{for all } \eta > 0,$$

where $\Gamma = \{x \in \Omega : w(x) > 0\}$. Then, $w = 0$ in $W_0^{1,q}(\Omega)$.

Now, let $k \ge 1$ and define

$$v_p = (qk)^{\frac{1}{q}} w_p.$$

Then, note that

$$v_p \rightharpoonup 0 \quad \text{in } W_0^{1,q}(\Omega)$$

and

$$v_p \to 0 \quad \text{in } L^r(\Omega) \text{ for } 1 \leq r < \frac{Nq}{N-q}.$$

For each $p \in (1,q)$, let $t_p \in [0,1]$ such that

$$\Phi_p(t_p u_p) = \max_{0 \leq t \leq 1} \Phi_p(t u_p).$$

From (3.8), there exists $\overline{p} \in (1,q)$, such that

$$0 \leq \frac{(qk)^{\frac{1}{q}}}{\|u_p\|_{1,q}} \leq 1 \quad \text{for all } p \in (1,\overline{p}).$$

Then, taking (f_1) and (f_2) into account, there exists a constant $C > 0$ such that

$$F(x,s) \geq -C \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}.$$

Using this, we obtain

$$\begin{aligned}
\Phi_p(t_p u_p) &\geq \Phi_p\left(\frac{(qk)^{\frac{1}{q}}}{\|u_p\|_{1,q}} u_p\right) \\
&= \Phi_p(v_p) \\
&= \frac{1}{p}(qk)^{\frac{p}{q}} \|\nabla w_p\|_p^p + k\|\nabla w_p\|_q^q - \int_\Omega F(x,v_p)\,\mathrm{d}x \\
&\geq \min\left\{\frac{1}{p}(qk)^{\frac{p}{q}}, 1\right\} k^{\frac{p}{q}} \left(\|\nabla w_p\|_p^p + \|\nabla w_p\|_q^q\right) - \int_\Omega F(x,v_p)\,\mathrm{d}x \\
&\geq \min\left\{\frac{1}{p}(qk)^{\frac{p}{q}}, 1\right\} k^{\frac{p}{q}} \|\nabla w_p\|_q^q - C \\
&\geq \min\left\{\frac{1}{p}(qk)^{\frac{p}{q}}, 1\right\} k^{\frac{p}{q}} \|w_p\|_{1,q}^q - C \\
&= \min\left\{\frac{1}{p}(qk)^{\frac{p}{q}}, 1\right\} k^{\frac{p}{q}} - C.
\end{aligned}$$

Then, it follows that

$$\lim_{p \to 1^+} \Phi_p(t_p u_p) = +\infty. \tag{3.12}$$

Since $\Phi_p(0) = \frac{p-1}{p}|\Omega|$ and, from Lemma 3.1, $(\Phi_p(u_p))_p$ is bounded, there exists $\tilde{p} > 1$ such that

$$0 < t_p < 1 \quad \text{for all } p \in (1,\tilde{p}).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_p(t u_p)\Big|_{t=t_p} = 0.$$

Then, for $1 < p < \tilde{p}$,

$$\|t_p \nabla u_p\|_p^p + \|t_p \nabla u_p\|_q^q = \int_\Omega f(x, t_p u_p) t_p u_p \,\mathrm{d}x. \tag{3.13}$$

From Lemma 3.1 there exists $C > 0$ such that

$$q\Phi_p(u_p) \leq C. \tag{3.14}$$

Moreover, we have that

$$\|\nabla u_p\|_p^p + \|\nabla u_p\|_q^q = \int_\Omega f(x,u_p)u_p\,\mathrm{d}x. \tag{3.15}$$

Subtracting (3.15) from (3.14) yields

$$\left(\frac{q}{p} - 1\right)\|\nabla u_p\|_p^p + \int_\Omega \left(f(x,u_p)u_p - qF(x,u_p)\right)\,\mathrm{d}x \leq C. \tag{3.16}$$

On the other hand, by (f$_4$) and (3.16),

$$\begin{aligned}&\left(\frac{q}{p}-1\right)\|t_p\nabla u_p\|_p^p+\int_\Omega\left(f(x,t_pu_p)t_pu_p-qF(x,t_pu_p)\right)\,\mathrm{d}x\\&\le\left(\frac{q}{p}-1\right)\|\nabla u_p\|_p^p+\int_\Omega\left(f(x,u_p)u_p-qF(x,u_p)\right)\,\mathrm{d}x\\&\le C.\end{aligned}\tag{3.17}$$

From (3.13) and (3.17), for $1<p<\tilde p$, we have

$$\frac{q}{p}\|t_p\nabla u_p\|_p^p+\|t_p\nabla u_p\|_q^q-q\int_\Omega F(x,t_pu_p)\,\mathrm{d}x\le C,$$

which implies that

$$q\Phi_p(t_pu_p)\le C\quad\text{for all } p\in(1,\tilde p).$$

But this contradicts (3.12), so the assertion of the lemma follows. □

Using Lemma 3.2 we can find $u_0\in W_0^{1,q}(\Omega)$ such that

$$u_p\rightharpoonup u_0\quad\text{in } W_0^{1,q}(\Omega),\tag{3.18}$$

$$u_p\to u_0\quad\text{in } L^r(\Omega)\text{ for all } 1\le r<\frac{qN}{N-q},\tag{3.19}$$

$$u_p\to u_0\quad\text{a.e. in } \Omega,$$

as $p\to1^+$. Note that this implies that $u_0\ge0$.

Now let us find out what kind of problem the limit function u_0 satisfies. In a first step, we are going to prove that u_0 satisfies (1.5) in a very weak sense. Actually, after the next results, we prove that u_0 is a solution of bounded variation of (1.5), *i.e.*, there exists $\mathbf{z}\in L^\infty(\Omega,\mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty\le1$ and

$$\begin{cases}-\operatorname{div}\mathbf{z}-\Delta_qu_0&=&f(x,u_0)\text{ in }\mathcal{D}'(\Omega),\\ \mathbf{z}\cdot\nabla u_0&=&|\nabla u_0|\text{ a.e. in }\Omega,\\ u_0&=&0,\text{ on }\partial\Omega.\end{cases}$$

Lemma 3.3. *There exists* $\mathbf{z}\in L^\infty(\Omega,\mathbb{R}^N)$ *such that* $\|\mathbf{z}\|_\infty\le1$ *and*

$$|\nabla u_p|^{p-2}\nabla u_p\rightharpoonup\mathbf{z}\quad\text{in } L^s(\Omega,\mathbb{R}^N)\text{ for } 1\le s<+\infty,\tag{3.20}$$

as $p\to1^+$.

Proof. Let us fix $s\in[1,+\infty)$. By Hölder's inequality, for $1<p<s/(s-1)$, one has

$$\begin{aligned}\||\nabla u_p|^{p-2}\nabla u_p\|_s^s&=\int_\Omega|\nabla u_p|^{(p-1)s}\,\mathrm{d}x\\&\le\left(\int_\Omega|\nabla u_p|^p\,\mathrm{d}x\right)^{\frac{(p-1)s}{p}}|\Omega|^{1-\frac{s}{p'}}\\&\le\left(\left(\int_\Omega|\nabla u_p|^q\,\mathrm{d}x\right)^{\frac{p}{q}}|\Omega|^{\frac{(q-p)}{q}}\right)^{\frac{(p-1)s}{p}}|\Omega|^{1-\frac{s}{p'}}\\&\le\|u_p\|_{1,q}^{(p-1)s}|\Omega|^{1-\frac{(p-1)s}{q}}\\&\le C^{(p-1)s}|\Omega|^{1-\frac{(p-1)s}{q}},\end{aligned}$$

where $C>0$ is as in (3.7). Then

$$\||\nabla u_p|^{p-2}\nabla u_p\|_s\le C^{(p-1)}|\Omega|^{\frac{1}{s}-\frac{(p-1)}{q}}.$$

Hence, $(|\nabla u_p|^{p-2}\nabla u_p)_{p>1}$ is bounded in $L^s(\Omega)$. Then, there exists $\mathbf{z}_s\in L^s(\Omega,\mathbb{R}^N)$, such that $|\nabla u_p|^{p-2}\nabla u_p\rightharpoonup\mathbf{z}_s$ in $L^s(\Omega,\mathbb{R}^N)$. Through a diagonal argument, it is possible to show that $\mathbf{z}_s$ does not depend on s and then we denote it simply by $\mathbf{z}$. By making $p\to1^+$, from the last inequality and the weak semicontinuity of the norm in $L^s(\Omega,\mathbb{R}^N)$,

$$\|\mathbf{z}\|_s\le\liminf_{p\to1^+}\||\nabla u_p|^{p-2}\nabla u_p\|_s\le|\Omega|^{\frac{1}{s}}.$$

Finally, letting $s\to+\infty$, we have that

$$\|\mathbf{z}\|_\infty\le1,$$

which finishes the proof. □

From (3.20), we get

$$-\Delta_p = \mathrm{div}(|\nabla u_p|^{p-2}\nabla u_p) \to \mathrm{div}\,\mathbf{z} \quad \text{in } \mathcal{D}'(\Omega) \tag{3.21}$$

as $p \to 1^+$, what follows from taking $\nabla\varphi$ as a test function in (3.20), where $\varphi \in \mathcal{D}(\Omega)$.

Note that, from (3.1), (3.19) and (3.21) and applying Lebesgue's Dominated Convergence Theorem, it follows that

$$-\,\mathrm{div}\,\mathbf{z} - \Delta_q u_0 = f(x,u_0) \quad \text{in } \mathcal{D}'(\Omega), \tag{3.22}$$

i.e.,

$$\int_\Omega \mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega |\nabla u_0|^{q-2}\nabla u_0\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega f(x,u_0)\varphi\,\mathrm{d}x, \quad \forall\varphi\in\mathcal{D}(\Omega).$$

Lemma 3.4. *The function u_0 and the vector field $\mathbf{z}$ satisfy the following equality*

$$\mathbf{z}\cdot\nabla u_0 = |\nabla u_0| \quad \text{a.e. in } \Omega.$$

Proof. First, note that, since $\|\mathbf{z}\|_\infty \le 1$, it follows that,

$$\mathbf{z}\cdot\nabla u_0 \le |\mathbf{z}\cdot\nabla u_0| \le \|\mathbf{z}\|_\infty|\nabla u_0| \le |\nabla u_0|, \quad \text{a.e. in } \Omega.$$

Hence, it is enough to show the opposite inequality. For this, it is enough to prove that

$$\int_\Omega \varphi\,\mathbf{z}\cdot\nabla u_0\,\mathrm{d}x \ge \int_\Omega \varphi|\nabla u_0|\,\mathrm{d}x \tag{3.23}$$

for all $\varphi \in C_0^1(\Omega)$ with $\varphi \ge 0$.

Let $\varphi \in C_0^1(\Omega)$, such that $\varphi \ge 0$ and let us consider $(\rho_\epsilon)_{\epsilon>0}$ a family of mollifiers. By taking $(u_0\varphi)*\rho_\epsilon$ as a test function in (3.22), we have that

$$\begin{aligned}
\int_\Omega \mathbf{z}\cdot\nabla u_0\varphi*\rho_\epsilon\,\mathrm{d}x \;&=\; \int_\Omega \mathbf{z}\cdot\nabla(u_0\varphi*\rho_\epsilon)\,\mathrm{d}x - \int_\Omega \mathbf{z}\cdot\nabla\varphi u_0*\rho_\epsilon\,\mathrm{d}x\\
&= -\int_\Omega |\nabla u_0|^{q-2}\nabla u_0\cdot\nabla(u_0\varphi*\rho_\epsilon)\,\mathrm{d}x + \int_\Omega f(x,u_0)u_0\varphi*\rho_\epsilon\,\mathrm{d}x\\
&\quad - \int_\Omega \mathbf{z}\cdot\nabla\varphi u_0*\rho_\epsilon\,\mathrm{d}x\\
&= -\int_\Omega |\nabla u_0|^q\varphi*\rho_\epsilon\,\mathrm{d}x - \int_\Omega |\nabla u_0|^{q-2}\nabla u_0\cdot\nabla\varphi u_0*\rho_\epsilon\,\mathrm{d}x\\
&\quad + \int_\Omega f(x,u_0)u_0\varphi*\rho_\epsilon\,\mathrm{d}x - \int_\Omega \mathbf{z}\cdot\nabla\varphi u_0*\rho_\epsilon\,\mathrm{d}x.
\end{aligned}$$

By doing $\epsilon \to 0^+$, we have that

$$\begin{aligned}
\int_\Omega \mathbf{z}\cdot\nabla u_0\varphi\,\mathrm{d}x \;&=\; \int_\Omega f(x,u_0)u_0\varphi\,\mathrm{d}x - \int_\Omega \varphi|\nabla u_0|^q\,\mathrm{d}x\\
&\quad - \int_\Omega u_0|\nabla u_0|^{q-2}\nabla u_0\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega u_0\mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x.
\end{aligned} \tag{3.24}$$

Now, let us consider $u_p\varphi \in W_0^{1,q}(\Omega)$ as a test function in (3.1). Then we obtain,

$$\begin{aligned}
&\int_\Omega \varphi|\nabla u_p|^p\,\mathrm{d}x + \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x\\
&+ \int_\Omega \varphi|\nabla u_p|^q\,\mathrm{d}x + \int_\Omega u_p|\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\varphi\,\mathrm{d}x\\
&= \int_\Omega f(x,u_p)u_p\varphi\,\mathrm{d}x.
\end{aligned} \tag{3.25}$$

Now we calculate the $\liminf$ as $p \to 1^+$ in both sides of (3.25). Before, note that,

$$\int_\Omega \varphi|\nabla u_0|\,\mathrm{d}x \le \liminf_{p\to1^+}\int_\Omega |\nabla u_p|^p\varphi\,\mathrm{d}x. \tag{3.26}$$

Indeed, by Young's inequality,

$$\begin{aligned}
\int_\Omega \varphi|\nabla u_0|\,\mathrm{d}x &\le \liminf_{p\to1^+}\int_\Omega \varphi|\nabla u_p|\,\mathrm{d}x\\
&\le \liminf_{p\to1^+}\left(\frac{1}{p}\int_\Omega \varphi|\nabla u_p|^p\,\mathrm{d}x + \frac{p-1}{p}\int_\Omega \varphi\,\mathrm{d}x\right)
\end{aligned}$$

$$= \liminf_{p\to 1^+} \int_\Omega \varphi|\nabla u_p|^p \, dx.$$

Moreover, by (3.19) and (3.20), it follows that

$$\lim_{p\to 1^+} \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi \, dx = \int_\Omega u_0 \mathbf{z} \cdot \nabla\varphi \, dx. \tag{3.27}$$

Finally, Lebesgue's Dominated Convergence Theorem and (3.19) imply that

$$\lim_{p\to 1^+} \int_\Omega f(x,u_p)u_p\varphi \, dx = \int_\Omega f(x,u_0)u_0\varphi \, dx. \tag{3.28}$$

Then, from (3.18), (3.24), (3.25), (3.26), (3.27) and (3.28), we obtain

$$\begin{aligned}
&\int_\Omega \mathbf{z}\cdot\nabla u_0\varphi \, dx \\
&= \int_\Omega f(x,u_0)u_0\varphi \, dx - \int_\Omega \varphi|\nabla u_0|^q \, dx - \int_\Omega u_0|\nabla u_0|^{q-2}\nabla u_0 \cdot\nabla\varphi \, dx - \int_\Omega u_0\mathbf{z}\cdot\nabla\varphi \, dx \\
&\geq \liminf_{p\to 1^+} \Bigg(\int_\Omega f(x,u_p)u_p\varphi \, dx - \int_\Omega \varphi|\nabla u_p|^q \, dx - \int_\Omega u_p|\nabla u_p|^{q-2}\nabla u_p\cdot\nabla\varphi \, dx \\
&\qquad - \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi \, dx\Bigg) \\
&= \liminf_{p\to 1^+}\int_\Omega \varphi|\nabla u_p|^p \, dx \\
&\geq \int_\Omega \varphi|\nabla u_0| \, dx,
\end{aligned}$$

which implies (3.23). □

Note that, up to now, from (3.22) and Lemma 3.4, we have found $u_0 \in W_0^{1,q}(\Omega)$, for which there exists $\mathbf{z}\in X_N(\Omega)$ such that $\|\mathbf{z}\|_\infty \leq 1$ and

$$\begin{cases} -\operatorname{div}\mathbf{z} - \Delta_q u_0 = f(x,u_0) \text{ in } \mathcal{D}'(\Omega), \\ \mathbf{z}\cdot\nabla u_0 = |\nabla u_0| \text{ a.e. in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.29}$$

Now, what is left to do is to show that $u_0 \neq 0$. For this purpose, we introduce the energy functional $\Phi : W_0^{1,q}(\Omega)\to\mathbb{R}$ given by

$$\Phi(u) = \int_\Omega |\nabla u| \, dx + \frac{1}{q}\int_\Omega |\nabla u|^q \, dx - \int_\Omega F(x,u) \, dx. \tag{3.30}$$

First of all, note that by Young's inequality, $\Phi(u) \leq \Phi_p(u)$ for every $u \in W_0^{1,q}(\Omega)$. Moreover,

$$\lim_{p\to 1^+} \Phi_p(u_p) = \Phi(u_0). \tag{3.31}$$

Indeed, since u_0 satisfies (3.29) and u_p fulfills (3.1), by a regularizing argument which allows us to use u_0 as test function in (3.29), note that, as $p \to 1^+$,

$$\begin{aligned}
\int_\Omega |\nabla u_0| \, dx + \int_\Omega |\nabla u_0|^q \, dx &= \int_\Omega \mathbf{z}\cdot\nabla u_0 + \int_\Omega |\nabla u_0|^q \, dx \\
&= -\int_\Omega u_0 \operatorname{div}\mathbf{z} \, dx + \int_\Omega |\nabla u_0|^q \, dx \\
&= \int_\Omega f(x,u_0)u_0 \, dx \\
&= \int_\Omega f(x,u_p)u_p \, dx + o_p(1) \\
&= \int_\Omega |\nabla u_p|^p \, dx + \int_\Omega |\nabla u_p|^q \, dx + o_p(1).
\end{aligned} \tag{3.32}$$

On the other hand, from (3.18) and since $W_0^{1,q}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$, for every $1 \leq r < q$, it follows from the weak lower semicontinuity of the norm and Young's inequality that

$$\int_\Omega |\nabla u_0| \, dx \leq \liminf_{p\to 1^+}\int_\Omega |\nabla u_p| \, dx \leq \liminf_{p\to 1^+}\int_\Omega |\nabla u_p|^p \, dx. \tag{3.33}$$

In the same way,

$$\int_\Omega |\nabla u_0|^q \, dx \leq \liminf_{p\to 1^+}\int_\Omega |\nabla u_p|^q \, dx. \tag{3.34}$$

Then, from (3.32), (3.33) and (3.34), it follows that

$$\int_\Omega |\nabla u_0|\,\mathrm{d}x = \int_\Omega |\nabla u_p|^p\,\mathrm{d}x + o_p(1) \tag{3.35}$$

and

$$\int_\Omega |\nabla u_0|^q\,\mathrm{d}x = \int_\Omega |\nabla u_p|^q\,\mathrm{d}x + o_p(1). \tag{3.36}$$

Moreover, by (f_1), (3.19) and the Lebesgue's Dominated Convergence Theorem, as $p \to 1^+$,

$$\int_\Omega F(x,u_0)\,\mathrm{d}x = \int_\Omega F(x,u_p)\,\mathrm{d}x + o(1). \tag{3.37}$$

Then, (3.35), (3.36) and (3.37) imply (3.31).

Note also that, by (f_1), (f_3) and the Sobolev embedding, for all $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$ such that

$$|f(x,s)s| \le \varepsilon|s| + C_\varepsilon |s|^r \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}.$$

Then,

$$\Phi(u) \ge (1-\varepsilon)\|u\|_{1,q}^q\,\mathrm{d}x - C_\varepsilon \|u\|_{1,q}^r.$$

Let us consider $\varepsilon > 0$ small enough such that $1-\varepsilon > 1/2$. Then, if $\|u\|_{1,q} \le \rho$, where $0 < \rho < \left(\frac{(1-\varepsilon)-1/2}{C_\varepsilon}\right)^{\frac{1}{r-q}}$, it follows that

$$\Phi(u) \ge \frac{\|u\|_{1,q}^q}{2}. \tag{3.38}$$

Then, for all $p \in (1,\overline{p})$,

$$\Phi_p(u_p) \ge \Phi_p\left(\frac{\rho u_p}{\|u_p\|_{1,q}}\right) \ge \Phi\left(\frac{\rho u_p}{\|u_p\|_{1,q}}\right) \ge \frac{\rho^q}{2}.$$

Hence

$$\begin{aligned}
\frac{\rho^q}{2} &\le \Phi_p(u_p)\\
&\le \int_\Omega |\nabla u_p|^p\,\mathrm{d}x + \int_\Omega |\nabla u_p|^q\,\mathrm{d}x - \int_\Omega F(x,u_p)\,\mathrm{d}x\\
&= \int_\Omega f(x,u_p)u_p\,\mathrm{d}x - \int_\Omega F(x,u_p)\,\mathrm{d}x.
\end{aligned}$$

for all $p \in (1,\overline{p})$. Then, if $u_p \to 0$ in $L^r(\Omega)$, for $1 \le r < qN/(N-q)$, we have a contradiction. Hence, $u_0 \ne 0$.

So far, we have proved that $u_0 \in W_0^{1,q}(\Omega)$ is a solution of bounded variation of (1.5). To be more precise, we have proved that u_0 satisfies (3.29). Now, let us prove that u_0 is in fact a weak solution of (1.5), *i.e.*,

$$\int_\Omega \frac{\nabla u_0 \cdot \nabla v}{|\nabla u_0|}\,\mathrm{d}x + \int_\Omega |\nabla u_0|^{q-2}\nabla u_0 \cdot \nabla v\,\mathrm{d}x = \int_\Omega f(x,u_0)v\,\mathrm{d}x$$

holds for all $v \in W_0^{1,q}(\Omega)$. To this end, let us consider the functional Φ defined in (3.30). Note that

$$\Phi = \mathcal{J} - \mathcal{F},$$

where

$$\mathcal{J}(u) = \int_\Omega |\nabla u|\,\mathrm{d}x + \frac{1}{q}\int_\Omega |\nabla u|^q\,\mathrm{d}x \quad \text{and} \quad \mathcal{F}(u) = \int_\Omega F(x,u)\,\mathrm{d}x.$$

Since $\mathcal{J}$ is a convex and locally Lipschitz and $\mathcal{F} \in C^1(W_0^{1,q}(\Omega))$, the subdifferential of $\Phi(u_0)$, given by $\partial\Phi(u_0) \subset W^{-1,q'}(\Omega)$, is well defined. Moreover, $u_0 \in \partial\Phi(u_0)$ if and only if

$$\mathcal{J}(v) - \mathcal{J}(u_0) \ge \int_\Omega f(x,u_0)(v-u_0)\,\mathrm{d}x \quad \text{for all } v \in W_0^{1,q}(\Omega). \tag{3.39}$$

This, in turn, is equivalent to $\mathcal{F}'(u_0) \in \partial\mathcal{J}(u_0)$.

Furthermore, note that $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_q$, where

$$\mathcal{J}_1(u) = \int_\Omega |\nabla u|\,\mathrm{d}x \quad \text{and} \quad \mathcal{J}_q(u) = \frac{1}{q}\int_\Omega |\nabla u|^q\,\mathrm{d}x.$$

Since $\mathcal{J}_q \in C^1\left(W_0^{1,q}(\Omega),\mathbb{R}\right)$ and it is convex, we infer

$$\mathcal{J}_q'(u_0)(v-u_0) \le \mathcal{J}_q(v) - \mathcal{J}_q(u_0) \quad \text{for all } v \in W_0^{1,q}(\Omega). \tag{3.40}$$

Lemma 3.5. *The solution u_0 of bounded variation is such that*

$$\mathcal{F}'(u_0) \in \partial\mathcal{J}(u_0).$$

Proof. For $v \in W_0^{1,q}(\Omega)$, let us take $(v - u_0)$ as test function in (3.29). Then it follows

$$-\int_\Omega \operatorname{div} \mathbf{z}(v-u_0)\,dx + \int_\Omega |\nabla u_0|^{q-2}\nabla u_0 \cdot \nabla(v-u_0)\,dx = \int_\Omega f(x,u_0)(v-u_0)\,dx.$$

The last equality, Green's formula (2.2), (2.1), Lemma 3.4 and (3.40), imply that

$$\begin{aligned}
&\mathcal{F}'(u_0)(v-u_0)\\
&= -\int_\Omega v \operatorname{div}\mathbf{z}\,dx + \int_\Omega u_0 \operatorname{div}\mathbf{z}\,dx + \int_\Omega |\nabla u_0|^{q-2}\nabla u_0 \cdot \nabla v\,dx - \int_\Omega |\nabla u_0|^q\,dx\\
&= \int_\Omega \mathbf{z}\cdot\nabla v\,dx - \int_\Omega \mathbf{z}\cdot\nabla u_0\,dx + \int_\Omega |\nabla u_0|^{q-2}\nabla u_0\cdot\nabla v\,dx - \int_\Omega |\nabla u_0|^q\,dx\\
&\le \int_\Omega |\nabla v|\,dx - \int_\Omega |\nabla u_0|\,dx + \mathcal{J}_q'(u_0)(v-u_0)\\
&\le \int_\Omega |\nabla v|\,dx - \int_\Omega |\nabla u_0|\,dx + \mathcal{J}_q(v) - \mathcal{J}_q(u_0)\\
&= \mathcal{J}(v) - \mathcal{J}(u_0).
\end{aligned}$$

Hence, $\mathcal{F}'(u_0) \in \partial\mathcal{J}(u_0)$. □

Since we know that (3.39) holds for all $v \in W_0^{1,q}(\Omega)$, by considering $v = u_0 + tw$ as test function and making $t \to 0^\pm$, we find that

$$\int_\Omega \frac{\nabla u_0\cdot\nabla w}{|\nabla u_0|}\,dx + \int_\Omega |\nabla u_0|^{q-2}\nabla u_0\cdot\nabla w\,dx = \int_\Omega f(x,u_0)w\,dx.$$

Then, u_0 is a nonnegative weak nontrivial solution of (1.5).

The existence of a nonpositive weak solution v_0 of (1.5) can be shown in the same way, just dealing with the functional defined truncating the negative part of $f(x,\cdot)$. The Theorem 1.1 is proved.

4. Existence of nodal solutions

In the proof of Theorem 1.2, as in the previous section, we approximate the nodal solution we are looking for by the solutions of the (p,q)-Laplacian problem (3.1).

Let us consider the energy functional $I_p : W_0^{1,q}(\Omega) \to \mathbb{R}$ given by

$$I_p(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla u|^q\,dx - \int_\Omega F(x,u)\,dx.$$

Again, as in the previous section, we consider

$$\Phi_p(u) := I_p(u) + \frac{p-1}{p}|\Omega|$$

and note that Φ_p and I_p have the very same critical points.

In order to get nodal solutions of (3.1), let us consider the so called nodal Nehari set

$$\mathcal{M}_p = \left\{u \in W_0^{1,q}(\Omega) : u^\pm \neq 0 \text{ and } \langle \Phi_p'(u), u^\pm\rangle = 0\right\}.$$

As before, we denote by $\mathcal{N}_p$ the usual Nehari manifold associated to (3.1). From Gasiński–Winkert [23], for every $u \in W_0^{1,q}(\Omega)\setminus\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_p$. This implies that for every $u \in W_0^{1,q}(\Omega)$ such that $u^\pm \neq 0$, there exist a unique pair $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$, such that $tu^+ + su^- \in \mathcal{M}_p$. Moreover, again by Gasiński–Winkert [23], if $u \in \mathcal{N}_p$,

$$\Phi_p(u) = \max_{t>0} \Phi_p(tu).$$

Hence, if $u \in \mathcal{M}_p$, then

$$\Phi_p(u) = \max_{t,s>0} \Phi_p(tu^+ + su^-). \tag{4.1}$$

By Gasiński–Winkert [23], there exists a nodal solution $v_p \in W_0^{1,q}(\Omega)$ of (3.1) such that

$$\Phi_p(v_p) = \min_{\mathcal{M}_p} \Phi_p. \tag{4.2}$$

Hence, since by its definition $\mathcal{M}_p$ contains all sign-changing solutions, it follows that v_p is a nodal solution with the lowest energy level among all the sign-changing ones.

Lemma 4.1. *The family $(\Phi_p(v_p))_p$ is nondecreasing for $p \in (1,q)$.*

Proof. Let $1 < p_1 \le p_2 < q$ and $v_{p_1}, v_{p_2} \in W_0^{1,q}(\Omega)$ satisfying (4.2). Since $v_{p_2}^{\pm} \neq 0$, there exist $t, s > 0$ such that

$$tv_{p_2}^+ + sv_{p_2}^- \in \mathcal{N}_{p_1}^{\pm}. \tag{4.3}$$

Then, from (3.2), (4.1), (4.2) and (4.3), it follows that

$$\Phi_{p_2}(v_{p_2}) \ge \Phi_{p_2}(tv_{p_2}^+ + sv_{p_2}^-) \ge \Phi_{p_1}(tv_{p_2}^+ + sv_{p_2}^-) \ge \Phi_{p_1}(v_{p_1}). \quad \square$$

Lemma 4.2. *The family* $(v_p)_{1<p<q}$ *is bounded in* $W_0^{1,q}(\Omega)$.

Proof. The proof is analogous to Lemma 3.2. $\square$

As in Section 3, it follows that there exists $v_0 \in W_0^{1,q}(\Omega)$ such that

$$\begin{aligned}
&v_p \rightharpoonup v_0 \quad \text{in } W_0^{1,q}(\Omega),\\
&v_p \to v_0 \quad \text{in } L^r(\Omega) \text{ for all } 1 \le r < \frac{qN}{N-q},\\
&v_p \to v_0 \quad \text{a.e. in } \Omega,
\end{aligned}$$

as $p \to 1^+$. Moreover, with the same arguments, one can prove that v_0 is a weak solution of (3.1), *i.e.*,

$$\int_\Omega \frac{\nabla v_0 \cdot \nabla w}{|\nabla v_0|}\,\mathrm{d}x + \int_\Omega |\nabla v_0|^{q-2}\nabla v_0 \cdot \nabla w\,\mathrm{d}x = \int_\Omega f(x, v_0) w\,\mathrm{d}x,$$

for all $w \in W_0^{1,q}(\Omega)$.

Then, in order to complete the proof of Theorem 1.2, we just should prove that $v_0^{\pm} \neq 0$. To this end, let us remember that from (3.38), there exists $\rho > 0$ sufficiently small such that, if $\|v\|_{1,q} \le \rho$, then

$$\Phi(v) \ge \frac{\|v\|_{1,q}^q}{2}, \tag{4.4}$$

where Φ is given by (3.30). Note that since v_p belongs to the Nehari nodal set $\mathcal{M}_p$, then $v_p^{\pm} \in \mathcal{N}_p$. Hence, $s = 1$ is the maximum of the function $s \mapsto \Phi_p(sv_p^{\pm})$ and so, for all $p \in (1, q)$, from (4.4),

$$\Phi_p(v_p^{\pm}) \ge \Phi_p\left(\frac{\rho v_p^{\pm}}{\|v_p^{\pm}\|}\right) \ge \Phi\left(\frac{\rho v_p^{\pm}}{\|v_p^{\pm}\|}\right) \ge \frac{\rho}{2}. \tag{4.5}$$

Then, from (4.5), we obtain

$$\begin{aligned}
\frac{\rho}{2} &\le \Phi_p(v_p^{\pm})\\
&\le \|\nabla v_p^{\pm}\|_p^p + \|\nabla v_p^{\pm}\|_q^q - \int_\Omega F(x, v_p^{\pm})\,\mathrm{d}x\\
&= \int_\Omega \left(f(x, v_p^{\pm})v_p^{\pm} - F(x, v_p^{\pm})\right)\mathrm{d}x.
\end{aligned} \tag{4.6}$$

Then, from (4.6), if $v_p^{\pm} \to 0$ in $L^r(\Omega)$ as $p \to 1^+$, we see that (4.5) would not hold. Therefore, $v_0^{\pm} \neq 0$ and this finishes the proof of Theorem 1.2.

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors would like to deeply thank the anonymous referees, for valuable comments and suggestions. Marcos T.O. Pimenta is partially supported by FAPESP 2021/04158-4 and 2023/06617-1, CNPq 304765/2021-0, Brazil. Giovany M. Figueiredo is partially supported by FAPESP 2023/10287-7, CNPq and FAPDF. Marcos T.O. Pimenta thanks the University of Technology Berlin for the kind hospitality during a research stay in February/March 2024. Marcos T.O. Pimenta and Patrick Winkert were financially supported by TU Berlin-FAPESP Mobility Promotion.

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