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On the first eigenvalue of the (p, q)-Laplacian and some related problems

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Abstract

The aim of this paper is to establish the existence of the first (smallest) eigenvalue λ_1 for a nonlinear elliptic problem driven by the nonhomogeneous (p, q)-Laplace operator $-\Delta_p - \Delta_q$ in a bounded domain with a source term involving the exponent γ with $q < \gamma \leq p$. We show that λ_1 is simple and associated to a unique and bounded eigenfunction $u_1 > 0$. In the second part, using variational arguments, we study two types of nonlinear problems involving the nonhomogeneous (p, q)-Laplace operator, in particular we study two classes of sublinear and superlinear (p, q)-Laplacian problems with parameters.

Keywords (p, q)-Laplace operator · Smooth bounded domains · The first eigenvalue · Simplicity · Sublinear and superlinear problems

Mathematics Subject Classification Primary 35E05 ; Secondary 35E20 · 37L25 · 46E35

1 Introduction

In this paper, we are concerned with the study of a nonlinear eigenvalue problem involving a differential operator of (p, q)-Laplacian type of the form

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$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{\gamma-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{PQ}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $1 < q < \gamma \leq p < +\infty$, λ is a real parameter and $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ denotes the classical r-Laplacian for $1 < r < \infty$. Besides this, we pay our attention to the study of two kinds of problems involving the (p, q)-Laplacian. Namely, we study some sublinear and superlinear problems with parameters where the parameter is present only in the source term in the sublinear case and in the differential operator for the superlinear problem.

When $q = p = \gamma$, problem (PQ) reduces to the well-known eigenvalue problem $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2} u = 0$ in which the study of the properties of the first eigenvalue problem and the associated eigenfunction has been extensively studied by several authors both in regular and irregular domains, see e.g. Anane [1], Lê [17], Lindqvist [19, 20], Kawohl and Lindqvist [16] for a detailed study. For the case of nonlinear elliptic systems of two second order quasilinear partial differential equations, in particular, de Thélin [10] obtained the existence of the first eigenvalue associated to a unique and bounded eigenfunction for a weakly eigenvalue coupled system where the interaction of variables is present only in the source terms, while in both equations the differential terms involved the differential operators (Δ_p, Δ_q), and have only one dependent variable each.

Let us also mention that several studies have been devoted recently to the investigation of related problems and a lot of papers have appeared dealing with problems involving (p, q)-Laplacian in both bounded and unbounded domains. For the references and therein, see e.g. Baldelli and Filippucci [2], Baldelli et al. [3, 4], Bobkov and Tanaka [6–8], Candito et al. [9], El Manouni et al. [14], Motreanu and Tanaka [23]. Let us point out that the (p, q)-Laplacian has great background in applications, we mention e.g., biophysics, plasma physics, reaction-diffusion equations, and models of elementary particles, etc.

Concerning problem (PQ), we show that there is a smallest eigenvalue $\lambda_1 > 0$ associated to a unique eigenfunction $u_1 > 0$ in Ω such that $\int_\Omega u_1^\gamma \, dx = 1$. Moreover, the regularity result in terms of global L^∞-estimates is obtained via a technique based on a construction argument of exponent sequences and an iteration scheme as well as truncation arguments to bound the maximal norm of the solution. As far as the uniqueness result is concerned, it should be noted that this appears to be very interesting and is proven differently in the two cases $q < \gamma < p$ and $\gamma = p$.

Regarding the sublinear case, let us point out that more works have been done in this direction in the case of nonlinear problems involving the p-Laplacian. We can cite Maya and Shivaji [22] in the semilinear case $p = 2$ and Perera [24] for scalar equations and El Manouni and Perera [13] when $p > 1$ and $p \neq 2$ for systems of two second order quasilinear equations. Recently, El Manouni et al. [14] have studied the existence and nonexistence of nontrivial solutions for some quasilinear elliptic problems driven by the nonhomogeneous (p, q)-Laplace operator depending on two parameters in bounded and unbounded domains.

Regarding the superlinear case, we consider a different type of problem with one parameter for the (p, q)-Laplacian. In particular a parameter μ appears in the differential operator side, that is, we consider $-\Delta_p u - \mu \Delta_q u$, $\mu > 0$. The idea is to construct a problem involving $-\Delta_p u - \Delta_q u$, from which we deduce the existence of a real number $\mu > 0$ corresponding to a unique solution up to multiplication with constants.

2 The first eigenvalue and associated positive eigenfunction of $-\Delta_p - \Delta_q$

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider the following Dirichlet problem

$$\begin{aligned} -\Delta_p u - \Delta_q u &= \lambda |u|^{\gamma-2}u \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $1 < q < \gamma \leq p < \infty$ and λ is a real number.

In this section, we are interested to the first eigenvalue of the nonhomogeneous operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u)$ as the least real number λ_1 for which the Eq. (2.1) has a nontrivial solution u_1 with homogeneous Dirichlet boundary value conditions. Namely, λ_1 will be obtained as the minimum of a slight variant of so called Rayleigh quotient

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\dfrac{\gamma}{p}\displaystyle\int_\Omega |\nabla u|^p \, dx + \dfrac{\gamma}{q}\displaystyle\int_\Omega |\nabla u|^q \, dx}{\displaystyle\int_\Omega |u|^\gamma \, dx}. \tag{2.2}$$

In this case, we say that λ_1 is the first eigenvalue and the corresponding eigenfunction u_1 is called the first eigenfunction.

Remark that in the case $\gamma = p$, the first eigenvalue λ_1 given in (2.2) coincides with the first eigenvalue of the p-Laplacian. Indeed, it is clear that λ_1 is greater than or equal to the first eigenvalue of the p-Laplacian. Taking $u = s\varphi_1$, where φ_1 is a first eigenfunction of the p-Laplacian with $s > 0$, we get

$$\lambda_1 \leq \frac{\displaystyle\int_\Omega |\nabla \varphi_1|^p \, dx + s^{q-p}\frac{p}{q}\int_\Omega |\nabla \varphi_1|^q \, dx}{\displaystyle\int_\Omega |\varphi_1|^p \, dx}.$$

Letting $s \to \infty$ shows that λ_1 is less than or equal to the first eigenvalue of the p-Laplacian.

Next, we define the following functionals I and J on $W_0^{1,p}(\Omega)$ by

$$\begin{aligned} I(u) &= \frac{\gamma}{p}\int_\Omega |\nabla u|^p \, dx + \frac{\gamma}{q}\int_\Omega |\nabla u|^q \, dx, \\ J(u) &= \int_\Omega |u|^\gamma \, dx. \end{aligned} \tag{2.3}$$

Consider the minimization problem

$$\inf_{u \in W_0^{1,p}(\Omega)} I(u), \quad J(u) = 1. \tag{P}$$

By a weak solution of (2.1), we mean any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi \, dx = \lambda \int_\Omega |u|^{\gamma-2}u\varphi \, dx \tag{2.4}$$

is satisfied for all $\varphi \in W_0^{1,p}(\Omega)$. The corresponding real number λ is called an eigenvalue and u is an associated eigenfunction. Note that here we obtain solutions of (P) that allow to find $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ satisfying (2.4). We point out that the assumptions on p, q and γ guarantee that the integrals in (2.4) are well-defined if $u, \varphi \in W_0^{1,p}(\Omega)$.

Before we state the first main theorem of this section, we recall the following result due to Berger [5, Theorem 6.3.2, p. 325].

Theorem 2.1 *Suppose that the C^1-functionals $\mathcal{A}$ and $\mathcal{B}$ defined on the reflexive Banach space X have the following properties:*

(i) $\mathcal{A}(x)$ is weakly lower semicontinuous and coercive on $X \cap \{\mathcal{B}(x) \le const.\}$;
(ii) $\mathcal{B}(x)$ is continuous with respect to weak sequential convergence and $\mathcal{B}'(x) = 0$ only at $x = 0$.

Then the equation $\mathcal{A}'(x) = \lambda \mathcal{B}'(x)$ has a one-parameter family of nontrivial solutions (x_R, λ_R) for all R in the range of $\mathcal{B}(x)$ such that $\mathcal{B}(x_R) = R$ and x_R is characterized as the minimum of $\mathcal{A}(x)$ over the set $\mathcal{B}(x) = R$.

Theorem 2.2 *Let $1 < q < \gamma \le p$. Assume that for $\gamma \ne p$, there is $\mathcal{C} \gg 1$ large enough such that*

$$q < \gamma \le \frac{qp(1+\mathcal{C})}{q + \mathcal{C}p} < p. \tag{2.5}$$

Then there exists a least eigenvalue λ_1 associated to an eigenfunction $u_1 \ne 0$ with $u_1 \ge 0$ satisfying (2.4). Moreover, $u_1 > 0$ and $u_1 \in L^\infty(\Omega) \cap C^{1,\eta}(\overline{\Omega})$ for some $\eta > 0$.

Proof It is well known that I and J are of class C^1. Furthermore, the set $\{u \in W_0^{1,p}(\Omega) : J(u) = 1\}$ is closed for the weak sequential convergence. Indeed, let $u_n \in W_0^{1,p}(\Omega)$ be a sequence such that $J(u_n) = 1$ and $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. Since $q < \gamma \le p$, write $\gamma = tp + (1-t)q$ for $t \in (0, 1]$. Then by the Rellich-Kondrachov compact embeddings, we get for a subsequence, still denoted by u_n, that $|u_n|^{tp} \mapsto |u|^{tp}$ strongly in $L^{\frac{1}{t}}(\Omega)$ and $|u_n|^{(1-t)q} \mapsto |u|^{(1-t)q}$ strongly in $L^{\frac{1}{1-t}}(\Omega)$. Now using the dominated convergence theorem and Hölder's inequality, we get $|u_n|^\gamma \mapsto |u|^\gamma$ strongly in $L^1(\Omega)$, and so $J(u) = 1$. Also, $J'(u) = 0$ implies that $J(u) = 0$, and so $u = 0$. Hence, in view of Theorem 2.1, problem (P) has a solution $u_1 \not\equiv 0$ and there exists λ_1 such that $I'(u_1) = \lambda_1 J'(u_1)$. Thus we obtain the existence of a non-trivial solution of (2.1) for $\lambda = \lambda_1$. Also, we have $u_1 \ge 0$ since $I(u_1) = I(|u_1|)$.

Moreover, we show that λ_1 is the least eigenvalue of (P). Indeed, for $\gamma \ne p$, remark first that (2.5) implies $\frac{p-\gamma}{p} \ge \mathcal{C}\frac{\gamma-q}{q}$. Let u be a solution of (2.1) associated to λ satisfying $J(u) = 1$. We have

$$\begin{aligned} &\frac{p-\gamma}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\gamma-q}{q}\int_\Omega |\nabla u|^q \,\mathrm{d}x \\ &\quad \ge \frac{\gamma-q}{q}\left(\mathcal{C}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega |\nabla u|^q \,\mathrm{d}x\right) \ge 0. \end{aligned}$$

Testing the weak formulation (2.4) by u and using the last inequality leads to

$$\lambda = \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |\nabla u|^q \,\mathrm{d}x \ge I(u) \ge I(u_1).$$

Hence, $\lambda \ge \lambda_1$. On the other hand, as mentioned before for the case $\gamma = p$, λ_1 coincides with the first eigenvalue of the p-Laplacian, and we have

$$\lambda = \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |\nabla u|^q \,\mathrm{d}x \ge \int_\Omega |\nabla u|^p \,\mathrm{d}x \ge \lambda_1.$$

Hence λ_1 is the smallest eigenvalue of problem (2.1). This shows the first assertion of the theorem.

Now we prove the second assertion. For this purpose, let $1 < c < \frac{p^*}{p}$ such that $W_0^{1,p}(\Omega) \hookrightarrow L^{cp}(\Omega)$. Fix the following sequences of numbers

$$p_k = pc^k, \quad q_k = qc^k, \quad m_k = (c^k - 1)p.$$

Let us choose $u_1^{1+m_k}$ as a test function in (2.4). Then for more restrictions on c, obviously $u_1^{1+m_k} \in W_0^{1,p}(\Omega)$, and it follows that

$$\begin{aligned}
&\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla(u_1^{1+m_k})\,\mathrm{d}x + \int_\Omega |\nabla u_1|^{q-2}\nabla u_1 \cdot \nabla(u_1^{1+m_k})\,\mathrm{d}x \\
&\quad = \lambda_1 \int_\Omega |u_1|^{\gamma-2} u_1 u_1^{1+m_k}\,\mathrm{d}x.
\end{aligned}$$

On the one hand, we have

$$\begin{aligned}
&\left\| u_1^{c^k} \right\|_{cp}^p \le C_1 \left\| \nabla(u_1^{c^k}) \right\|_p^p \\
&= C_1 c^{kp} \int_\Omega |\nabla u_1|^p u_1^{m_k}\,\mathrm{d}x \\
&\le C_1 c^{kp} \left(\int_\Omega |\nabla u_1|^p u_1^{m_k}\,\mathrm{d}x + \int_\Omega |\nabla u_1|^q u_1^{m_k}\,\mathrm{d}x \right) \\
&= C_1 \frac{c^{kp}}{1+m_k} \left(\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla(u_1^{1+m_k})\,\mathrm{d}x + \int_\Omega |\nabla u_1|^{q-2}\nabla u_1 \nabla(u_1^{1+m_k})\,\mathrm{d}x \right) \\
&= \lambda_1 C_1 \frac{c^{kp}}{1+m_k} \int_\Omega u_1^{\gamma+m_k}\,\mathrm{d}x,
\end{aligned}$$

where C_1 is a positive constant. Since $c^k - 1 \le m_k$ and $\gamma = tp + (1-t)q$ with $t \in (0, 1]$, it follows that

$$\left(\|u_1\|_{p_{k+1}}^{p_{k+1}} \right)^{\frac{1}{c}} = \left\| u_1^{c^k} \right\|_{cp}^p \le \lambda_1 C_1 c^{k(p-1)} \int_\Omega u_1^{tp+(1-t)q+m_k}\,\mathrm{d}x.$$

Since

$$\frac{tp + m_k}{p_k} + \frac{(1-t)q}{q_k} = 1,$$

we get

$$\left(\|u_1\|_{p_{k+1}}^{p_{k+1}} \right)^{\frac{1}{c}} \le \lambda_1 C_1 c^{k(p-1)} \left(\|u_1\|_{p_k}^{p_k} \right)^{\frac{tp+m_k}{p_k}} \left(\|u_1\|_{q_k}^{q_k} \right)^{\frac{(1-t)q}{q_k}}.$$

On the other hand, since $q < p$ and $q_k = \frac{qp_k}{p}$ there is $C_2 > 0$ such that

$$\|u_1\|_{q_k}^{q_k} = \int_\Omega u_1^{q_k}\,\mathrm{d}x = \int_\Omega u_1^{\frac{qp_k}{p}}\,\mathrm{d}x \le C_2 \left(\int_\Omega u_1^{p_k}\,\mathrm{d}x \right)^{\frac{q}{p}} = C_2 \left(\|u_1\|_{p_k}^{p_k} \right)^{\frac{q}{p}}.$$

Consequently, we deduce

$$\left(\|u_1\|_{p_{k+1}}^{p_{k+1}} \right)^{\frac{1}{c}} \le \lambda_1 C_1 C_2^{\frac{(1-t)q}{q_k}} c^{k(p-1)} \left(\|u_1\|_{p_k}^{p_k} \right)^{\frac{tp+m_k}{p_k}} \left(\|u_1\|_{p_k}^{p_k} \right)^{\frac{(1-t)q}{p_k}}.$$

Put $E_k = p_k \log \max\{1, \|u_1\|_{p_k}\}$ and

$$a = c(p-1)\log c, \quad b = c \log \lambda_1 C_1, \quad \alpha_k = \frac{1-t}{c^{k-1}} \log C_2, \quad \beta_k = c - \frac{(1-t)(p-q)}{pc^{k-1}}.$$

This yields the relation

$$E_{k+1} \le \beta_k E_k + \alpha_k + b + ak.$$

Remark that $\beta_k \ge 0,\ \beta_k \le c$ for all k and α_k is a bounded sequence. Then we have

$$E_{k+1} \le cE_k + \mu + ak,$$

where μ is a real number such that $\mu > 1$. Set $r_k = \mu + ak$, we obtain

$$E_{k+1} \le cE_k + r_k \le c^k E_1 + r_k + cr_{k-1} + \cdots + c^{k-1} r_1.$$

Thus

$$E_{k+1} \le c^k \left(E_1 + \frac{r_1}{c} + \frac{r_2}{c^2} + \cdots + \frac{r_k}{c^k} \right).$$

This implies that

$$\begin{aligned} E_{k+1} &\le c^k \left(E_1 + \frac{\mu + a}{c} + \frac{\mu + 2a}{c^2} + \cdots + \frac{\mu + ak}{c^k} \right) \\ &= c^k \left(E_1 + \mu \left(\frac{1}{c} + \frac{1}{c^2} + \cdots + \frac{1}{c^k} \right) + a \left(\frac{1}{c} + \frac{2}{c^2} + \cdots + \frac{k}{c^k} \right) \right) \\ &= c^k \left(E_1 + \frac{\mu}{c-1} + \frac{a}{(c-1)^2} \right). \end{aligned}$$

Finally we derive

$$\|u_1\|_\infty \le \limsup_{k\to\infty} e^{\frac{E_k}{p_k}} \le \limsup_{k\to\infty} e^{\frac{dc^{k-1}}{p_k}} = e^{\frac{d}{pc}},$$

where $d = E_1 + \frac{\mu}{c-1} + \frac{a}{(c-1)^2}$. Therefore $u_1 \in L^\infty(\Omega)$. Furthermore, since $u_1 \in L^\infty(\Omega)$, due to the regularity result of Lieberman [18], u_1 belongs to $C^{1,\eta}(\overline{\Omega})$ with some $\eta \in (0,1)$ and the maximum principle of Vázquez [26] shows that $u_1 > 0$ in Ω. □

The next theorem states the uniqueness of the first eigenfunction associated to the first eigenvalue λ_1.

Theorem 2.3 *If $1 < q < \gamma < p$ satisfying the following condition*

$$\begin{aligned} &\left(\left(\frac{\gamma}{p} - 1)(1 - s^\gamma \right) + \frac{\gamma}{p} s^{p-\gamma} \right) a_1^p + \left(\left(\frac{\gamma}{p} - 1 \right) \left(1 - \left(\frac{1}{s} \right)^\gamma \right) + \frac{\gamma}{p} \left(\frac{1}{s} \right)^{p-\gamma} \right) a_2^p \\ &\quad + \left(\left(\frac{\gamma}{q} - 1 \right) (1 - s^\gamma) + \frac{\gamma}{q} \left(\frac{1}{s} \right)^{\gamma - q} \right) a_1^q + \left(\left(\frac{\gamma}{q} - 1 \right) \left(1 - \left(\frac{1}{s} \right)^\gamma \right) + \frac{\gamma}{q} s^{\gamma - q} \right) a_2^q \\ &\quad < \frac{\gamma}{p}(a_1^p + a_2^p) + \frac{\gamma}{q}(a_1^q + a_2^q), \end{aligned} \tag{2.6}$$

where s, a_1, a_2 are positive real numbers with $s \ne 1$, or if $\gamma = p$, then the first eigenfunction u_1 associated to λ_1 is unique.

Proof We will distinguish between two cases: $q < \gamma < p$ with condition (2.6) and $\gamma = p$. Let us first consider the case $q < \gamma < p$ satisfying (2.6) and let u_1, u_2 be two eigenfunctions associated to λ_1 with $u_1 \neq u_2$. We will use the two functions

$$u_1 - \frac{u_2^\gamma}{u_1^{\gamma-1}} \quad \text{and} \quad u_2 - \frac{u_1^\gamma}{u_2^{\gamma-1}}$$

as test functions in (2.4). Then

$$\begin{aligned}
&\int_\Omega -\Delta_p u_1\left(u_1 - \frac{u_2^\gamma}{u_1^{\gamma-1}}\right)\mathrm{d}x + \int_\Omega -\Delta_p u_2\left(u_2 - \frac{u_1^\gamma}{u_2^{\gamma-1}}\right)\mathrm{d}x\\
&\quad + \int_\Omega -\Delta_q u_1\left(u_1 - \frac{u_2^\gamma}{u_1^{\gamma-1}}\right)\mathrm{d}x + \int_\Omega -\Delta_q u_2\left(u_2 - \frac{u_1^\gamma}{u_2^{\gamma-1}}\right)\mathrm{d}x\\
&\quad = \lambda_1 \int_\Omega \left[u_1^{\gamma-1}\left(u_1 - \frac{u_2^\gamma}{u_1^{\gamma-1}}\right) + u_2^{\gamma-1}\left(u_2 - \frac{u_1^\gamma}{u_2^{\gamma-1}}\right)\right]\mathrm{d}x.
\end{aligned}$$

Clearly, the right-hand side is equal to 0 and a simple calculation shows that

$$\nabla\left(u_1 - \frac{u_2^\gamma}{u_1^{\gamma-1}}\right) = \left(1 + (\gamma-1)\left(\frac{u_2}{u_1}\right)^\gamma\right)\nabla u_1 - \gamma\left(\frac{u_2}{u_1}\right)^{\gamma-1}\nabla u_2.$$

Similarly for $\nabla\left(u_2 - \frac{u_1^\gamma}{u_2^{\gamma-1}}\right)$ by interchanging u_1 and u_2. Thus we obtain

$$\begin{aligned}
&\int_\Omega \left[\left(1 + (\gamma-1)\left(\frac{u_2}{u_1}\right)^\gamma\right)|\nabla u_1|^p + \left(1 + (\gamma-1)\left(\frac{u_1}{u_2}\right)^\gamma\right)|\nabla u_2|^p\right]\mathrm{d}x\\
&- \int_\Omega \left[\gamma\left(\frac{u_2}{u_1}\right)^{\gamma-1}|\nabla u_1|^{p-2}\nabla u_1\cdot\nabla u_2 + \gamma\left(\frac{u_1}{u_2}\right)^{\gamma-1}|\nabla u_2|^{p-2}\nabla u_2\cdot\nabla u_1\right]\mathrm{d}x\\
&+ \int_\Omega \left[\left(1 + (\gamma-1)\left(\frac{u_2}{u_1}\right)^\gamma\right)|\nabla u_1|^q + \left(1 + (\gamma-1)\left(\frac{u_1}{u_2}\right)^\gamma\right)|\nabla u_2|^q\right]\mathrm{d}x\\
&- \int_\Omega \left[\gamma\left(\frac{u_2}{u_1}\right)^{\gamma-1}|\nabla u_1|^{q-2}\nabla u_1\cdot\nabla u_2 + \gamma\left(\frac{u_1}{u_2}\right)^{\gamma-1}|\nabla u_2|^{q-2}\nabla u_2\cdot\nabla u_1\right]\mathrm{d}x\\
&=: \Gamma_1 + \Gamma_2\\
&= 0,
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_1 = &\int_\Omega (\xi_1 - \xi_2)(|\nabla \log u_1|^p - |\nabla \log u_2|^p)\,\mathrm{d}x\\
&- \int_\Omega p\xi_2|\nabla \log u_1|^{p-2}\nabla \log u_1\cdot(\nabla \log u_2 - \nabla \log u_1)\,\mathrm{d}x\\
&- \int_\Omega p\xi_1|\nabla \log u_2|^{p-2}\nabla \log u_2\cdot(\nabla \log u_1 - \nabla \log u_2)\,\mathrm{d}x\\
&+ \int_\Omega (\xi_1' - \xi_2')(|\nabla \log u_1|^q - |\nabla \log u_2|^q)\,\mathrm{d}x\\
&- \int_\Omega q\xi_2'|\nabla \log u_1|^{q-2}\nabla \log u_1\cdot(\nabla \log u_2 - \nabla \log u_1)\,\mathrm{d}x\\
&- \int_\Omega q\xi_1'|\nabla \log u_2|^{q-2}\nabla \log u_2\cdot(\nabla \log u_1 - \nabla \log u_2)\,\mathrm{d}x,
\end{aligned}$$

with

$$\xi_1 = \frac{\gamma}{p}\frac{u_1^\gamma}{u_2^{\gamma-p}}, \quad \xi_2 = \frac{\gamma}{p}\frac{u_2^\gamma}{u_1^{\gamma-p}}, \quad \xi_1' = \frac{\gamma}{q}\frac{u_1^\gamma}{u_2^{\gamma-q}} \quad \text{and} \quad \xi_2' = \frac{\gamma}{q}\frac{u_2^\gamma}{u_1^{\gamma-q}},$$

and

$$\begin{aligned}
\Gamma_2 = &\int_\Omega \Big[\Big(1-(1-\frac{\gamma}{p})\Big(\frac{u_2}{u_1}\Big)^\gamma - \frac{\gamma}{p}\Big(\frac{u_2}{u_1}\Big)^{p-\gamma}\Big)|\nabla u_1|^p \\
&+ \Big(1-(1-\frac{\gamma}{p})\Big(\frac{u_1}{u_2}\Big)^\gamma - \frac{\gamma}{p}\Big(\frac{u_1}{u_2}\Big)^{p-\gamma}\Big)|\nabla u_2|^p\Big]\,\mathrm{d}x \\
&+ \int_\Omega \Big[\Big(1-(1-\frac{\gamma}{q})\Big(\frac{u_2}{u_1}\Big)^\gamma - \frac{\gamma}{q}\Big(\frac{u_2}{u_1}\Big)^{q-\gamma}\Big)|\nabla u_1|^q \\
&+ \Big(1-(1-\frac{\gamma}{q})\Big(\frac{u_1}{u_2}\Big)^\gamma - \frac{\gamma}{q}\Big(\frac{u_1}{u_2}\Big)^{q-\gamma}\Big)|\nabla u_2|^q\Big]\,\mathrm{d}x \\
= &\int_\Omega \Big[\Big(\frac{\gamma}{p}+(1-\frac{\gamma}{p})(1-\Big(\frac{u_2}{u_1}\Big)^\gamma) - \frac{\gamma}{p}\Big(\frac{u_2}{u_1}\Big)^{p-\gamma}\Big)|\nabla u_1|^p \\
&+ \Big(\frac{\gamma}{p}+(1-\frac{\gamma}{p})(1-\Big(\frac{u_1}{u_2}\Big)^\gamma) - \frac{\gamma}{p}\Big(\frac{u_1}{u_2}\Big)^{p-\gamma}\Big)|\nabla u_2|^p\Big]\,\mathrm{d}x \\
&+ \int_\Omega \Big[\Big(\frac{\gamma}{q}+(1-\frac{\gamma}{q})(1-\Big(\frac{u_2}{u_1}\Big)^\gamma) - \frac{\gamma}{q}\Big(\frac{u_2}{u_1}\Big)^{q-\gamma}\Big)|\nabla u_1|^q \\
&+ \Big(\frac{\gamma}{q}+(1-\frac{\gamma}{q})(1-\Big(\frac{u_1}{u_2}\Big)^\gamma) - \frac{\gamma}{q}\Big(\frac{u_1}{u_2}\Big)^{q-\gamma}\Big)|\nabla u_2|^q\Big]\,\mathrm{d}x.
\end{aligned}$$

On the one hand, by using the inequality $|b|^r \geq |a|^r + p|a|^{r-2}a(b-a)$ for $r \geq 1$, we deduce that $\Gamma_1 \geq 0$. Hence it follows that $\Gamma_2 \leq 0$.

On the other hand, in view of (2.6), we get

$$\begin{aligned}
\lambda_1 \leq &\frac{\int_\Omega \Big((\frac{\gamma}{p}-1)(1-\Big(\frac{u_2}{u_1}\Big)^\gamma) + \frac{\gamma}{p}\Big(\frac{u_2}{u_1}\Big)^{p-\gamma}\Big)|\nabla u_1|^p\,\mathrm{d}x}{\int_\Omega (u_1^\gamma + u_2^\gamma)\,\mathrm{d}x} \\
&+ \frac{\int_\Omega \Big((\frac{\gamma}{p}-1)(1-\Big(\frac{u_1}{u_2}\Big)^\gamma) + \frac{\gamma}{p}\Big(\frac{u_1}{u_2}\Big)^{p-\gamma}\Big)|\nabla u_2|^p\,\mathrm{d}x}{\int_\Omega (u_1^\gamma + u_2^\gamma)\,\mathrm{d}x} \\
&+ \frac{\int_\Omega \Big((\frac{\gamma}{q}-1)(1-\Big(\frac{u_2}{u_1}\Big)^\gamma) + \frac{\gamma}{q}\Big(\frac{u_2}{u_1}\Big)^{q-\gamma}\Big)|\nabla u_1|^q\,\mathrm{d}x}{\int_\Omega (u_1^\gamma + u_2^\gamma)\,\mathrm{d}x} \\
&+ \frac{\int_\Omega \Big((\frac{\gamma}{q}-1)(1-\Big(\frac{u_1}{u_2}\Big)^\gamma) + \frac{\gamma}{q}\Big(\frac{u_1}{u_2}\Big)^{q-\gamma}\Big)|\nabla u_2|^q\,\mathrm{d}x}{\int_\Omega (u_1^\gamma + u_2^\gamma)\,\mathrm{d}x} \\
< &\frac{\int_\Omega \big(\frac{\gamma}{p}|\nabla u_1|^p + \frac{\gamma}{p}|\nabla u_2|^p + \frac{\gamma}{q}|\nabla u_1|^q + \frac{\gamma}{q}|\nabla u_2|^q\big)\,\mathrm{d}x}{\int_\Omega (u_1^\gamma + u_2^\gamma)\,\mathrm{d}x}
\end{aligned}$$

$$= \frac{\lambda_1 \int_\Omega u_1^\gamma \,\mathrm{d}x + \lambda_1 \int_\Omega u_2^\gamma \,\mathrm{d}x}{\int_\Omega (u_1^\gamma + u_2^\gamma)\,\mathrm{d}x}$$
$$= \lambda_1,$$

which is a contradiction. Thus the first assertion is proved.

Suppose $\gamma = p$ and let u_1 and u_2 be two positive eigenfunctions associated to λ_1. Using $v = (u_1^p + u_2^p)^{\frac{1}{p}}$ as test function in (2.4), we have for $i = 1, 2$

$$\lambda_1 = \frac{\int_\Omega |\nabla u_i|^p \,\mathrm{d}x + \frac{p}{q}\int_\Omega |\nabla u_i|^q \,\mathrm{d}x}{\int_\Omega |u_i|^p \,\mathrm{d}x} \le \frac{\int_\Omega |\nabla v|^p \,\mathrm{d}x + \frac{p}{q}\int_\Omega |\nabla v|^q \,\mathrm{d}x}{\int_\Omega |v|^p \,\mathrm{d}x}. \tag{2.7}$$

A simple calculation shows that

$$\nabla v = v\left(\frac{u_1^p \nabla \log u_1 + u_2^p \nabla \log u_2}{u_1^p + u_2^p}\right).$$

By Jensen's inequality for convex functions, it follows that

$$\begin{aligned} |\nabla v|^p &\le v^p \left(\frac{u_1^p |\nabla \log u_1|^p}{u_1^p + u_2^p} + \frac{u_2^p |\nabla \log u_2|^p}{u_1^p + u_2^p}\right) \\ &= v^p \left(\frac{|\nabla u_1|^p}{u_1^p + u_2^p} + \frac{|\nabla u_2|^p}{u_1^p + u_2^p}\right) \\ &= |\nabla u_1|^p + |\nabla u_2|^p. \end{aligned} \tag{2.8}$$

Similarly, since $u_1, u_2 \le v$ and $q < p$, we have

$$\begin{aligned} |\nabla v|^q &\le v^q \left(\frac{u_1^p |\nabla \log u_1|^q}{u_1^p + u_2^p} + \frac{u_2^p |\nabla \log u_2|^q}{u_1^p + u_2^p}\right) \\ &\le v^{q-p}\left(u_1^{p-q}|\nabla u_1|^q + u_2^{p-q}|\nabla u_2|^q\right) \\ &\le |\nabla u_1|^q + |\nabla u_2|^q. \end{aligned} \tag{2.9}$$

Note that the inequalities (2.8) and (2.9) are strict when $\nabla \log u_1 \ne \nabla \log u_2$. Integrating (2.8) and (2.9), using (2.7), and if $\nabla \log u_1 \ne \nabla \log u_2$ on a set of positive measure, we get the following contradiction

$$\begin{aligned} \lambda_1 &\le \frac{\int_\Omega |\nabla v|^p \,\mathrm{d}x + \frac{p}{q}\int_\Omega |\nabla v|^q \,\mathrm{d}x}{\int_\Omega (u_1^p + u_2^p)\,\mathrm{d}x} \\ &< \frac{\int_\Omega |\nabla u_1|^p \,\mathrm{d}x + \frac{p}{q}\int_\Omega |\nabla u_1|^q \,\mathrm{d}x}{\int_\Omega (u_1^p + u_2^p)\,\mathrm{d}x} + \frac{\int_\Omega |\nabla u_2|^p \,\mathrm{d}x + \frac{p}{q}\int_\Omega |\nabla u_2|^q \,\mathrm{d}x}{\int_\Omega (u_1^p + u_2^p)\,\mathrm{d}x} = \lambda_1. \end{aligned}$$

Therefore, $\nabla \log u_1 = \nabla \log u_2$, so that $u_2 = \sigma u_1$ for some positive constant σ. Moreover, since u_2 is an eigenfunction associated to λ_1, we have

$$\int_\Omega |\nabla(\sigma u_1)|^p \,\mathrm{d}x + \int_\Omega |\nabla(\sigma u_1)|^q \,\mathrm{d}x = \lambda_1 \int_\Omega (\sigma u_1)^p \,\mathrm{d}x.$$

It follows that

$$\begin{aligned}\sigma^p \int_\Omega |\nabla u_1|^p \,\mathrm{d}x + \sigma^q \int_\Omega |\nabla u_1|^q \,\mathrm{d}x &= \sigma^p \lambda_1 \int_\Omega u_1^p \,\mathrm{d}x \\ &= \sigma^p \left(\int_\Omega |\nabla u_1|^p \,\mathrm{d}x + \int_\Omega |\nabla u_1|^q \,\mathrm{d}x \right),\end{aligned}$$

which gives $(\sigma^p - \sigma^q) \int_\Omega |\nabla u_1|^q \,\mathrm{d}x = 0$, so that $\sigma = 1$ since $q < p$, and we obtain $u_1 = u_2$. □

Remark 2.4 Note that in our case, due to the nonhomogeneous (p, q)-Laplace eigenvalue problem (2.1) with $\gamma = p$, the previous theorem gives the uniqueness of the first eigenfunction u_1 while in Kawohl and Lindqvist [16], the authors proved the uniqueness modulo scaling for the case of the p-Laplace eigenvalue problem

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \lambda |u|^{p-2} u = 0.$$

Note also that for $q < \gamma < p$, the estimation argument, applying Jensen's inequality, does not work here.

We have the following nonexistence result for positive solutions of problem (2.1).

Theorem 2.5 *For all λ satisfying $\dfrac{\gamma\lambda}{q} < \lambda_1$, problem (2.1) has no positive solutions.*

Proof Suppose that (2.1) has a positive solution $u \in W_0^{1,p}(\Omega)$. Testing the weak formulation of (2.1) by u, using (2.2) gives

$$\begin{aligned}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |\nabla u|^q \,\mathrm{d}x &= \lambda \int_\Omega u^\gamma \,\mathrm{d}x \\ &\leq \frac{\lambda}{\lambda_1} \left(\frac{\gamma}{p} \int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{\gamma}{q} \int_\Omega |\nabla u|^q \,\mathrm{d}x \right) \\ &\leq \frac{\lambda}{\lambda_1} \frac{\gamma}{q} \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega |\nabla u|^q \,\mathrm{d}x \right)\end{aligned}$$

contrary to the assumption. Hence the assertion of the theorem follows. □

3 Sublinear eigenvalue problems involving (p, q)-Laplacian

In this section, we consider the following nonhomogeneous eigenvalue problem

$$\begin{aligned} -\Delta_p u - \Delta_q u &= \lambda f(x, u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $\lambda > 0$, and $f : \Omega \times [0, \infty) \to \mathbb{R}$ is assumed to be a Carathéodory function satisfying

$$|f(x, t)| \leq C t^{\gamma - 1}, \quad \text{for all } (x, t) \in \Omega \times [0, \infty), \tag{3.2}$$

where $1 < q < \gamma \leq p$ and for some $C > 0$.

Throughout this section, we will denote by

$$\|u\| := \|\nabla u\|_p = \left(\int_\Omega |\nabla u|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$$

the equivalent norm defined on $W_0^{1,p}(\Omega)$.

We start by a nonexistence result for positive solutions of problem (3.1).

Theorem 3.1 *There exists $\underline{\lambda} > 0$ such that for all $\lambda < \underline{\lambda}$, problem (3.1) has no positive solutions.*

Proof Suppose that (3.1) has a positive solution $u \in W_0^{1,p}(\Omega)$. Testing the weak formulation of (3.1) by u, using (2.2) and (3.2) gives

$$\begin{aligned}\int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega |\nabla u|^q \, \mathrm{d}x &= \lambda \int_\Omega f(x, u)u \, \mathrm{d}x \\ &\leq \lambda C \int_\Omega |u|^\gamma \, \mathrm{d}x \\ &\leq \lambda \frac{C}{\lambda_1} \frac{\gamma}{q} \left(\int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega |\nabla u|^q \, \mathrm{d}x\right).\end{aligned}$$

The assertion of the theorem follows with $\underline{\lambda} = \dfrac{q\lambda_1}{C\gamma}$. □

The next theorem shows the existence of positive solutions for problem (3.1).

Theorem 3.2 *Assume the following conditions:*

(F1) there exists $\delta > 0$ such that

$$F(x, t) := \int_0^t f(x, \tau) \, \mathrm{d}\tau \leq 0 \quad \textit{for } 0 \leq t \leq \delta,$$

(F2) there exists $t_0 > 0$ such that $F(x, t_0) > 0$ for a.e. $x \in \Omega$;
(F3) it holds

$$\limsup_{t \to \infty} \frac{F(x, t)}{t^\gamma} \leq 0 \quad \textit{uniformly for a.e. } x \in \Omega.$$

Then there exists $\overline{\lambda}$ such that (3.1) has at least two positive solutions for $\lambda \geq \overline{\lambda}$.

Proof We set

$$f(x, t) = 0 \quad \text{if} \quad t < 0 \quad \text{for a.e. } x \in \Omega. \tag{3.3}$$

Consider the C^1-functional $\Phi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\Phi_\lambda(u) = \int_\Omega \frac{1}{p}|\nabla u|^p \, \mathrm{d}x + \int_\Omega \frac{1}{q}|\nabla u|^q \, \mathrm{d}x - \int_\Omega \lambda F(x, u) \, \mathrm{d}x$$

for all $u \in W_0^{1,p}(\Omega)$. If u is a critical point of Φ_λ, denoting by u^- the negative part of u, we have

$$0 = \langle \Phi'_{\lambda,\mu}(u)), u^- \rangle$$

$$
\begin{aligned}
&= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u^- \,\mathrm{d}x + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla u^- \,\mathrm{d}x - \int_\Omega \lambda f(x,u)u^- \,\mathrm{d}x \\
&= \|\nabla u^-\|_p^p + \|\nabla u^-\|_q^q \\
&\geq \|u^-\|^p.
\end{aligned}
$$

This implies that $u \geq 0$. Furthermore, we proceed as in the proof of Theorem 2.2 to show that $u \in L^\infty(\Omega)$ and the regularity results of Lieberman [18] give $u \in C^{1,\alpha}(\overline{\Omega})$, so the positivity of u now follows from the weak Harnack type inequality proved by Trudinger [25, Theorem 1.1], that is, either $u > 0$ or $u \equiv 0$. Thus, nontrivial critical points of Φ_λ are positive solutions of (3.1).

By the condition (3.2) we get

$$
|F(x,t)| \leq C'|t|^\gamma \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}, \tag{3.4}
$$

for some positive constant C'. By (F3) there is $B_\lambda > 0$ such that

$$
F(x,t) \leq \frac{\lambda_1}{2\gamma\lambda}|t|^\gamma \quad \text{for all } |t| \geq B_\lambda. \tag{3.5}
$$

Combining (3.4) and (3.5), there is a constant $C_\lambda > 0$ such that

$$
\lambda F(x,t) \leq \frac{\lambda_1}{2\gamma}|t|^\gamma + C_\lambda \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.6}
$$

Hence, since $q < p$, by applying (3.6) and (2.2), it follows that

$$
\begin{aligned}
\Phi_\lambda(u) &\geq \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla u|^q - \frac{\lambda_1}{2\gamma}|u|^\gamma - C_\lambda\right)\mathrm{d}x, \\
&\geq \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla u|^q - \frac{1}{2}\left(\frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla u|^q\right) - C_\lambda\right)\mathrm{d}x \\
&\geq \mu\|u\|^p - C_\lambda|\Omega|_N,
\end{aligned}
$$

where $\mu = \frac{1}{2p}$ and $|\cdot|_N$ is the Lebesgue measure in $\mathbb{R}^N$. Hence, Φ_λ is bounded from below and coercive. In addition, Φ_λ is sequentially weakly lower semicontinuous which implies the existence of a global minimizer $w_1 \in W_0^{1,p}(\Omega)$ of $\Phi_\lambda(u)$.

Claim 1: There exists $\overline{\lambda} > 0$ such that $\inf \Phi_\lambda < 0$ for $\lambda \geq \overline{\lambda}$.

In order to prove this, we take a sufficiently large compact subset Ω' of Ω and $u_0 \in W_0^{1,p}(\Omega)$ such that $u_0 = t_0$ on Ω' and $0 \leq u_0 \leq t_0$ on $\Omega \setminus \Omega'$, where t_0 is as in (F2). Then we have

$$
\int_\Omega F(x,u_0)\,\mathrm{d}x \geq \int_{\Omega'} F(x,t_0)\,\mathrm{d}x - C'|t_0|^\gamma|\Omega \setminus \Omega'|_N > 0,
$$

for $|\Omega \setminus \Omega'|_N$ sufficiently small. This yields

$$
\Phi_\lambda(u_0) \leq \int_\Omega \left(\frac{1}{p}|\nabla u_0|^p + \frac{1}{q}|\nabla u_0|^q\right)\mathrm{d}x - \lambda\int_\Omega F(x,u_0)\,\mathrm{d}x < 0
$$

for λ large enough. This proves the Claim 1.

From the Claim 1, choosing $\lambda \geq \overline{\lambda}$, we get $\Phi_\lambda(w_1) < 0 = \Phi_\lambda(0)$ and so $w_1 \neq 0$. Now, let us fix λ with $\lambda \geq \overline{\lambda}$ and consider

$$
\widetilde{f}(x,t) = \begin{cases} f(x,t), & \text{if } t \leq w_1(x), \\ f(x,w_1(x)), & \text{if } t > w_1(x), \end{cases} \quad \text{and} \quad \widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,\tau)\,\mathrm{d}\tau.
$$

Let

$$\widetilde{\Phi}_\lambda(u) = \int_\Omega \frac{1}{p}|\nabla u|^p \,\mathrm{d}x + \int_\Omega \frac{1}{q}|\nabla u|^q \,\mathrm{d}x - \int_\Omega \lambda \widetilde{F}(x,u)\,\mathrm{d}x.$$

If u is a critical point of $\widetilde{\Phi}_\lambda$, then $u \geq 0$ as before by the positivity of w_1 and (3.3). Further, we have

$$\begin{aligned}
0 &= \left(\widetilde{\Phi}'_\lambda(u) - \Phi'_\lambda(w_1), (u-w_1)^+\right)\\
&= \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla w_1|^{p-2}\nabla w_1\right)\cdot \nabla(u-w_1)^+\,\mathrm{d}x\\
&\quad + \int_\Omega \left(|\nabla u|^{q-2}\nabla u - |\nabla w_1|^{q-2}\nabla w_1\right)\cdot \nabla(u-w_1)^+\,\mathrm{d}x\\
&\quad - \lambda\int_\Omega \left(\widetilde{f}(x,u) - f(x,w_1)\right)(u-w_1)^+\,\mathrm{d}x\\
&= \int_{\{u>w_1\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla w_1|^{p-2}\nabla w_1\right)\cdot(\nabla u - \nabla w_1)\,\mathrm{d}x\\
&\quad + \int_{\{u>w_1\}} \left(|\nabla u|^{q-2}\nabla u - |\nabla w_1|^{q-2}\nabla w_1\right)\cdot(\nabla u - \nabla w_1)\,\mathrm{d}x.
\end{aligned}$$

Thus by the elementary inequality $\left(|b|^{r-2}b - |a|^{r-2}a, b-a\right) \geq 0$ for $r > 1$ and any $a, b \in \mathbb{R}^N$ (see e.g. Lindqvist[21]), it follows that

$$\begin{aligned}
0 &\leq \int_{\{u>w_1\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla w_1|^{p-2}\nabla w_1\right)\cdot(\nabla u - \nabla w_1)\,\mathrm{d}x\\
&= -\int_{\{u>w_1\}} \left(|\nabla u|^{q-2}\nabla u - |\nabla w_1|^{q-2}\nabla w_1\right)\cdot(\nabla u - \nabla w_1)\,\mathrm{d}x\\
&\leq 0.
\end{aligned}$$

Hence $u \leq w_1$. So u is a solution of (3.1) in the order interval $[0, w_1]$.

The second critical point w_2 with $\widetilde{\Phi}_\lambda(w_2) > 0$ will be obtained via the mountain-pass theorem, which would complete the proof since $\widetilde{\Phi}_\lambda(0) = 0 > \widetilde{\Phi}_\lambda(w_1)$.

Claim 2: The origin is a strict local minimizer of $\widetilde{\Phi}_\lambda$.

Let $u \in W_0^{1,p}(\Omega)$. We set $\Omega_u = \{x \in \Omega\colon u(x) > \min\{w_1(x), \delta\}\}$, where $\delta > 0$ is given in (F1). By hypothesis (F1), $\widetilde{F}(x,u) \leq 0$ on $\Omega \setminus \Omega_u$. Then we have

$$\begin{aligned}
\widetilde{\Phi}_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_p^q - \lambda\int_{\Omega_u} \widetilde{F}(x,u)\,\mathrm{d}x - \lambda\int_{\Omega\setminus\Omega_u} \widetilde{F}(x,u)\,\mathrm{d}x\\
&\geq \frac{1}{p}\|u\|^p - \lambda\int_{\Omega_u} \widetilde{F}(x,u)\,\mathrm{d}x.
\end{aligned} \tag{3.7}$$

Applying (3.4), Hölder's inequality, the Sobolev embedding theorem and since $q < \gamma \leq p$, it follows that

$$\begin{aligned}
\lambda\int_{\Omega_u} |\widetilde{F}(x,u)|\,\mathrm{d}x &\leq \lambda C' \int_{\Omega_u} |u|^\gamma\,\mathrm{d}x\\
&\leq \lambda C' |\Omega_u|_N^{1-\frac{\gamma}{p}} \int_{\Omega_u} |u|^p\,\mathrm{d}x\\
&\leq \lambda C'' |\Omega_u|_N^{1-\frac{\gamma}{p}} \|u\|^p,
\end{aligned} \tag{3.8}$$

for some positive constant C''. It suffices to show that $|\Omega_u|_N \to 0$ as $\|u\| \to 0$. Let $\varepsilon > 0$ be arbitrary and take a compact subset Ω_ε of Ω such that $|\Omega \setminus \Omega_\varepsilon|_N < \varepsilon$ and let $\Omega_{u,\varepsilon} = \Omega_u \cap \Omega_\varepsilon$. Then

$$\|u\|_p^p \geq \int_{\Omega_{u,\varepsilon}} u^p \,\mathrm{d}x \geq \Big(\min\big\{\min_{\Omega_\varepsilon} w_1, \delta\big\}\Big)^p |\Omega_{u,\varepsilon}|_N,$$

knowing that $\min\big\{\min_{\Omega_\varepsilon} w_1, \delta\big\} > 0$. Applying again the Sobolev embedding theorem and letting $\|u\|$ tend to 0, gives $|\Omega_{u,\varepsilon}|_N \to 0$. Now, since $\Omega_u \subset \Omega_{u,\varepsilon} \cup (\Omega \backslash \Omega_\varepsilon)$, we have

$$|\Omega_u|_N < |\Omega_{u,\varepsilon}| + \varepsilon,$$

for all $\varepsilon > 0$. Hence $|\Omega_u|_N \to 0$ as $\|u\| \to 0$ and Claim 2 follows from (3.7) and (3.8).

Note that $\widetilde{\Phi}_\lambda$ is also coercive by an argument similar to the one used for Φ_λ. So every Palais-Smale sequence of $\widetilde{\Phi}_\lambda$ is bounded and hence contains a convergent subsequence. Now the mountain-pass theorem gives a critical point w_2 of $\widetilde{\Phi}_\lambda$ at the level

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} \widetilde{\Phi}_\lambda(u) > 0,$$

where $\Gamma = \big\{\gamma \in C([0,1], W_0^{1,p}(\Omega)) \colon \gamma(0) = 0, \gamma(1) = w_1\big\}$ is the class of paths joining the origin to w_1. Then $w_2 \leq w_1$, and so w_2 is a critical point of Φ_λ since $\widetilde{f}(x, w_2) = f(x, w_2)$. Therefore there are two positive solutions w_1, w_2 such that

$$\Phi_\lambda(w_1) = \widetilde{\Phi}_\lambda(w_1) < 0 = \widetilde{\Phi}_\lambda(0) = \Phi_\lambda(0) < \widetilde{\Phi}_\lambda(w_2) = \Phi_\lambda(w_2).$$

This achieves the proof of the theorem. □

4 Superlinear problems involving (*p*, *q*)-Laplacian

In this section, we consider the following nonhomogeneous eigenvalue problem

$$\begin{aligned} -\Delta_p u - \mu \Delta_q u &= |u|^{\gamma-2} u \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p < N$, $1 < q < p < \gamma < p^* = \frac{Np}{N-p}$ and $\mu > 0$ is a real parameter. The associated energy functional to (4.1) is given by

$$I(u) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{\mu}{q}|\nabla u|^q\right) \mathrm{d}x - \int_\Omega \frac{1}{\gamma}|u|^\gamma \,\mathrm{d}x.$$

Let

$$C_\gamma = \inf_{u \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{2^p}\left[\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla u|^q \,\mathrm{d}x\right)^{\frac{1}{q}}\right]^p : \int_\Omega |u|^\gamma \,\mathrm{d}x = 1 \right\}.$$

By standard arguments using a minimizing sequence and compact embedding, this infimum is achieved. Indeed, for completeness and reader's convenience, we present it here with some details.

Proposition 4.1 *There exists* $u_\gamma \in W_0^{1,p}(\Omega)$ *such that*

$$\begin{cases} u_\gamma \geq 0 \quad in\ \Omega, \quad \int_\Omega |u_\gamma|^\gamma \,\mathrm{d}x = 1 \quad and \\ C_\gamma = \frac{1}{2^p}\left[\left(\int_\Omega |\nabla u_\gamma|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla u_\gamma|^q \,\mathrm{d}x\right)^{\frac{1}{q}}\right]^p. \end{cases} \tag{4.2}$$

Proof Let $\mathtt{M} = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^\gamma \,\mathrm{d}x = 1\right\}$ and consider the following functional

$$\mathtt{E}(u) = \frac{1}{2^p}\left[\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla u|^q \,\mathrm{d}x\right)^{\frac{1}{q}}\right]^p, \quad u \in \mathtt{M}.$$

Then $C_\gamma = \inf_{u \in \mathtt{M}} \mathtt{E}(u)$. Clearly, $C_\gamma \geq 0$. Let (u_j) be a minimizing sequence. Since $\mathtt{E}(u_j) \to C_\gamma$, (u_j) is bounded in $W_0^{1,p}(\Omega)$ and hence converges weakly in $W_0^{1,p}(\Omega)$ to some $u_\gamma \in W_0^{1,p}(\Omega)$, strongly in $L^\gamma(\Omega)$, and a.e. in Ω. In particular, $u_\gamma \in \mathtt{M}$. Since $\mathtt{E}$ is weakly lower semicontinuous, we have

$$C_\gamma \leq \mathtt{E}(u_\gamma) \leq \liminf \mathtt{E}(u_j) = C_\gamma,$$

and so $\mathtt{E}(u_\gamma) = C_\gamma$. Then $|u_\gamma| \in \mathtt{M}$ and $\mathtt{E}(|u_\gamma|) = C_\gamma$, so $|u_\gamma| \geq 0$ is also a minimizer. □

Clearly, we have

$$\left(\int_\Omega |u|^\gamma \,\mathrm{d}x\right)^{\frac{1}{\gamma}} \leq \frac{1}{2} C_\gamma^{\frac{-1}{p}}\left[\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla u|^q \,\mathrm{d}x\right)^{\frac{1}{q}}\right]$$

for any $u \in W_0^{1,p}(\Omega)$ and equality holds if and only if $u = u_s := s u_\gamma$ for some $s \in \mathbb{R}$.

Now, we state and prove the main result of this section.

Theorem 4.2 *Let* $1 < p < N$ *and* $1 < q < p < \gamma < p^*$. *Then there exists* $\mu > 0$ *such that problem* (4.1) *has a unique, up to multiplication with constants, positive weak solution* $u \in L^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha > 0$.

Proof Let $\eta = 2C_\gamma^{\frac{1}{p}}$ and consider the nonhomogeneous elliptic problem

$$-\Delta_p u - \Delta_q u = \eta |u|^{\gamma-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

By standard application of the Lagrange multiplier, u_s is a weak solution of (4.3). By Moser-type iterations (see, e.g., Drábek et al. [12]), the regularity results of Lieberman [18] and the Harnack-type inequality by Trudinger [25] imply that for $s > 0$, we have $u_s > 0$ in Ω and $u_s \in L^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$.

Regarding the uniqueness of the solution of (4.3), we define

$$\widetilde{I}(w) = \left(\int_\Omega |\nabla w|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla w|^q \,\mathrm{d}x\right)^{\frac{1}{q}} - \eta\left(\int_\Omega |w|^\gamma \,\mathrm{d}x\right)^{\frac{1}{\gamma}}, \quad w \in W_0^{1,p}(\Omega).$$

Let $u, v \in W_0^{1,p}(\Omega)$ satisfying (4.2). We have $u, v \in L^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$, $u, v > 0$ in Ω and $\widetilde{I}(u) = \widetilde{I}(v) = 0$. We use an argument similar to Drábek [11] and Idogawa and Ôtani [15]. Let $t > 0$, and set

$$\overline{u}(t, x) = \max\{u(x), tv(x)\}, \quad \underline{u}(t, x) = \min\{u(x), tv(x)\}.$$

Then, we have

$$
\begin{aligned}
0 &\le \widetilde{I}(\overline{u}) + \widetilde{I}(\underline{u}) \\
&= \Big(\int_\Omega |\nabla \overline{u}|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_\Omega |\nabla \overline{u}|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} - \eta\Big(\int_\Omega |\overline{u}|^\gamma \,\mathrm{d}x\Big)^{\frac{1}{\gamma}} \\
&\quad + \Big(\int_\Omega |\nabla \underline{u}|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_\Omega |\nabla \underline{u}|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} - \eta\Big(\int_\Omega |\underline{u}|^\gamma \,\mathrm{d}x\Big)^{\frac{1}{\gamma}} \\
&= \Big(\int_{u>tv} |\nabla \overline{u}|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{u>tv} |\nabla \overline{u}|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} - \eta\Big(\int_{u>tv} |\overline{u}|^\gamma \,\mathrm{d}x\Big)^{\frac{1}{\gamma}} \\
&\quad + \Big(\int_{u\le tv} |\nabla \overline{u}|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{u\le tv} |\nabla \overline{u}|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} - \eta\Big(\int_{u\le tv} |\overline{u}|^\gamma \,\mathrm{d}x\Big)^{\frac{1}{\gamma}} \\
&\quad + \Big(\int_{u>tv} |\nabla \underline{u}|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{u>tv} |\nabla \underline{u}|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} - \eta\Big(\int_{u>tv} |\underline{u}|^\gamma \,\mathrm{d}x\Big)^{\frac{1}{\gamma}} \\
&\quad + \Big(\int_{u\le tv} |\nabla \underline{u}|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{u\le tv} |\nabla \underline{u}|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} - \eta\Big(\int_{u\le tv} |\underline{u}|^\gamma \,\mathrm{d}x\Big)^{\frac{1}{\gamma}} \\
&= \widetilde{I}(u) + \widetilde{I}(tv) \\
&= \widetilde{I}(u) + t\widetilde{I}(v) \\
&= 0.
\end{aligned}
$$

It follows that $\widetilde{I}(\overline{u}) = \widetilde{I}(\underline{u}) = 0$. Then $\overline{u}$ and $\underline{u}$ are weak solutions of problem (4.3) and thus the first assertion of Proposition 4.1 applies to $\overline{u}$ and $\underline{u}$. Let $x_0 \in \Omega$ and set $t_0 = \frac{u(x_0)}{v(x_0)} > 0$. Let ξ be any unit vector. Since $\overline{u}(t_0, x_0) = u(x_0) = t_0 v(x_0)$ we have

$$
\begin{aligned}
u(x_0 + h\xi) - u(x_0) &\le \overline{u}(t_0, x_0 + h\xi) - \overline{u}(t_0, x_0), \\
t_0 v(x_0 + h\xi) - t_0 v(x_0) &\le \overline{u}(t_0, x_0 + h\xi) - \overline{u}(t_0, x_0).
\end{aligned}
$$

Dividing these inequalities by $h > 0$ and $h < 0$, then letting h tend to 0^+ and 0^-, we get

$$\nabla u(x_0) = \nabla \overline{u}(t_0, x_0) = t_0 \nabla v(x_0).$$

Hence,

$$
\begin{aligned}
\nabla\left(\frac{u}{v}\right)(x_0) &= \frac{v(x_0)\nabla u(x_0) - u(x_0)\nabla v(x_0)}{(v(x_0))^2} \\
&= \frac{v(x_0)(\nabla u(x_0) - t_0\nabla v(x_0))}{(v(x_0))^2} \\
&= 0.
\end{aligned}
$$

Thus, $\frac{u(x)}{v(x)}$ is constant in Ω, namely, $\frac{u(x)}{v(x)} = \theta > 0$. Due to (4.2), we have

$$1 = \int_\Omega |u|^\gamma \,\mathrm{d}x = \theta^\gamma \int_\Omega |v|^\gamma \,\mathrm{d}x = \theta^\gamma.$$

Hence $\theta = 1$, and therefore, the uniqueness of u_γ follows.

Since u_γ is a weak solution of (4.3), we have

$$\int_\Omega |\nabla u_\gamma|^{p-2}\nabla u_\gamma \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega |\nabla u_\gamma|^{q-2}\nabla u_\gamma \cdot \nabla\varphi \,\mathrm{d}x$$

$$= \eta \int_\Omega |u_\gamma|^{\gamma-2} u_\gamma \varphi \,\mathrm{d}x = \eta^{\frac{\gamma-1}{\gamma-p}} \eta^{\frac{1-p}{\gamma-p}} \int_\Omega |u_\gamma|^{\gamma-2} u_\gamma \varphi \,\mathrm{d}x,$$

for all $\varphi \in W_0^{1,p}(\Omega)$. This implies that

$$\begin{aligned}
&\int_\Omega |\nabla(\eta^{\frac{1}{\gamma-p}} u_\gamma)|^{p-2} \nabla(\eta^{\frac{1}{\gamma-p}} u_\gamma) \cdot \nabla\varphi \,\mathrm{d}x \\
&\quad + \eta^{\frac{p-q}{\gamma-p}} \int_\Omega |\nabla(\eta^{\frac{1}{\gamma-p}} u_\gamma)|^{q-2} \nabla(\eta^{\frac{1}{\gamma-p}} u_\gamma) \cdot \nabla\varphi \,\mathrm{d}x \\
&= \eta^{\frac{\gamma-1}{\gamma-p}} \int_\Omega |u_\gamma|^{\gamma-2} u_\gamma \varphi \,\mathrm{d}x \\
&= \int_\Omega |(\eta^{\frac{1}{\gamma-p}} u_\gamma)|^{\gamma-2} (\eta^{\frac{1}{\gamma-p}}) \varphi \,\mathrm{d}x,
\end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. Hence for $\mu = \eta^{\frac{p-q}{\gamma-p}}$, the function $u = \eta^{\frac{1}{\gamma-p}} u_\gamma$ is a positive weak solution of (4.1), u is unique up to multiplication with constants, and $u \in L^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for some $\alpha > 0$. □

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