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Parametric superlinear double phase problems with singular term and critical growth on the boundary

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In this paper, we study quasilinear elliptic equations driven by the double phase operator along with a reaction that has a singular and a parametric superlinear term and with a nonlinear Neumann boundary condition of critical growth. Based on a new equivalent norm for Musielak–Orlicz Sobolev spaces and the Nehari manifold along with the fibering method, we prove the existence of at least two weak solutions, provided the parameter is sufficiently small.

KEYWORDS

critical growth, double phase operator, equivalent norm, existence of solutions, fibering method, Musielak–Orlicz Sobolev space, Nehari manifold, singular problems

MSC CLASSIFICATION

35J15; 35J62; 35J75; 58J05

1 | INTRODUCTION

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we study the following singular double phase problem with critical growth on the boundary

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + \alpha(x)u^{p-1} &= \zeta(x)u^{-\kappa} + \lambda u^{q_1-1} \quad \text{in } \Omega, \\
\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \cdot \nu &= -\beta(x)u^{p_*-1} \quad \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $\lambda > 0$ and $\nu(x)$ is the outer unit normal of Ω at the point $x \in \partial\Omega$. The operator involved is the so-called double phase operator given by

$$
\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \text{ for } u \in W^{1,\mathcal{H}}(\Omega),
$$

which is related to the energy functional

$$
\omega \mapsto \int_\Omega \left(|\nabla\omega|^p + \mu(x)|\nabla\omega|^q\right) dx.
\tag{1.2}
$$

Functionals of type (1.2) have first been studied by Zhikov[1] in order to provide models for strongly anisotropic materials. The main characteristic of the functional defined in (1.2) is the change of ellipticity on the set where the weight function is zero, that is, on the set $\{x \in \Omega : \mu(x) = 0\}$. To be more precise, the energy density of (1.2) exhibits ellipticity in the gradient of order q on the points x where $\mu(x)$ is positive and of order p on the points x where $\mu(x)$ vanishes. Further results

on regularity of minimizers of (1.2) can be found in the papers of Baroni–Colombo–Mingione,[2,3] Colombo–Mingione,[4,5] De Filippis–Mingione,[6] Marcellini,[7,8] and Ragusa–Tachikawa.[9]

We suppose the following assumptions:

(H):

(i) $1 < p < N$, $p < q < p^*$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$;
(ii) $0 < \kappa < 1$ and $q_1 \in (\max\{q, p_*\}, p^*)$, where

$$p^* = \frac{Np}{N-p} \quad \text{and} \quad p_* = \frac{(N-1)p}{N-p} \tag{1.3}$$

are the critical exponents to p;
(iii) $\alpha \in L^\infty(\Omega)$ with $\alpha(x) \geq 0$ for a. a. $x \in \Omega$ and $\alpha \not\equiv 0$;
(iv) $\beta \in L^\infty(\partial\Omega)$ with $\beta(x) \geq 0$ for a. a. $x \in \partial\Omega$;
(v) $\zeta \in L^\infty(\Omega)$ and $\zeta(x) > 0$ for a. a. $x \in \Omega$.

We call a function $u \in W^{1,\mathcal{H}}(\Omega)$ a weak solution of problem (1.1) if $\zeta(\cdot)u^{-\kappa}h \in L^1(\Omega)$, $u(x) > 0$ for a. a. $x \in \Omega$ and

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u \right) \cdot \nabla h \, dx$$
$$+ \int_\Omega \alpha(x)u^{p-1}h \, dx + \int_{\partial\Omega} \beta(x)u^{p_*-1}h \, d\sigma \tag{1.4}$$
$$= \int_\Omega \zeta(x)u^{-\kappa}h \, dx + \lambda \int_\Omega u^{q_1-1}h \, dx$$

is satisfied for all test functions $h \in W^{1,\mathcal{H}}(\Omega)$. Based on (H), it is easy to see that the definition of a weak solution in (1.4) is well defined. Denoting by $\Theta_\lambda : W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$, the energy functional corresponding to problem (1.1), the main result in this paper is the following theorem.

Theorem 1.1. *If hypotheses (H) hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*]$ problem (1.1) has at least two weak solutions $u_\lambda, v_\lambda \in W^{1,\mathcal{H}}(\Omega)$ such that $\Theta_\lambda(u_\lambda) < 0 < \Theta_\lambda(v_\lambda)$.*

The proof of Theorem 1.1 relies on the properties of the Nehari manifold along with a new equivalent norm in the corresponding Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$. In contrast to other works dealing with double phase problems, we were able to weaken the usual condition

$$\frac{q}{p} < 1 + \frac{1}{N}. \tag{1.5}$$

Such assumption is standard for Dirichlet double phase problems in order to have the equivalent norm $\|\nabla \cdot \|_{\mathcal{H}}$ in $W_0^{1,\mathcal{H}}(\Omega)$. It also guarantees the density of smooth functions in $W^{1,\mathcal{H}}(\Omega)$. Condition (1.5) can be replaced in our paper by $q < p^*$ which is equivalent to $\frac{Nq}{N+q} < p$. Note that such condition is indeed weaker than (1.5). Furthermore, we can relax the assumptions on the weight function $\mu(\cdot)$. Instead of a Lipschitz condition, we only need $\mu(\cdot)$ to be bounded, not necessarily continuous.

Based on the new equivalent norm in $W^{1,\mathcal{H}}(\Omega)$, we were able to suppose critical growth on the boundary of Ω. To the best of our knowledge, there is only one paper concerning singular double phase problems with nonlinear boundary condition, namely, the paper of Farkas–Fiscella–Winkert[10] who studied the problem

$$-\text{div}(A(u)) + u^{p-1} + \mu(x)u^{q-1} = u^{p^*-1} + \lambda \left(u^{\gamma-1} + g_1(x, u) \right) \quad \text{in } \Omega,$$
$$A(u) \cdot v = u^{p_*-1} + g_2(x, u) \qquad \text{on } \partial\Omega, \tag{1.6}$$

where

$$\text{div}(A(u)) := \text{div} \left(F^{p-1}(\nabla u)\nabla F(\nabla u) + \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u) \right)$$

is the Finsler double phase operator with a Minkowski space $(\mathbb{R}^N, F)$. They obtain the existence of one weak solution of (1.6) by applying variational tools and truncation techniques. The treatment is completely different from ours and only

one solution is obtained. Also in the case of nonsingular Neumann double phase problems, there are only few works. We refer to El Manouni et al.,[11] Gasiński and Winkert,[12] Papageorgiou et al,[13] and Papageorgiou et al.[14] Gasiński and Winkert[12] study the problem

$$
\begin{aligned}
-\text{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &= f(x,u) - |u|^{p-2}u - \mu(x)|u|^{q-2}u \quad \text{in } \Omega, \\
\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot v &= g(x,u) \qquad\qquad\qquad \text{on } \partial\Omega.
\end{aligned}
\tag{1.7}
$$

Based on the Nehari manifold method, it is shown that problem (1.7) has at least three nontrivial solutions. We point out that the use of the Nehari manifold in Gasiński and Winkert[12] is different from ours. Indeed the idea in the current paper is the splitting of the Nehari manifold into three disjoint parts, and the two solutions in Theorem 1.1 turn out to be the global minimizers of Θ_λ restricted to two of them, provided the parameter is sufficiently small. The third one is the empty set.

In general, the use of the fibering method along with the Nehari manifold is a very powerful tool and was initiated by the works of Drábek and Pohozaev[15] and Sun et al.[16] Afterwards several authors applied this method to different problems of singular type and nonsingular type. We refer to the works of Alves et al.,[17] Lei,[18] Liu et al.,[19] Liu and Winkert,[20] Mukherjee and Sreenadh,[21] Papageorgiou et al.,[22] Papageorgiou and Winkert,[23] Wang et al,[24] and Yang and Bai.[25]

For existence results for double phase problems with homogeneous Dirichlet boundary condition, we refer to the papers of Colasuonno and Squassina[26] (eigenvalue problem for the double phase operator), Farkas and Winkert[27] (Finsler double phase problems), Gasiński and Papageorgiou[28] (locally Lipschitz right-hand side), Gasiński and Winkert[29,30] (convection and superlinear problems), Liu and Dai[31] (Nehari manifold approach), Perera and Squassina[32] (Morse theoretical approach), Zeng et al.[33,34] (multivalued obstacle problems), and the references therein. Finally, we mention the nice overview article of Mingione and Rădulescu[35] about recent developments for problems with nonstandard growth and nonuniform ellipticity.

2 | PRELIMINARIES

In this section, we will present the main properties of Musielak–Orlicz spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$, respectively, and equip the space $W^{1,\mathcal{H}}(\Omega)$ with a new equivalent norm.

The usual Lebesgue spaces $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ will be endowed with the norm $\|\cdot\|_r$ and the boundary Lebesgue spaces are denoted by $L^r(\partial\Omega)$ with norm $\|\cdot\|_{r,\partial\Omega}$ whenever $1 \leq r \leq \infty$. The corresponding Sobolev spaces are denoted by $W^{1,r}(\Omega)$ for $1 < r < \infty$ with the equivalent norm

$$
\|u\|_{1,r} = \left(\|\nabla u\|_r^r + \int_\Omega \alpha(x)|u|^r \mathrm{d}x\right)^{\frac{1}{r}},
$$

where α fulfills hypothesis (H)(iii). The proof for such result is similar to those proofs as in Papageorgiou and Winkert.[36] Proposition 2.8,[37] Proposition 4.5.34

Let hypothesis (H)(i) be satisfied and let $\mathcal{H} : \Omega \times [0,\infty) \to [0,\infty)$ be defined by

$$
\mathcal{H}(x,t) = t^p + \mu(x)t^q.
$$

Let $M(\Omega)$ be the space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual, we identify two such functions which differ on a Lebesgue-null set. Then, the Musielak–Orlicz Lebesgue space $L^{\mathcal{H}}(\Omega)$ is given by

$$
L^{\mathcal{H}}(\Omega) = \{u \in M(\Omega) : \varrho_{\mathcal{H}}(u) < +\infty\}
$$

equipped with the Luxemburg norm

$$
\|u\|_{\mathcal{H}} = \inf\left\{\tau > 0 : \varrho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1\right\},
$$

where the modular function is given by

$$
\varrho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\mathrm{d}x = \int_\Omega \left(|u|^p + \mu(x)|u|^q\right)\mathrm{d}x.
$$

In addition, we define the seminormed space

$$L_\mu^q(\Omega) = \left\{ u \in M(\Omega) \ : \ \int_\Omega \mu(x)|u|^q \mathrm{d}x < +\infty \right\}$$

endowed with the seminorm

$$\|u\|_{q,\mu} = \left(\int_\Omega \mu(x)|u|^q \mathrm{d}x \right)^{\frac{1}{q}}.$$

The Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \ : \ |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \| |\nabla u| \|_{\mathcal{H}}$. We know that $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$ are uniformly convex and so reflexive Banach spaces; see Colasuonno and Squassina[26], Proposition 2.14 or Harjulehto and Hästö.[38], Theorem 6.1.4

The following proposition states the main embeddings for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$; see Gasiński and Winkert[12], Proposition 2.2 or Crespo et al.[39], Propositions 2.17 and 2.19

Proposition 2.1. *Let (H)(i) be satisfied and let p^* as well as p_* be the critical exponents to p as given in (1.3). Then the following embeddings hold:*

(i) *$L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ are continuous for all $r \in [1, p]$;*

(ii) *$W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for all $r \in [1, p^*]$ and compact for all $r \in [1, p^*)$;*

(iii) *$W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is continuous for all $r \in [1, p_*]$ and compact for all $r \in [1, p_*)$;*

(iv) *$L^{\mathcal{H}}(\Omega) \hookrightarrow L_\mu^q(\Omega)$ is continuous;*

(v) *$L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous.*

Furthermore, we introduce the seminormed space

$$W_\mu^q(\Omega; \mathbb{R}^N) = \left\{ u \in L_\mu^q(\Omega) \ : \ \int_\Omega \mu(x)|\nabla u|^q \mathrm{d}x < +\infty \right\}$$

endowed with the seminorm

$$\|\nabla u\|_{q,\mu} = \left(\int_\Omega \mu(x)|\nabla u|^q \mathrm{d}x \right)^{\frac{1}{q}}.$$

See Proposition 2.1(iv).

Next, we prove the existence of two equivalent norms in $W^{1,\mathcal{H}}(\Omega)$ which will be useful in our treatment. In the following, we use the seminorms

$$\|u\|_{r_1,\theta_1} = \left(\int_\Omega \theta_1(x)|u|^{r_1} \mathrm{d}x \right)^{\frac{1}{r_1}} \text{ and } \|u\|_{r_2,\theta_2,\partial\Omega} = \left(\int_{\partial\Omega} \theta_2(x)|u|^{r_2} \mathrm{d}\sigma \right)^{\frac{1}{r_2}},$$

where we suppose
(H′):

(i) $1 < p < N$, $p < q < p^*$ and $0 \le \mu(\cdot) \in L^\infty(\Omega)$;

(ii) $1 \le r_1 \le p^*$ and $1 \le r_2 \le p_*$;

(iii) $\theta_1 \in L^\infty(\Omega), \theta_1(x) \ge 0$ for a. a. $x \in \Omega$;

(iv) $\theta_2 \in L^\infty(\partial\Omega), \theta_2(x) \ge 0$ for a. a. $x \in \partial\Omega$;

(v) $\theta_1 \not\equiv 0$ or $\theta_2 \not\equiv 0$.

Proposition 2.2. *If hypotheses (H′) hold, then*

$$\|u\|_{1,\mathcal{H}}^{\circ} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{r_1,\theta_1} + \|u\|_{r_2,\theta_2,\partial\Omega}$$

$$\|u\|_{1,\mathcal{H}}^{*} = \inf\left\{\tau > 0 \ : \ \int_{\Omega}\left(\left(\frac{|\nabla u|}{\tau}\right)^p + \mu(x)\left(\frac{|\nabla u|}{\tau}\right)^q\right)dx + \int_{\Omega}\theta_1(x)\left(\frac{|u|}{\tau}\right)^{r_1}dx + \int_{\partial\Omega}\theta_2(x)\left(\frac{|u|}{\tau}\right)^{r_2}d\sigma \leq 1\right\}$$

are both equivalent norms on $W^{1,\mathcal{H}}(\Omega)$.

Proof. We prove the result in the critical case, that is, $r_1 = p^*$ and $r_2 = p_*$, the other cases work similarly. First, it is straightforward to show that $\|\cdot\|_{1,\mathcal{H}}^{\circ}$ and $\|\cdot\|_{1,\mathcal{H}}^{*}$ are norms on $W^{1,\mathcal{H}}(\Omega)$.

Applying Proposition 2.1(ii), (iii) gives for $u \in W^{1,\mathcal{H}}(\Omega)$

$$\|u\|_{1,\mathcal{H}}^{\circ} \leq \|\nabla u\|_{\mathcal{H}} + \|\theta_1\|_{\infty}^{\frac{1}{p^*}}\|u\|_{p^*} + \|\theta_2\|_{\infty}^{\frac{1}{p_*}}\|u\|_{p_*,\partial\Omega}$$

$$\leq \|\nabla u\|_{\mathcal{H}} + C_{\Omega}\|\theta_1\|_{\infty}^{\frac{1}{p^*}}\|u\|_{1,\mathcal{H}} + C_{\partial\Omega}\|\theta_2\|_{\infty}^{\frac{1}{p_*}}\|u\|_{1,\mathcal{H}}$$

$$\leq C_1\|u\|_{1,\mathcal{H}},$$

where $C_{\Omega}, C_{\partial\Omega} > 0$ are the embedding constants and $C_1 > 0$.

Let us now show that

$$\|u\|_{\mathcal{H}} \leq c\|u\|_{1,\mathcal{H}}^{\circ} \tag{2.1}$$

for some $c > 0$. Arguing indirectly, we suppose that (2.1) is not true. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\Omega)$ such that

$$\|u_n\|_{\mathcal{H}} > n\|u_n\|_{1,\mathcal{H}}^{\circ} \quad \text{for all} \ n \in \mathbb{N}. \tag{2.2}$$

We set $y_n = \frac{u_n}{\|u_n\|_{\mathcal{H}}}$ which gives $\|y_n\|_{\mathcal{H}} = 1$. From (2.2), we then obtain

$$\frac{1}{n} > \|y_n\|_{1,\mathcal{H}}^{\circ}. \tag{2.3}$$

Since $\|\nabla \cdot\|_{\mathcal{H}} + \|\cdot\|_{\mathcal{H}}$ is the norm of $W^{1,\mathcal{H}}(\Omega)$, we see that $\{y_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\Omega)$ is bounded. Hence, we may assume that

$$y_n \rightharpoonup y \ \text{in} \ W^{1,\mathcal{H}}(\Omega) \ \text{and} \ y_n \rightharpoonup y \ \text{in} \ L^{p^*}(\Omega) \ \text{and} \ L^{p_*}(\partial\Omega). \tag{2.4}$$

See Proposition 2.1(ii), (iii). Moreover, from (H′)(i) we know that $q < p^*$. So from Proposition 2.1(ii), (v), we have $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^q(\Omega)$ compactly and $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ continuously. Therefore, $y_n \to y$ in $L^{\mathcal{H}}(\Omega)$ and since $\|y_n\|_{\mathcal{H}} = 1$, it is clear that $y \neq 0$. Passing to the limit in (2.3) as $n \to \infty$ and using (2.4) along with the weak lower semicontinuity of the norm $\|\nabla \cdot\|_{\mathcal{H}}$ and of the seminorms $\|\cdot\|_{p^*,\theta_1}, \|\cdot\|_{p_*,\theta_2,\partial\Omega}$ lead to

$$0 \geq \|\nabla y\|_{\mathcal{H}} + \|y\|_{p^*,\theta_1} + \|y\|_{p_*,\theta_2,\partial\Omega}. \tag{2.5}$$

From (2.5), we conclude that $y \equiv \tau \neq 0$ is a constant and so we have

$$0 \geq |\tau|^{\frac{1}{p^*}}\left(\int_{\Omega}\theta_1(x)dx\right)^{\frac{1}{p^*}} + |\tau|^{\frac{1}{p_*}}\left(\int_{\partial\Omega}\theta_2(x)d\sigma\right)^{\frac{1}{p_*}} > 0,$$

since $\theta_1 \not\equiv 0$ or $\theta_2 \not\equiv 0$ by hypothesis (H′)(v). This is a contradiction, and so (2.1) is true. From this, we directly have that

$$\|u\|_{1,\mathcal{H}} \leq C_2\|u\|_{1,\mathcal{H}}^{\circ}$$

for some $C_2 > 0$.

Let us now prove that $\|\cdot\|_{1,\mathcal{H}}^\circ$ and $\|\cdot\|_{1,\mathcal{H}}^*$ are equivalent. For $u \in W^{1,\mathcal{H}}(\Omega)$, we have

$$
\int_\Omega \left(\left(\frac{|\nabla u|}{\|u\|_{1,\mathcal{H}}^\circ} \right)^p + \mu(x) \left(\frac{|\nabla u|}{\|u\|_{1,\mathcal{H}}^\circ} \right)^q \right) dx + \int_\Omega \theta_1(x) \left(\frac{|u|}{\|u\|_{1,\mathcal{H}}^\circ} \right)^{p^*} dx
$$
$$
+ \int_{\partial\Omega} \theta_2(x) \left(\frac{|u|}{\|u\|_{1,\mathcal{H}}^\circ} \right)^{p_*} d\sigma
$$
$$
\leq \varrho_{\mathcal{H}} \left(\frac{\nabla u}{\|\nabla u\|_{\mathcal{H}}} \right) + \int_\Omega \theta_1(x) \left(\frac{|u|}{\|u\|_{p^*,\theta_1}} \right)^{p^*} dx + \int_{\partial\Omega} \theta_2(x) \left(\frac{|u|}{\|u\|_{p_*,\theta_2,\partial\Omega}} \right)^{p_*} d\sigma
$$
$$
= 3.
$$

Therefore, $\|u\|_{1,\mathcal{H}}^* \leq 3\|u\|_{1,\mathcal{H}}^\circ$. Similarly, we obtain

$$
\int_\Omega \left(\left(\frac{|\nabla u|}{\|u\|_{1,\mathcal{H}}^*} \right)^p + \mu(x) \left(\frac{|\nabla u|}{\|u\|_{1,\mathcal{H}}^*} \right)^q \right) dx + \int_\Omega \theta_1(x) \left(\frac{|u|}{\|u\|_{1,\mathcal{H}}^*} \right)^{p^*} dx
$$
$$
+ \int_{\partial\Omega} \theta_2(x) \left(\frac{|u|}{\|u\|_{1,\mathcal{H}}^*} \right)^{p_*} d\sigma \tag{2.6}
$$
$$
\leq \rho_{1,\mathcal{H}}^* \left(\frac{u}{\|u\|_{1,\mathcal{H}}^*} \right),
$$

where $\rho_{1,\mathcal{H}}^*$ is the corresponding modular to $\|\cdot\|_{1,\mathcal{H}}^*$ given by

$$
\rho_{1,\mathcal{H}}^*(u) = \int_\Omega \left(|\nabla u|^p + \mu(x)|\nabla u|^q \right) dx + \int_\Omega \theta_1(x)|u|^{p^*} dx + \int_{\partial\Omega} \theta_2(x)|u|^{p_*} d\sigma.
$$

Note that, for $u \in W^{1,\mathcal{H}}(\Omega)$, the function $\tau \mapsto \rho_{1,\mathcal{H}}^*(\tau u)$ is continuous, convex, and even, and it is strictly increasing when $\tau \in [0, +\infty)$. So, by definition, we directly obtain

$$
\|u\|_{1,\mathcal{H}}^* = \tau \text{ if and only if } \rho_{1,\mathcal{H}}^* \left(\frac{u}{\tau} \right) = 1.
$$

From this and (2.6), we conclude that $\|\nabla u\|_{\mathcal{H}} \leq \|u\|_{1,\mathcal{H}}^*$, $\|u\|_{p^*,\theta_1} \leq \|u\|_{1,\mathcal{H}}^*$ and $\|u\|_{r_2,\theta_2,\partial\Omega} \leq \|u\|_{1,\mathcal{H}}^*$. Therefore, $\frac{1}{3}\|u\|_{1,\mathcal{H}}^\circ \leq \|u\|_{1,\mathcal{H}}^*$. $\square$

Assuming hypotheses (H), we know from Proposition 2.2 that

$$
\|u\| = \inf \left\{ \tau > 0 : \int_\Omega \left(\left(\frac{|\nabla u|}{\tau} \right)^p + \mu(x) \left(\frac{|\nabla u|}{\tau} \right)^q \right) dx + \int_\Omega \alpha(x) \left(\frac{|u|}{\tau} \right)^p dx + \int_{\partial\Omega} \beta(x) \left(\frac{|u|}{\tau} \right)^{p_*} d\sigma \leq 1 \right\}
$$

is a norm on $W^{1,\mathcal{H}}(\Omega)$ which is equivalent to $\|\cdot\|_{1,\mathcal{H}}$ and $\|\cdot\|_{1,\mathcal{H}}^\circ$. The corresponding modular ρ to $\|\cdot\|$ is given by

$$
\rho(u) = \int_\Omega \left(|\nabla u|^p + \mu(x)|\nabla u|^q \right) dx + \int_\Omega \alpha(x)|u|^p dx + \int_{\partial\Omega} \beta(x)|u|^{p_*} d\sigma \tag{2.7}
$$

for $u \in W^{1,\mathcal{H}}(\Omega)$.

The norm $\|\cdot\|$ and the modular function ρ are related as follows.

Proposition 2.3. *Let (H)(i), (H)(iii), and (H)(iv) be satisfied, let $y \in W^{1,\mathcal{H}}(\Omega)$, and let ρ be defined by (2.7). Then the following hold:*

(i) *If $y \neq 0$, then $\|y\| = \lambda$ if and only if $\rho\left(\frac{y}{\lambda}\right) = 1$;*
(ii) *$\|y\| < 1$ (resp. > 1, $= 1$) if and only if $\rho(y) < 1$ (resp. > 1, $= 1$);*
(iii) *If $\|y\| < 1$, then $\|y\|^q \leq \rho(y) \leq \|y\|^p$;*
(iv) *If $\|y\| > 1$, then $\|y\|^p \leq \rho(y) \leq \|y\|^q$;*
(v) *$\|y\| \to 0$ if and only if $\rho(y) \to 0$;*
(vi) *$\|y\| \to +\infty$ if and only if $\rho(y) \to +\infty$.*

The proof of Proposition 2.3 can be done as in Liu and Dai,[31] Proposition 2.1 or Crespo et al.,[39] Proposition 2.16

Let $A : W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ be the nonlinear mapping defined by

$$\langle A(u), \varphi \rangle_{\mathcal{H}} = \int_\Omega \left(|\nabla u|^{p-2} \nabla u + \mu(x)|\nabla u|^{q-2} \nabla u \right) \cdot \nabla \varphi \, dx$$
$$+ \int_\Omega \alpha(x)|u|^{p-2} u \varphi \, dx + \int_{\partial\Omega} \beta(x)|u|^{p_*-2} u \varphi \, d\sigma \tag{2.8}$$

for all $u, \varphi \in W^{1,\mathcal{H}}(\Omega)$ with $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ being the duality pairing between $W^{1,\mathcal{H}}(\Omega)$ and its dual space $W^{1,\mathcal{H}}(\Omega)^*$. The properties of the operator $A : W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ are summarized in the next proposition.

Proposition 2.4. *Let hypotheses (H)(i), (H)(iii), and (H)(iv) be satisfied. Then, the operator A defined by (2.8) is bounded (i.e., it maps bounded sets into bounded sets), continuous, strictly monotone (hence maximal monotone), and it is of type (S_+).*

The proof of Proposition 2.4 is similar to those in Liu and Dai[31], Proposition 3.1 or Crespo et al.,[39] Proposition 3.5

3 | PROOF OF THE MAIN RESULT

This section is concerned with the proof of Theorem 1.1. For this purpose, we introduce the energy functional $\Theta_\lambda : W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ of problem (1.1) given by

$$\Theta_\lambda(u) = \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|\nabla u\|_{q,\mu}^q + \frac{1}{p_*} \|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{1}{1-\kappa} \int_\Omega \zeta(x)|u|^{1-\kappa} dx - \frac{\lambda}{q_1} \|u\|_{q_1}^{q_1}.$$

It is clear that Θ_λ is not a C^1-functional because of the singular term. Next, for $u \in W^{1,\mathcal{H}}(\Omega)$, we introduce the fibering function $\psi_u : [0, +\infty) \to \mathbb{R}$ given by

$$\psi_u(t) = \Theta_\lambda(tu) \text{ for all } t \geq 0.$$

It is easy to see that $\psi_u \in C^\infty((0, \infty))$. Based on this, we can introduce the so-called Nehari manifold related to problem (1.1) which is defined by

$$\mathcal{N}_\lambda = \left\{ u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\} : \|u\|_{1,p}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p_*,\beta,\partial\Omega}^{p_*} = \int_\Omega \zeta(x)|u|^{1-\kappa} dx + \lambda \|u\|_{q_1}^{q_1} \right\}$$
$$= \left\{ u \in W^{1,\mathcal{H}}(\Omega) \setminus \{0\} : \psi_u'(1) = 0 \right\}.$$

We know that $\mathcal{N}_\lambda$ contains all weak solutions of problem (1.1), but it is smaller than the whole space $W^{1,\mathcal{H}}(\Omega)$. The advantage of $\mathcal{N}_\lambda$ is the fact that our energy functional Θ_λ has nice properties restricted to $\mathcal{N}_\lambda$ which fail globally. Next, we split the manifold $\mathcal{N}_\lambda$ into three disjoint parts in the following way:

$$
\begin{aligned}
\mathcal{N}_\lambda^+ = \Big\{ u \in \mathcal{N}_\lambda \ : \ &(p + \kappa - 1)\|u\|_{1,p}^p + (q + \kappa - 1)\|\nabla u\|_{q,\mu}^q + (p_* + \kappa - 1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&- \lambda(q_1 + \kappa - 1)\|u\|_{q_1}^{q_1} > 0 \Big\} \\
= \Big\{ & u \in \mathcal{N}_\lambda \ : \ \psi_u''(1) > 0 \Big\}, \\
\mathcal{N}_\lambda^\circ = \Big\{ u \in \mathcal{N}_\lambda \ : \ &(p + \kappa - 1)\|u\|_{1,p}^p + (q + \kappa - 1)\|\nabla u\|_{q,\mu}^q + (p_* + \kappa - 1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&= \lambda(q_1 + \kappa - 1)\|u\|_{q_1}^{q_1} \Big\} \\
= \Big\{ & u \in \mathcal{N}_\lambda \ : \ \psi_u''(1) = 0 \Big\}, \\
\mathcal{N}_\lambda^- = \Big\{ u \in \mathcal{N}_\lambda \ : \ &(p + \kappa - 1)\|u\|_{1,p}^p + (q + \kappa - 1)\|\nabla u\|_{q,\mu}^q + (p_* + \kappa - 1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&- \lambda(q_1 + \kappa - 1)\|u\|_{q_1}^{q_1} < 0 \Big\} \\
= \Big\{ & u \in \mathcal{N}_\lambda \ : \ \psi_u''(1) < 0 \Big\}.
\end{aligned}
$$

In general, the energy functional Θ_λ is not coercive on $W^{1,\mathcal{H}}(\Omega)$, but it is on the manifold $\mathcal{N}_\lambda$ as stated in the next proposition.

Proposition 3.1. *If hypotheses (H) hold, then $\Theta_\lambda|_{\mathcal{N}_\lambda}$ is coercive.*

Proof. Let $u \in \mathcal{N}_\lambda$ be such that $\|u\| > 1$. By the definition of $\mathcal{N}_\lambda$ we have

$$
-\frac{\lambda}{q_1}\|u\|_{q_1}^{q_1} = -\frac{1}{q_1}\|u\|_{1,p}^p - \frac{1}{q_1}\|\nabla u\|_{q,\mu}^q - \frac{1}{q_1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \frac{1}{q_1}\int_\Omega \zeta(x)|u|^{1-\kappa}\mathrm{d}x. \tag{3.1}
$$

Applying (3.1), Proposition 2.3(iv), and Theorem 13.17 of Hewitt and Stromberg,[40] p. 196 we obtain

$$
\begin{aligned}
\Theta_\lambda(u) = & \left[\frac{1}{p} - \frac{1}{q_1}\right]\|u\|_{1,p}^p + \left[\frac{1}{q} - \frac{1}{q_1}\right]\|\nabla u\|_{q,\mu}^q + \left[\frac{1}{p_*} - \frac{1}{q_1}\right]\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
& + \left[\frac{1}{q_1} - \frac{1}{1-\kappa}\right]\int_\Omega \zeta(x)|u|^{1-\kappa}\mathrm{d}x \\
\geq & \ c_1\rho(u) + \left[\frac{1}{q_1} - \frac{1}{1-\kappa}\right]\int_\Omega \zeta(x)|u|^{1-\kappa}\mathrm{d}x \\
\geq & \ c_1\|u\|^p - c_2\|u\|^{1-\kappa},
\end{aligned} \tag{3.2}
$$

since $p < q < q_1$ and $p_* < q_1$ by hypothesis (H)(ii) and for some positive constants c_1, c_2. Thus, the coercivity of Θ_λ on $\mathcal{N}_\lambda$ follows from (3.2) as $1 - \kappa < 1 < p$. □

Now we are going to prove that the global minimum of Θ_λ on $\mathcal{N}_\lambda^+$ is negative provided $\mathcal{N}_\lambda^+ \neq \emptyset$. The nonemptiness of $\mathcal{N}_\lambda^+$ will be proved later in Proposition 3.4. To this end, let $m_\lambda^+ = \inf_{\mathcal{N}_\lambda^+} \Theta_\lambda$.

Proposition 3.2. *If hypotheses (H) hold and if $\mathcal{N}_\lambda^+ \neq \emptyset$, then $\Theta_\lambda|_{\mathcal{N}_\lambda^+} < 0$. In particular, $m_\lambda^+ < 0$.*

Proof. Let $u \in \mathcal{N}_\lambda^+ \neq \emptyset$. By the definition of $\mathcal{N}_\lambda^+$, we get

$$
\lambda\|u\|_{q_1}^{q_1} < \frac{p + \kappa - 1}{q_1 + \kappa - 1}\|u\|_{1,p}^p + \frac{q + \kappa - 1}{q_1 + \kappa - 1}\|\nabla u\|_{q,\mu}^q + \frac{p_* + \kappa - 1}{q_1 + \kappa - 1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}. \tag{3.3}
$$

Since $\mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we have by the definition of $\mathcal{N}_\lambda$

$$
-\frac{1}{1-\kappa}\int_\Omega \zeta(x)|u|^{1-\kappa}\mathrm{d}x = -\frac{1}{1-\kappa}\left(\|u\|_{1,p}^p + \|\nabla u\|_{q,\mu}^q + \|u\|_{p_*,\beta,\partial\Omega}^{p_*}\right) + \frac{\lambda}{1-\kappa}\|u\|_{q_1}^{q_1}. \tag{3.4}
$$

Applying (3.4) and (3.3) leads to

$$
\begin{aligned}
\Theta_\lambda(u) &= \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|\nabla u\|_{q,\mu}^q + \frac{1}{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{1}{1-\kappa}\int_\Omega \zeta(x)|u|^{1-\kappa}dx - \frac{\lambda}{q_1}\|u\|_{q_1}^{q_1} \\
&= \left[\frac{1}{p} - \frac{1}{1-\kappa}\right]\|u\|_{1,p}^p + \left[\frac{1}{q} - \frac{1}{1-\kappa}\right]\|\nabla u\|_{q,\mu}^q + \left[\frac{1}{p_*} - \frac{1}{1-\kappa}\right]\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&\quad + \lambda\left[\frac{1}{1-\kappa} - \frac{1}{q_1}\right]\|u\|_{q_1}^{q_1} \\
&\leq \left[\frac{-(p+\kappa-1)}{p(1-\kappa)} + \frac{p+\kappa-1}{q_1+\kappa-1}\cdot\frac{q_1+\kappa-1}{q_1(1-\kappa)}\right]\|u\|_{1,p}^p \\
&\quad + \left[\frac{-(q+\kappa-1)}{q(1-\kappa)} + \frac{q+\kappa-1}{q_1+\kappa-1}\cdot\frac{q_1+\kappa-1}{q_1(1-\kappa)}\right]\|\nabla u\|_{q,\mu}^q \\
&\quad + \left[\frac{-(p_*+\kappa-1)}{p_*(1-\kappa)} + \frac{p_*+\kappa-1}{q_1+\kappa-1}\cdot\frac{q_1+\kappa-1}{q_1(1-\kappa)}\right]\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&= \frac{p+\kappa-1}{1-\kappa}\left[\frac{1}{q_1} - \frac{1}{p}\right]\|u\|_{1,p}^p + \frac{q+\kappa-1}{1-\kappa}\left[\frac{1}{q_1} - \frac{1}{q}\right]\|\nabla u\|_{q,\mu}^q \\
&\quad + \frac{p_*+\kappa-1}{1-\kappa}\left[\frac{1}{q_1} - \frac{1}{p_*}\right]\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&< 0,
\end{aligned}
$$

since $p < q < q_1$ and $p_* < q_1$; see hypotheses (H)(i) and (H)(ii). This shows that $\Theta_\lambda|_{\mathcal{N}_\lambda^+} < 0$ and so $m_\lambda^+ < 0$. $\qquad\square$

The next proposition shows that $\mathcal{N}_\lambda^\circ$ is empty provided the parameter $\lambda > 0$ is sufficiently small.

Proposition 3.3. *If hypotheses (H) hold, then there exists $\hat\lambda > 0$ such that $\mathcal{N}_\lambda^\circ = \varnothing$ for all $\lambda \in (0, \hat\lambda)$.*

Proof. We argue indirectly and assume that for every $\hat\lambda > 0$ we can find $\lambda \in (0, \hat\lambda)$ such that $\mathcal{N}_\lambda^\circ \neq \varnothing$. This means that for such $\lambda > 0$ there exists $u \in \mathcal{N}_\lambda^\circ$ such that

$$(p+\kappa-1)\|u\|_{1,p}^p + (q+\kappa-1)\|\nabla u\|_{q,\mu}^q + (p_*+\kappa-1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} = \lambda(q_1+\kappa-1)\|u\|_{q_1}^{q_1}. \tag{3.5}$$

We know that $\mathcal{N}_\lambda^\circ \subset \mathcal{N}_\lambda$ and so $u \in \mathcal{N}_\lambda$, that is,

$$
\begin{aligned}
&(q_1+\kappa-1)\|u\|_{1,p}^p + (q_1+\kappa-1)\|\nabla u\|_{q,\mu}^q + (q_1+\kappa-1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} \\
&= (q_1+\kappa-1)\int_\Omega \zeta(x)|u|^{1-\kappa}dx + \lambda(q_1+\kappa-1)\|u\|_{q_1}^{q_1}.
\end{aligned} \tag{3.6}
$$

Subtracting (3.5) from (3.6), we obtain

$$(q_1-p)\|u\|_{1,p}^p + (q_1-q)\|\nabla u\|_{q,\mu}^q + (q_1-p_*)\|u\|_{p_*,\beta,\partial\Omega}^{p_*} = (q_1+\kappa-1)\int_\Omega \zeta(x)|u|^{1-\kappa}dx. \tag{3.7}$$

Since $0 < 1-\kappa < 1 < p < q < q_1$ and $p_* < q_1$, we can use Proposition 2.3(iii), (iv) to the left-hand side of (3.7) and Theorem 13.17 of Hewitt and Stromberg[40], p. 196 along with Proposition 2.1(ii) to the right-hand side of (3.7) in order to get

$$\min\left\{\|u\|^p, \|u\|^q\right\} \leq c_3\|u\|^{1-\kappa}$$

for some constant $c_3 > 0$. As $0 < 1-\kappa < 1 < p < q$, this implies

$$\|u\| \leq c_4 \tag{3.8}$$

for some $c_4 > 0$. However, from Equation (3.5), we deduce

$$\min\left\{\|u\|^p, \|u\|^q\right\} \leq \lambda c_5 \|u\|^{q_1} \tag{3.9}$$

for some $c_5 > 0$ where we have used Propositions 2.3(iii), (iv) and 2.1(ii). From (3.9), we then conclude that

$$\|u\| \geq \left(\frac{1}{\lambda c_5}\right)^{\frac{1}{q_1-p}} \text{ or } \|u\| \geq \left(\frac{1}{\lambda c_5}\right)^{\frac{1}{q_1-q}}.$$

Letting $\lambda \to 0^+$, it follows $\|u\| \to +\infty$ since $p < q < q_1$ contradicting (3.8). This proves the emptiness of $\mathcal{N}_\lambda^\circ$ for all $\lambda \in (0, \hat{\lambda})$. $\square$

Next, we are going to prove the nonemptiness of the sets $\mathcal{N}_\lambda^\pm$ for small values of the parameter $\lambda > 0$, and we will show that the functional Θ_λ achieves its global minimum restricted to the set $\mathcal{N}_\lambda^+$.

Proposition 3.4. *If hypotheses (H) hold, then there exists $\tilde{\lambda} \in (0, \hat{\lambda}]$ such that $\mathcal{N}_\lambda^\pm \neq \emptyset$ for all $\lambda \in (0, \tilde{\lambda})$ and for any $\lambda \in (0, \tilde{\lambda})$, there exists $u_\lambda \in \mathcal{N}_\lambda^+$ such that $\Theta_\lambda(u_\lambda) = m_\lambda^+ < 0$ and $u_\lambda(x) \geq 0$ for a. a. $x \in \Omega$.*

Proof. Let $u \in W^{1,\mathcal{H}}(\Omega)$ be such that $u \not\equiv 0$. We consider the function $\tilde{\eta}_u : (0, +\infty) \to \mathbb{R}$ defined by

$$\tilde{\eta}_u(t) = t^{p-q_1} \|u\|_{1,p}^p - t^{-q_1-\kappa+1} \int_\Omega \zeta(x)|u|^{1-\kappa} dx. \tag{3.10}$$

Recall that $q_1 - p < q_1 + \kappa - 1$. Hence, there exists a unique $\tilde{t}_u^\circ > 0$ such that

$$\tilde{\eta}_u\left(\tilde{t}_u^\circ\right) = \max_{t>0} \tilde{\eta}_u(t).$$

This means

$$(p - q_1)\left(\tilde{t}_u^\circ\right)^{p-q_1-1} \|u\|_{1,p}^p + (q_1 + \kappa - 1)\left(\tilde{t}_u^\circ\right)^{-q_1-\kappa} \int_\Omega \zeta(x)|u|^{1-\kappa} dx = 0$$

and so

$$\tilde{t}_u^\circ = \left[\frac{(q_1 + \kappa - 1)\int_\Omega \zeta(x)|u|^{1-\kappa} dx}{(q_1 - p)\|u\|_{1,p}^p}\right]^{\frac{1}{p+\kappa-1}}. \tag{3.11}$$

Using (3.11) into the definition of $\tilde{\eta}_u$ in (3.10) gives

$$
\begin{aligned}
\tilde{\eta}_u\left(\tilde{t}_u^\circ\right) &= \frac{\left[(q_1 - p)\|u\|_{1,p}^p\right]^{\frac{q_1-p}{p+\kappa-1}}}{\left[(q_1 + \kappa - 1)\int_\Omega \zeta(x)|u|^{1-\kappa} dx\right]^{\frac{q_1-p}{p+\kappa-1}}} \|u\|_{1,p}^p \\
&\quad - \frac{\left[(q_1 - p)\|u\|_{1,p}^p\right]^{\frac{q_1+\kappa-1}{p+\kappa-1}}}{\left[(q_1 + \kappa - 1)\int_\Omega \zeta(x)|u|^{1-\kappa} dx\right]^{\frac{q_1+\kappa-1}{p+\kappa-1}}} \int_\Omega \zeta(x)|u|^{1-\kappa} dx \\
&= \frac{(q_1 - p)^{\frac{q_1-p}{p+\kappa-1}} \|u\|_{1,p}^{\frac{p(q_1+\kappa-1)}{p+\kappa-1}}}{(q_1 + \kappa - 1)^{\frac{q_1-p}{p+\kappa-1}} \left[\int_\Omega \zeta(x)|u|^{1-\kappa} dx\right]^{\frac{q_1-p}{p+\kappa-1}}} \\
&\quad - \frac{(q_1 - p)^{\frac{q_1+\kappa-1}{p+\kappa-1}} \|u\|_{1,p}^{\frac{p(q_1+\kappa-1)}{p+\kappa-1}}}{(q_1 + \kappa - 1)^{\frac{q_1+\kappa-1}{p+\kappa-1}} \left[\int_\Omega \zeta(x)|u|^{1-\kappa} dx\right]^{\frac{q_1-p}{p+\kappa-1}}} \\
&= \frac{p + \kappa - 1}{q_1 - p}\left[\frac{q_1 - p}{q_1 + \kappa - 1}\right]^{\frac{q_1+\kappa-1}{p+\kappa-1}} \frac{\|u\|_{1,p}^{\frac{p(q_1+\kappa-1)}{p+\kappa-1}}}{\left[\int_\Omega \zeta(x)|u|^{1-\kappa} dx\right]^{\frac{q_1-p}{p+\kappa-1}}}.
\end{aligned}
\tag{3.12}
$$

Applying Theorem 13.17 of Hewitt and Stromberg,[40], p. 196 we have

$$\int_\Omega \zeta(x)|u|^{1-\kappa}\mathrm{d}x \leq c_6 \|u\|_{p^*}^{1-\kappa} \tag{3.13}$$

for some $c_6 > 0$. Furthermore, we denote by S the best constant of the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, that is,

$$S\|u\|_{p^*}^p \leq \|u\|_{1,p}^p. \tag{3.14}$$

From (3.12), (3.13), and (3.14), it follows that

$$
\begin{aligned}
\tilde{\eta}_u\left(\tilde{t}_u^\circ\right) &- \lambda\|u\|_{q_1}^{q_1}\\
&= \frac{p+\kappa-1}{q_1-p}\left[\frac{q_1-p}{q_1+\kappa-1}\right]^{\frac{q_1+\kappa-1}{p+\kappa-1}} \frac{\|u\|_{1,p}^{\frac{p(q_1+\kappa-1)}{p+\kappa-1}}}{\left[\int_\Omega\zeta(x)|u|^{1-\kappa}\mathrm{d}x\right]^{\frac{q_1-p}{p+\kappa-1}}} - \lambda\|u\|_{q_1}^{q_1}\\
&\geq \frac{p+\kappa-1}{q_1-p}\left[\frac{q_1-p}{q_1+\kappa-1}\right]^{\frac{q_1+\kappa-1}{p+\kappa-1}} \frac{S^{\frac{q_1+\kappa-1}{p+\kappa-1}}\left(\|u\|_{p^*}^p\right)^{\frac{q_1+\kappa-1}{p+\kappa-1}}}{\left(c_6\|u\|_{p^*}^{1-\kappa}\right)^{\frac{q_1-p}{p+\kappa-1}}} - \lambda c_7\|u\|_{p^*}^{q_1}\\
&= [c_8 - \lambda c_7]\,\|u\|_{p^*}^{q_1}
\end{aligned}
\tag{3.15}
$$

for some constants $c_7, c_8 > 0$. From (3.15), we conclude that there exists $\tilde{\lambda} \in (0, \hat{\lambda}]$ independent of u such that

$$\tilde{\eta}_u\left(\tilde{t}_u^\circ\right) - \lambda\|u\|_{q_1}^{q_1} > 0 \ \text{ for all } \ \lambda \in \left(0, \tilde{\lambda}\right). \tag{3.16}$$

Now we introduce the function $\eta_u : (0, +\infty) \to \mathbb{R}$ defined by

$$\eta_u(t) = t^{p-q_1}\|u\|_{1,p}^p + t^{q-q_1}\|\nabla u\|_{q,\mu}^q + t^{p_*-q_1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - t^{-q_1-\kappa+1}\int_\Omega\zeta(x)|u|^{1-\kappa}\mathrm{d}x.$$

Note that the equation

$$
\begin{aligned}
0 = \eta_u'(t) &= (p-q_1)t^{p-q_1-1}\|u\|_{1,p}^p + (q-q_1)t^{q-q_1-1}\|\nabla u\|_{q,\mu}^q\\
&\quad + (p_*-q_1)t^{p_*-q_1-1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - (-q_1-\kappa+1)t^{-q_1-\kappa}\int_\Omega\zeta(x)|u|^{1-\kappa}\mathrm{d}x
\end{aligned}
\tag{3.17}
$$

is equivalent to

$$
\begin{aligned}
(q_1-p)t^{p+\kappa-1}\|u\|_{1,p}^p &+ (q_1-q)t^{q+\kappa-1}\|\nabla u\|_{q,\mu}^q + (q_1-p_*)t^{p_*+\kappa-1}\|u\|_{p_*,\beta,\partial\Omega}^{p_*}\\
&= (q_1+\kappa-1)\int_\Omega\zeta(x)|u|^{1-\kappa}\mathrm{d}x.
\end{aligned}
\tag{3.18}
$$

Since $0 < q_1 - q < q_1 - p < q_1 + \kappa - 1$, $0 < q_1 - p_*$, $0 < p + \kappa - 1 < q + \kappa - 1$, and $0 < p_* + \kappa - 1$, we can observe that the left-hand side of (3.18), which we shall denote as $\xi_u(t)$, fulfills

$$\lim_{t\to 0^+}\xi_u(t) = 0, \ \lim_{t\to+\infty}\xi_u(t) = +\infty \ \text{and} \ \xi_u'(t) > 0 \ \text{ for all } \ t > 0.$$

From the two limits and via the intermediate value theorem, one can derive that there exists $t_u^\circ > 0$ such that (3.18) holds, and from the remaining claim that this value is unique due to the injectivity of $\xi_u(t)$. Furthermore, if we consider $\eta_u'(t) > 0$ ($\eta_u'(t) < 0$), then (3.18) holds with a sign $<$ ($>$), and as $\xi_u(t)$ is strictly increasing, this holds for $t < t_u^\circ$ ($t > t_u^\circ$).

Hence, $\eta_u(t)$ is strictly increasing in $(0, t_u^\circ)$, strictly decreasing in (t_u°, ∞), and

$$\eta_u(t_u^\circ) = \max_{t>0} \eta_u(t).$$

It is easy to see that $\eta_u \geq \hat{\eta}_u$. Thus, from (3.16), there exists $\tilde{\lambda} \in (0, \hat{\lambda}]$ independent of u such that

$$\eta_u(t_u^\circ) - \lambda \|u\|_{q_1}^{q_1} > 0 \text{ for all } \lambda \in \left(0, \tilde{\lambda}\right).$$

On the other hand, as $0 < q_1 - q < q_1 - p < q_1 + \kappa - 1$, $0 < q_1 - p_*$, there holds

$$\lim_{t \to 0^+} \eta_u(t) = -\infty \text{ and } \lim_{t \to +\infty} \eta_u(t) = 0.$$

By the intermediate value theorem and the injectivity of $\eta_u(t)$ in $(0, t_u^\circ)$ and (t_u°, ∞), there exist unique numbers $t_u^1 < t_u^\circ < t_u^2$ such that

$$\eta_u\left(t_u^1\right) = \lambda \|u\|_{q_1}^{q_1} = \eta_u\left(t_u^2\right) \text{ and } \eta_u'\left(t_u^2\right) < 0 < \eta_u'\left(t_u^1\right). \tag{3.19}$$

Recall that the fibering function $\psi_u : [0, +\infty) \to \mathbb{R}$ is given by

$$\psi_u(t) = \Theta_\lambda(tu) \text{ for all } t \geq 0.$$

We have

$$\psi_u'\left(t_u^1\right) = \left(t_u^1\right)^{p-1} \|u\|_{1,p}^p + \left(t_u^1\right)^{q-1} \|\nabla u\|_{q,\mu}^q + \left(t_u^1\right)^{p_*-1} \|u\|_{p_*,\beta,\partial\Omega}^{p_*}$$
$$- \left(t_u^1\right)^{-\kappa} \int_\Omega \zeta(x)|u|^{1-\kappa} dx - \lambda\left(t_u^1\right)^{q_1-1} \|u\|_{q_1}^{q_1}$$

and

$$\psi_u''\left(t_u^1\right) = (p-1)\left(t_u^1\right)^{p-2} \|u\|_{1,p}^p + (q-1)\left(t_u^1\right)^{q-2} \|\nabla u\|_{q,\mu}^q$$
$$+ (p_*-1)\left(t_u^1\right)^{p_*-2} \|u\|_{p_*,\beta,\partial\Omega}^{p_*} + \kappa\left(t_u^1\right)^{-\kappa-1} \int_\Omega \zeta(x)|u|^{1-\kappa} dx \tag{3.20}$$
$$- \lambda(q_1-1)\left(t_u^1\right)^{q_1-2} \|u\|_{q_1}^{q_1}.$$

The first relation in (3.19) gives

$$\left(t_u^1\right)^{p-q_1} \|u\|_{1,p}^p + \left(t_u^1\right)^{q-q_1} \|\nabla u\|_{q,\mu}^q + \left(t_u^1\right)^{p_*-q_1} \|u\|_{p_*,\beta,\partial\Omega}^{p_*}$$
$$- \left(t_u^1\right)^{-q_1-\kappa+1} \int_\Omega \zeta(x)|u|^{1-\kappa} dx = \lambda \|u\|_{q_1}^{q_1}. \tag{3.21}$$

Now we multiply (3.21) with $\kappa\left(t_u^1\right)^{q_1-2}$ and $-(q_1-1)\left(t_u^1\right)^{q_1-2}$, respectively. It follows

$$\kappa\left(t_u^1\right)^{p-2} \|u\|_{1,p}^p + \kappa\left(t_u^1\right)^{q-2} \|\nabla u\|_{q,\mu}^q + \kappa\left(t_u^1\right)^{p_*-2} \|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \kappa\lambda\left(t_u^1\right)^{q_1-2} \|u\|_{q_1}^{q_1}$$
$$= \kappa\left(t_u^1\right)^{-\kappa-1} \int_\Omega \zeta(x)|u|^{1-\kappa} dx \tag{3.22}$$

and

$$- (q_1-1)\left(t_u^1\right)^{p-2} \|u\|_{1,p}^p - (q_1-1)\left(t_u^1\right)^{q-2} \|\nabla u\|_{q,\mu}^q - (q_1-1)\left(t_u^1\right)^{p_*-2} \|u\|_{p_*,\beta,\partial\Omega}^{p_*}$$
$$+ (q_1-1)\left(t_u^1\right)^{-\kappa-1} \int_\Omega \zeta(x)|u|^{1-\kappa} dx = -\lambda(q_1-1)\left(t_u^1\right)^{q_1-2} \|u\|_{q_1}^{q_1}. \tag{3.23}$$

Now we use (3.22) in (3.20) which leads to

$$
\begin{aligned}
\psi_u''\left(t_u^1\right) &= (p + \kappa - 1)\left(t_u^1\right)^{p-2}\|u\|_{1,p}^p + (q + \kappa - 1)\left(t_u^1\right)^{q-2}\|\nabla u\|_{q,\mu}^q \\
&\quad + (p_* + \kappa - 1)\left(t_u^1\right)^{p_*-2}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda(q_1 + \kappa - 1)\left(t_u^1\right)^{q_1-2}\|u\|_{q_1}^{q_1} \\
&= \left(t_u^1\right)^{-2}\left[(p + \kappa - 1)\left(t_u^1\right)^p\|u\|_{1,p}^p + (q + \kappa - 1)\left(t_u^1\right)^q\|\nabla u\|_{q,\mu}^q \right. \\
&\quad \left. + (p_* + \kappa - 1)\left(t_u^1\right)^{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda(q_1 + \kappa - 1)\left(t_u^1\right)^{q_1}\|u\|_{q_1}^{q_1}\right].
\end{aligned}
\tag{3.24}
$$

On the other hand, applying (3.23) in (3.20) along with the representation in (3.17) yields

$$
\begin{aligned}
\psi_u''\left(t_u^1\right) &= (p - q_1)\left(t_u^1\right)^{p-2}\|u\|_{1,p}^p + (q - q_1)\left(t_u^1\right)^{q-2}\|\nabla u\|_{q,\mu}^q \\
&\quad + (p_* - q_1)\left(t_u^1\right)^{p_*-2}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} + (q_1 + \kappa - 1)\left(t_u^1\right)^{-\kappa-1}\int_\Omega \zeta(x)|u|^{1-\kappa}\mathrm{d}x \\
&= \left(t_u^1\right)^{q_1-1}\eta_u'\left(t_u^1\right) > 0;
\end{aligned}
\tag{3.25}
$$

see (3.17). From (3.24) and (3.25), we conclude that

$$
\begin{aligned}
&(p + \kappa - 1)\left(t_u^1\right)^p\|u\|_{1,p}^p + (q + \kappa - 1)\left(t_u^1\right)^q\|\nabla u\|_{q,\mu}^q \\
&\quad + (p_* + \kappa - 1)\left(t_u^1\right)^{p_*}\|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda(q_1 + \kappa - 1)\left(t_u^1\right)^{q_1}\|u\|_{q_1}^{q_1} > 0,
\end{aligned}
$$

since $t_u^1 > 0$. This shows that

$$
t_u^1 u \in \mathcal{N}_\lambda^+ \text{ for all } \lambda \in \left(0, \tilde\lambda\right].
$$

Hence, $\mathcal{N}_\lambda^+ \neq \emptyset$. The same treatment can be done for the point t_u^2 in order to prove that $\mathcal{N}_\lambda^- \neq \emptyset$.

Now, we are going to show the second assertion of the proposition. To this end, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ be a minimizing sequence, that is,

$$
\Theta_\lambda(u_n) \searrow m_\lambda^+ < 0 \text{ as } n \to \infty.
\tag{3.26}
$$

First, we know that $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\Omega)$ is bounded since $\mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$ and by applying Proposition 3.1. Hence, we may assume that

$$
u_n \rightharpoonup u_\lambda \text{ in } W^{1,\mathcal{H}}(\Omega) \ u_n \to u_\lambda \text{ in } L^{q_1}(\Omega) \text{ and } u_n \rightharpoonup u_\lambda \text{ in } L^{p_*}(\partial\Omega);
\tag{3.27}
$$

see Proposition 2.1(ii), (iii). From (3.26) and (3.27), it follows that

$$
\Theta_\lambda(u_\lambda) \leq \liminf_{n\to+\infty} \Theta_\lambda(u_n) < 0 = \Theta_\lambda(0).
$$

Therefore, $u_\lambda \neq 0$.

Next, we want to prove that

$$
\lim_{n\to+\infty} \rho(u_n) = \rho(u_\lambda)
\tag{3.28}
$$

for a subsequence (still denoted by u_n).

Claim 1: $\liminf_{n\to+\infty}\|u_n\|_{1,p}^p = \|u_\lambda\|_{1,p}^p$

Let us suppose Claim 1 is not true. Then we have

$$
\liminf_{n\to+\infty} \|u_n\|_{1,p}^p > \|u_\lambda\|_{1,p}^p.
$$

Using this and (3.19) along with the weak lower semicontinuity of the corresponding norms and seminorms results in

$$
\liminf_{n\to+\infty} \psi'_{u_n}\left(t^1_{u_\lambda}\right)
$$

$$
= \liminf_{n\to+\infty}\left[\left(t^1_{u_\lambda}\right)^{p-1}\|u_n\|^p_{1,p} + \left(t^1_{u_\lambda}\right)^{q-1}\|\nabla u_n\|^q_{q,\mu} + \left(t^1_{u_\lambda}\right)^{p_*-1}\|u_n\|^{p_*}_{p_*,\beta,\partial\Omega}\right.
$$

$$
\left. - \left(t^1_{u_\lambda}\right)^{-\kappa}\int_\Omega \zeta(x)|u_n|^{1-\kappa}\mathrm{d}x - \lambda\left(t^1_{u_\lambda}\right)^{q_1-1}\|u_n\|^{q_1}_{q_1}\right]
$$

$$
> \left(t^1_{u_\lambda}\right)^{p-1}\|u_\lambda\|^p_{1,p} + \left(t^1_{u_\lambda}\right)^{q-1}\|\nabla u_\lambda\|^q_{q,\mu} + \left(t^1_{u_\lambda}\right)^{p_*-1}\|u_\lambda\|^{p_*}_{p_*,\beta,\partial\Omega}
$$

$$
- \left(t^1_{u_\lambda}\right)^{-\kappa}\int_\Omega \zeta(x)|u_\lambda|^{1-\kappa}\mathrm{d}x - \lambda\left(t^1_{u_\lambda}\right)^{q_1-1}\|u_\lambda\|^{q_1}_{q_1}
$$

$$
= \psi'_{u_\lambda}\left(t^1_{u_\lambda}\right) = \left(t^1_{u_\lambda}\right)^{q_1-1}\left[\eta_{u_\lambda}\left(t^1_{u_\lambda}\right) - \lambda\|u_\lambda\|^{q_1}_{q_1}\right] = 0.
$$

Hence, there exists a number $n_0 \in \mathbb{N}$ such that $\psi'_{u_n}(t^1_{u_\lambda}) > 0$ for all $n > n_0$. We know that $u_n \in \mathcal{N}^+_\lambda \subset \mathcal{N}_\lambda$ and $\psi'_{u_n}(t) = t^{q_1-1}\left[\eta_{u_n}(t) - \lambda\|u_n\|^{q_1}_{q_1}\right]$. Therefore, we conclude that $\psi'_{u_n}(t) < 0$ for all $t \in (0,1)$ and $\psi'_{u_n}(1) = 0$ which implies $t^1_{u_\lambda} > 1$.

Recall that ψ_{u_λ} is decreasing on $(0, t^1_{u_\lambda}]$. This implies

$$
\Theta_\lambda\left(t^1_{u_\lambda}u_\lambda\right) \leq \Theta_\lambda(u_\lambda) < m^+_\lambda.
$$

Since $t^1_{u_\lambda}u_\lambda \in \mathcal{N}^+_\lambda$, we have

$$
m^+_\lambda \leq \Theta_\lambda\left(t^1_{u_\lambda}u_\lambda\right) < m^+_\lambda,
$$

a contradiction. So Claim 1 is proved.

From Claim 1, we find a subsequence (still denoted by u_n) such that

$$
\|u_n\|^p_{1,p} \to \|u_\lambda\|^p_{1,p}. \tag{3.29}
$$

Claim 2: $\liminf_{n\to+\infty}\|\nabla u_n\|^q_{q,\mu} = \|\nabla u_\lambda\|^q_{q,\mu}$ for the subsequence in (3.29).

As before, let us suppose Claim 2 is not true. So we have

$$
\liminf_{n\to+\infty}\|\nabla u_n\|^q_{q,\mu} > \|\nabla u_\lambda\|^q_{q,\mu}.
$$

Then we can argue exactly as in the proof of Claim 1. This shows Claim 2.

From Claim 2, we find a subsequence (still denoted by u_n) such that

$$
\|\nabla u_n\|^q_{q,\mu} \to \|\nabla u_\lambda\|^q_{q,\mu}. \tag{3.30}
$$

Claim 3: $\liminf_{n\to+\infty}\|u_n\|^{p_*}_{p_*,\beta,\partial\Omega} = \|u_\lambda\|^{p_*}_{p_*,\beta,\partial\Omega}$ for the subsequence in (3.30).

The proof is the same as in Claims 1 and 2. So we find again a subsequence (still denoted by u_n) such that

$$
\|u_n\|^{p_*}_{p_*,\beta,\partial\Omega} \to \|u_\lambda\|^{p_*}_{p_*,\beta,\partial\Omega}. \tag{3.31}
$$

For the sequence in (3.31), we know that the convergences in (3.29) and (3.30) hold true. So combining (3.29)–(3.31) for this sequence, we see that (3.28) is satisfied. Since the integrand corresponding to the modular function $\rho(\cdot)$ is uniformly convex, this implies that $u_n \to u_\lambda$ in $W^{1,\mathcal{H}}(\Omega)$; see also Fan and Guan[41] Theorems 3.2 and 3.5. By the continuity of Θ_λ, we have $\Theta_\lambda(u_n) \to \Theta_\lambda(u_\lambda)$, and thus, $\Theta_\lambda(u_\lambda) = m^+_\lambda$. We know that $u_n \in \mathcal{N}^+_\lambda$ for all $n \in \mathbb{N}$, that is,

$$
(p+\kappa-1)\|u_n\|^p_{1,p} + (q+\kappa-1)\|\nabla u_n\|^q_{q,\mu} + (p_*+\kappa-1)\|u_n\|^{p_*}_{p_*,\beta,\partial\Omega}
$$

$$
- \lambda(q_1+\kappa-1)\|u_n\|^{q_1}_{q_1} > 0. \tag{3.32}
$$

Passing to the limit in (3.32) as $n \to +\infty$, we obtain

$$(p + \kappa - 1)\|u_\lambda\|_{1,p}^p + (q + \kappa - 1)\|\nabla u_\lambda\|_{q,\mu}^q + (p_* + \kappa - 1)\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda(q_1 + \kappa - 1)\|u_\lambda\|_{q_1}^{q_1} \geq 0. \tag{3.33}$$

Since $\lambda \in (0, \tilde{\lambda})$ and $\tilde{\lambda} \leq \hat{\lambda}$, we conclude from Proposition 3.3 that we have a strict inequality in (3.33). Hence, $u_\lambda \in \mathcal{N}_\lambda^+$. Note that we can always use $|u_\lambda|$ instead of u_λ, so we may assume that $u_\lambda(x) \geq 0$ for a. a. $x \in \Omega$ such that $u_\lambda \neq 0$. □

The next proposition plays a key role in order to prove that u_λ is a weak solution of problem (1.1).

Proposition 3.5. *If hypotheses (H) hold, $h \in W^{1,\mathcal{H}}(\Omega)$ and $\lambda \in (0, \tilde{\lambda}]$, then we can find $\delta > 0$ such that $\Theta_\lambda(u_\lambda) \leq \Theta_\lambda(u_\lambda + th)$ for all $t \in [0, \delta]$.*

Proof. Taking $u \in \mathcal{N}_\lambda^+$, we introduce the function $\varphi : W^{1,\mathcal{H}}(\Omega) \times (0, \infty) \to \mathbb{R}$ defined by

$$\varphi(y,t) = t^{p+\kappa-1}\|u + y\|_{1,p}^p + t^{q+\kappa-1}\|\nabla(u + y)\|_{q,\mu}^q + t^{p_*+\kappa-1}\|u + y\|_{p_*,\beta,\partial\Omega}^{p_*}$$
$$- \int_\Omega \zeta(x)|u + y|^{1-\kappa}dx - \lambda t^{q_1+\kappa-1}\|u + y\|_{q_1}^{q_1} \quad \text{for all} \ y \in W^{1,\mathcal{H}}(\Omega).$$

Due to $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we easily see that $\varphi(0, 1) = 0$. Moreover, since $u \in \mathcal{N}_\lambda^+$, we obtain

$$\varphi_t'(0, 1) = (p + \kappa - 1)\|u\|_{1,p}^p + (q + \kappa - 1)\|\nabla u\|_{q,\mu}^q + (p_* + \kappa - 1)\|u\|_{p_*,\beta,\partial\Omega}^{p_*}$$
$$- \lambda(q_1 + \kappa - 1)\|u\|_{q_1}^{q_1} > 0.$$

Now we can apply the implicit function theorem; see, for example, Berger,[42] p. 115 which ensures the existence of $\varepsilon > 0$ and a continuous function $\chi : B_\varepsilon(0) \to (0, \infty)$ such that

$$\chi(0) = 1 \text{ and } \chi(y)(u + y) \in \mathcal{N}_\lambda \ \text{ for all } \ y \in B_\varepsilon(0),$$

where $B_\varepsilon(0) = \{u \in W^{1,\mathcal{H}}(\Omega) \ : \ \|u\| < \varepsilon\}$. Choosing $\varepsilon > 0$ small enough, we have

$$\chi(0) = 1 \text{ and } \chi(y)(u + y) \in \mathcal{N}_\lambda^+ \ \text{ for all } \ y \in B_\varepsilon(0). \tag{3.34}$$

We define now the function $\Xi_h : [0, +\infty) \to \mathbb{R}$ given by

$$\Xi_h(t) = (p - 1)\|u_\lambda + th\|_{1,p}^p + (q - 1)\|\nabla u_\lambda + t\nabla h\|_{q,\mu}^q + (p_* - 1)\|u_\lambda + th\|_{p_*,\beta,\partial\Omega}^{p_*}$$
$$+ \kappa \int_\Omega \zeta(x)|u_\lambda + th|^{1-\kappa}dx - \lambda(q_1 - 1)\|u_\lambda + th\|_{q_1}^{q_1}. \tag{3.35}$$

We know that $u_\lambda \in \mathcal{N}_\lambda$ and $u_\lambda \in \mathcal{N}_\lambda^+$. This gives

$$\kappa \int_\Omega \zeta(x)|u_\lambda|^{1-\kappa}dx = \kappa\|u_\lambda\|_{1,p}^p + \kappa\|\nabla u_\lambda\|_{q,\mu}^q + \kappa\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*} - \lambda\kappa\|u_\lambda\|_{q_1}^{q_1} \tag{3.36}$$

and

$$(p + \kappa - 1)\|u_\lambda\|_{1,p}^p + (q + \kappa - 1)\|\nabla u_\lambda\|_{q,\mu}^q + (p_* + \kappa - 1)\|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}$$
$$- \lambda(q_1 + \kappa - 1)\|u_\lambda\|_{q_1}^{q_1} > 0. \tag{3.37}$$

Using (3.36) and (3.37) in (3.35), we infer that $\Xi_h(0) > 0$, and since $\Xi_h : [0, +\infty) \to \mathbb{R}$ is continuous, there exists a number $\delta_0 > 0$ such that

$$\Xi_h(t) > 0 \ \text{ for all } \ t \in [0, \delta_0].$$

From the first part of the proof (see 3.34), we know that for every $t \in [0, \delta_0]$, there exists $\chi(t) > 0$ such that

$$\chi(t)(u_\lambda + th) \in \mathcal{N}_\lambda^+ \text{ and } \chi(t) \to 1 \text{ as } t \to 0^+.$$

From Proposition 3.4, it follows that

$$m_\lambda^+ = \Theta_\lambda(u_\lambda) \leq \Theta_\lambda(\chi(t)(u_\lambda + th)) \quad \text{for all } t \in [0, \delta_0]. \tag{3.38}$$

Let $u \in W^{1,\mathcal{H}}(\Omega)$ be arbitrary and recall that $\psi_u(t) = \Theta_\lambda(tu)$ for all $t \geq 0$. Note that for $t > 0$, we have $t\psi_u'(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$, that is, $t = t_u^1$ or $t = t_u^2$ by (3.19). From (3.25), we also know that $\psi_u''(t_u^1) > 0 > \psi_u''(t_u^2)$. On the other hand, there exists a unique $t_u^* > 0$ such that $\psi_u''(t_u^*) = 0$. Hence, $t_u^1 < t_u^* < t_u^2$. Altogether, if $\psi_u''(t) > 0$, then $t \in (0, t_u^*) \subseteq (0, t_u^2)$ and $\psi_u(t) \geq \psi_u(t_u^1)$, see Figure 1.

Now, from $\psi_{u_\lambda}''(1) > 0$ and the continuity in t, we have $\psi_{u_\lambda+th}''(1) > 0$ for $t \in [0, \delta]$ with $\delta \in (0, \delta_0]$. Therefore, using (3.38),

$$m_\lambda^+ = \Theta_\lambda(u_\lambda) \leq \Theta_\lambda(\chi(t)(u_\lambda + th)) = \psi_{u_\lambda+th}(\chi(t)) \leq \psi_{u_\lambda+th}(1) = \Theta_\lambda(u_\lambda + th)$$

for all $t \in [0, \delta]$. □

Now we are ready to show that u_λ is indeed a weak solution for problem (1.1).

Proposition 3.6. *If hypotheses (H) hold and $\lambda \in (0, \tilde{\lambda})$, then u_λ is a weak solution of problem (1.1) such that $\Theta_\lambda(u_\lambda) < 0$.*

Proof. First, we mention that $u_\lambda \geq 0$ for a. a. $x \in \Omega$ and $\Theta_\lambda(u_\lambda) < 0$; see Proposition 3.4.

Claim: $u_\lambda > 0$ for a. a. $x \in \Omega$

Arguing indirectly and suppose there exists a set B with positive measure such that $u_\lambda = 0$ in B. Let $h \in W^{1,\mathcal{H}}(\Omega)$ with $h > 0$ and let $t \in (0, \delta)$ (see Proposition 3.5); then $(u_\lambda + th)^{1-\kappa} > u_\lambda^{1-\kappa}$ a. e. in $\Omega \setminus B$. Applying Proposition 3.5 yields

$$
\begin{aligned}
0 &\leq \frac{\Theta_\lambda(u_\lambda + th) - \Theta_\lambda(u_\lambda)}{t} \\
&= \frac{1}{p} \frac{\|u_\lambda + th\|_{1,p}^p - \|u_\lambda\|_{1,p}^p}{t} + \frac{1}{q} \frac{\|\nabla(u_\lambda + th)\|_{q,\mu}^q - \|\nabla u_\lambda\|_{q,\mu}^q}{t} \\
&\quad + \frac{1}{p_*} \frac{\|u_\lambda + th\|_{p_*,\beta,\partial\Omega}^{p_*} - \|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}}{t} - \frac{1}{(1-\kappa)t^\kappa} \int_B \zeta(x)h^{1-\kappa}\,dx \\
&\quad - \frac{1}{1-\kappa} \int_{\Omega \setminus B} \zeta(x) \frac{(u_\lambda + th)^{1-\kappa} - (u_\lambda)^{1-\kappa}}{t}\,dx - \frac{\lambda}{q_1} \frac{\|u_\lambda + th\|_{q_1}^{q_1} - \|u_\lambda\|_{q_1}^{q_1}}{t} \\
&< \frac{1}{p} \frac{\|u_\lambda + th\|_{1,p}^p - \|u_\lambda\|_{1,p}^p}{t} + \frac{1}{q} \frac{\|\nabla(u_\lambda + th)\|_{q,\mu}^q - \|\nabla u_\lambda\|_{q,\mu}^q}{t} \\
&\quad + \frac{1}{p_*} \frac{\|u_\lambda + th\|_{p_*,\beta,\partial\Omega}^{p_*} - \|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}}{t} - \frac{1}{(1-\kappa)t^\kappa} \int_B \zeta(x)h^{1-\kappa}\,dx \\
&\quad - \frac{\lambda}{q_1} \frac{\|u_\lambda + th\|_{q_1}^{q_1} - \|u_\lambda\|_{q_1}^{q_1}}{t}.
\end{aligned}
$$

From this, we conclude by hypothesis (H)(v) that

$$0 \leq \frac{\Theta_\lambda(u_\lambda + th) - \Theta_\lambda(u_\lambda)}{t} \to -\infty \text{ as } t \to 0^+,$$

which is a contradiction. This shows the claim and so $u_\lambda > 0$ a. e. in Ω.

Let us now show that

$$\zeta(\cdot)(u_\lambda)^{-\kappa}h \in L^1(\Omega) \text{ for all } h \in W^{1,\mathcal{H}}(\Omega) \tag{3.39}$$

and

$$
\int_\Omega \left(|\nabla u_\lambda|^{p-2}\nabla u_\lambda + \mu(x)|\nabla u_\lambda|^{q-2}\nabla u_\lambda\right) \cdot \nabla h \mathrm{d}x
$$
$$
+ \int_\Omega \alpha(x)(u_\lambda)^{p-1}h\mathrm{d}x + \int_{\partial\Omega} \beta(x)(u_\lambda)^{p_*-1}h\mathrm{d}\sigma \tag{3.40}
$$
$$
\geq \int_\Omega \zeta(x)(u_\lambda)^{-\kappa}h\mathrm{d}x + \lambda \int_\Omega (u_\lambda)^{q_1-1}h\mathrm{d}x
$$

for all $h \in W^{1,\mathcal{H}}(\Omega)$ with $h \geq 0$.

Taking $h \in W^{1,\mathcal{H}}(\Omega)$ with $h \geq 0$ and choosing a decreasing sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,1]$ such that $\lim_{n\to\infty} t_n = 0$, we see that for $n \in \mathbb{N}$, the functions

$$\omega_n(x) = \zeta(x)\frac{(u_\lambda(x) + t_n h(x))^{1-\kappa} - u_\lambda(x)^{1-\kappa}}{t_n}$$

are measurable and nonnegative, and we have

$$\lim_{n\to\infty} \omega_n(x) = (1-\kappa)\zeta(x)u_\lambda(x)^{-\kappa}h(x) \text{ for a.a. } x \in \Omega.$$

From Fatou's lemma, we obtain

$$\int_\Omega \zeta(x)(u_\lambda)^{-\kappa}h\mathrm{d}x \leq \frac{1}{1-\kappa}\liminf_{n\to\infty}\int_\Omega \omega_n\mathrm{d}x. \tag{3.41}$$

Applying again Proposition 3.5, we get for $n \in \mathbb{N}$ sufficiently large that

$$
0 \leq \frac{\Theta_\lambda(u_\lambda + t_n h) - \Theta_\lambda(u_\lambda)}{t_n}
$$
$$
= \frac{1}{p}\frac{\|u_\lambda + t_n h\|_{1,p}^p - \|u_\lambda\|_{1,p}^p}{t_n} + \frac{1}{q}\frac{\|\nabla(u_\lambda + t_n h)\|_{q,\mu}^q - \|\nabla u_\lambda\|_{q,\mu}^q}{t_n}
$$
$$
+ \frac{1}{p_*}\frac{\|u_\lambda + t_n h\|_{p_*,\beta,\partial\Omega}^{p_*} - \|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}}{t_n} - \frac{1}{1-\kappa}\int_\Omega \omega_n\mathrm{d}x - \frac{\lambda}{q_1}\frac{\|u_\lambda + t_n h\|_{q_1}^{q_1} - \|u_\lambda\|_{q_1}^{q_1}}{t_n}.
$$

Passing to the limit as $n \to \infty$ and applying (3.41), we obtain (3.39) and

$$
\int_\Omega \zeta(x)(u_\lambda)^{-\kappa}h\mathrm{d}x
$$
$$
\leq \int_\Omega \left(|\nabla u_\lambda|^{p-2}\nabla u_\lambda + \mu(x)|\nabla u_\lambda|^{q-2}\nabla u_\lambda\right) \cdot \nabla h \mathrm{d}x
$$
$$
+ \int_\Omega \alpha(x)(u_\lambda)^{p-1}h\mathrm{d}x + \int_{\partial\Omega} \beta(x)(u_\lambda)^{p_*-1}h\mathrm{d}\sigma + \lambda \int_\Omega (u_\lambda)^{q_1-1}h\mathrm{d}x.
$$

This shows (3.40). Note that it is sufficient to prove the integrability in (3.39) for nonnegative test functions $h \in W^{1,\mathcal{H}}(\Omega)$.

Now, we can prove that u_λ is a weak solution of (1.1). For this purpose, let $v \in W^{1,\mathcal{H}}(\Omega)$ and $\varepsilon > 0$. Taking $h = (u_\lambda + \varepsilon v)_+$ as test function in (3.40) and using the fact that $u_\lambda \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$ such that $u_\lambda \geq 0$, we obtain

$$
\begin{aligned}
0 \leq & \int_\Omega \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla(u_\lambda + \varepsilon v)_+ \mathrm{d}x \\
& + \int_\Omega \alpha(x)(u_\lambda)^{p-1}(u_\lambda + \varepsilon v)_+ \mathrm{d}x + \int_{\partial\Omega} \beta(x)(u_\lambda)^{p_*-1}(u_\lambda + \varepsilon v)_+ \mathrm{d}\sigma \\
& - \int_\Omega \left(\zeta(x)(u_\lambda)^{-\kappa} + \lambda(u_\lambda)^{q_1-1} \right)(u_\lambda + \varepsilon v)_+ \mathrm{d}x \\
= & \int_\Omega \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla(u_\lambda + \varepsilon v) \mathrm{d}x \\
& - \int_{\{u_\lambda + \varepsilon v < 0\}} \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla(u_\lambda + \varepsilon v) \mathrm{d}x \\
& + \int_\Omega \alpha(x)(u_\lambda)^{p-1}(u_\lambda + \varepsilon v) \mathrm{d}x - \int_{\{u_\lambda + \varepsilon v < 0\}} \alpha(x)(u_\lambda)^{p-1}(u_\lambda + \varepsilon v) \mathrm{d}x \\
& + \int_{\partial\Omega} \beta(x)(u_\lambda)^{p_*-1}(u_\lambda + \varepsilon v) \mathrm{d}\sigma - \int_{\{u_\lambda + \varepsilon v < 0\}} \beta(x)(u_\lambda)^{p_*-1}(u_\lambda + \varepsilon v) \mathrm{d}\sigma \\
& - \int_\Omega \left(\zeta(x)(u_\lambda)^{-\kappa} + \lambda(u_\lambda)^{q_1-1} \right)(u_\lambda + \varepsilon v) \mathrm{d}x \\
& + \int_{\{u_\lambda + \varepsilon v < 0\}} \left(\zeta(x)(u_\lambda)^{-\kappa} + \lambda(u_\lambda)^{q_1-1} \right)(u_\lambda + \varepsilon v) \mathrm{d}x \\
= & \|u_\lambda\|_{1,p}^p + \|\nabla u_\lambda\|_{q,\mu}^q + \|u_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*} - \int_\Omega \zeta(x)|u_\lambda|^{1-\kappa} \mathrm{d}x - \lambda \|u_\lambda\|_{q_1}^{q_1} \\
& + \varepsilon \int_\Omega \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla v \mathrm{d}x \\
& + \varepsilon \int_\Omega \alpha(x)(u_\lambda)^{p-1} v \mathrm{d}x + \varepsilon \int_{\partial\Omega} \beta(x)(u_\lambda)^{p_*-1} v \mathrm{d}\sigma \\
& - \varepsilon \int_\Omega \left(\zeta(x)(u_\lambda)^{-\kappa} + \lambda(u_\lambda)^{q_1-1} \right) v \mathrm{d}x \\
& - \int_{\{u_\lambda + \varepsilon v < 0\}} \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla(u_\lambda + \varepsilon v) \mathrm{d}x \\
& - \int_{\{u_\lambda + \varepsilon v < 0\}} \alpha(x)(u_\lambda)^{p-1}(u_\lambda + \varepsilon v) \mathrm{d}x - \int_{\{u_\lambda + \varepsilon v < 0\}} \beta(x)(u_\lambda)^{p_*-1}(u_\lambda + \varepsilon v) \mathrm{d}\sigma \\
& + \int_{\{u_\lambda + \varepsilon v < 0\}} \left(\zeta(x)(u_\lambda)^{-\kappa} + \lambda(u_\lambda)^{q_1-1} \right)(u_\lambda + \varepsilon v) \mathrm{d}x \\
\leq & \varepsilon \int_\Omega \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla v \mathrm{d}x \\
& + \varepsilon \int_\Omega \alpha(x)(u_\lambda)^{p-1} v \mathrm{d}x + \varepsilon \int_{\partial\Omega} \beta(x)(u_\lambda)^{p_*-1} v \mathrm{d}\sigma \\
& - \varepsilon \int_\Omega \left(\zeta(x)(u_\lambda)^{-\kappa} + \lambda(u_\lambda)^{q_1-1} \right) v \mathrm{d}x \\
& - \varepsilon \int_{\{u_\lambda + \varepsilon v < 0\}} \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla v \mathrm{d}x \\
& - \varepsilon \int_{\{u_\lambda + \varepsilon v < 0\}} \alpha(x)(u_\lambda)^{p-1} v \mathrm{d}x - \varepsilon \int_{\{u_\lambda + \varepsilon v < 0\}} \beta(x)(u_\lambda)^{p_*-1} v \mathrm{d}\sigma.
\end{aligned}
$$

Note that the measure of the domain $\{u_\lambda + \varepsilon h < 0\}$ tends to zero as $\varepsilon \to 0$. Hence,

$$\int_{\{u_\lambda + \varepsilon v < 0\}} \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla v \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

$$\int_{\{u_\lambda + \varepsilon v < 0\}} \alpha(x)(u_\lambda)^{p-1} v \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

$$\int_{\{u_\lambda + \varepsilon v < 0\}} \beta(x)(u_\lambda)^{p_*-1} v \, d\sigma \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

So, we can divide the inequality above with $\varepsilon > 0$ and let $\varepsilon \to 0$. We obtain

$$\int_\Omega \left(|\nabla u_\lambda|^{p-2} \nabla u_\lambda + \mu(x) |\nabla u_\lambda|^{q-2} \nabla u_\lambda \right) \cdot \nabla v \, dx$$
$$+ \int_\Omega \alpha(x)(u_\lambda)^{p-1} v \, dx + \int_{\partial\Omega} \beta(x)(u_\lambda)^{p_*-1} v \, d\sigma$$
$$\geq \int_\Omega \zeta(x)(u_\lambda)^{-\kappa} v \, dx + \lambda \int_\Omega (u_\lambda)^{\nu-1} v \, dx.$$

Since $v \in W^{1,\mathcal{H}}(\Omega)$ is arbitrary, equality must hold. It follows that u_λ is a weak solution of problem (1.1) such that $\Theta_\lambda(u_\lambda) < 0$; see Propositions 3.2 and 3.4. □

Now we are interested in a second weak solution which turns out to be the global minimizer of Θ_λ restricted to $\mathcal{N}_\lambda^-$. First, we show that this minimum is nonnegative.

Proposition 3.7. *If hypotheses (H) hold, then there exists $\lambda^* \in (0, \tilde{\lambda}]$ such that $\Theta_\lambda|_{\mathcal{N}_\lambda^-} > 0$ for all $\lambda \in (0, \lambda^*]$.*

Proof. First we take $u \in \mathcal{N}_\lambda^-$ which is possible by Proposition 3.4. Applying the definition of $\mathcal{N}_\lambda^-$ and the embedding $W^{1,p}(\Omega) \hookrightarrow L^{q_1}(\Omega)$, we have

$$\lambda(q_1 + \kappa - 1) c_9^{q_1} \|u\|_{1,p}^{q_1} \geq \lambda(q_1 + \kappa - 1) \|u\|_{q_1}^{q_1}$$
$$> (p + \kappa - 1) \|u\|_{1,p}^p + (q + \kappa - 1) \|\nabla u\|_{q,\mu}^q + (p_* + \kappa - 1) \|u\|_{p_*,\beta,\partial\Omega}^{p_*}$$
$$\geq (p + \kappa - 1) \|u\|_{1,p}^p$$

for some constant $c_9 > 0$. This implies

$$\|u\|_{1,p} \geq \left[\frac{p + \kappa - 1}{\lambda c_9^{q_1}(q_1 + \kappa - 1)} \right]^{\frac{1}{q_1 - p}}. \tag{3.42}$$

We argue by contradiction and suppose that the statement of the proposition is not true. Then there exists $u \in \mathcal{N}_\lambda^-$ such that $\Theta_\lambda(u) \leq 0$, that is,

$$\frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|\nabla u\|_{q,\mu}^q + \frac{1}{p_*} \|u\|_{p_*,\beta,\partial\Omega}^{p_*} - \frac{1}{1-\kappa} \int_\Omega \zeta(x) |u|^{1-\kappa} dx - \frac{\lambda}{q_1} \|u\|_{q_1}^{q_1} \leq 0. \tag{3.43}$$

As $\mathcal{N}_\lambda^- \subseteq \mathcal{N}_\lambda$, we obtain from the definition of $\mathcal{N}_\lambda$

$$\frac{1}{q_1} \|\nabla u\|_{q,\mu}^q = \frac{1}{q_1} \int_\Omega \zeta(x) |u|^{1-\kappa} dx + \frac{\lambda}{q_1} \|u\|_{q_1}^{q_1} - \frac{1}{q_1} \|u\|_{1,p}^p - \frac{1}{q_1} \|u\|_{p_*,\beta,\partial\Omega}^{p_*}. \tag{3.44}$$

Combining (3.44) and (3.43), one has

$$
\left(\frac{1}{p} - \frac{1}{q_1}\right) \|u\|_{1,p}^p + \left(\frac{1}{q} - \frac{1}{q_1}\right) \|\nabla u\|_{q,\mu}^q + \left(\frac{1}{p_*} - \frac{1}{q_1}\right) \|u\|_{p_*,\beta,\partial\Omega}^{p_*}
$$
$$
+ \left(\frac{1}{q_1} - \frac{1}{1-\kappa}\right) \int_\Omega \zeta(x)|u|^{1-\kappa} dx \leq 0.
$$

Since $p_* < q_1$ and $q < q_1$, this implies

$$
\frac{q_1 - p}{pq_1} \|u\|_{1,p}^p \leq \frac{q_1 + \kappa - 1}{q_1(1-\kappa)} \int_\Omega \zeta(x)|u|^{1-\kappa} dx \leq \frac{q_1 + \kappa - 1}{q_1(1-\kappa)} c_{10} \|u\|_{1,p}^{1-\kappa}
$$

for some constant $c_{10} > 0$. Therefore,

$$
\|u\|_{1,p} \leq c_{11} \tag{3.45}
$$

for some $c_{11} > 0$. Now we use (3.45) in (3.42) and get

$$
0 < \frac{c_{12}}{c_{11}} \leq \lambda^{\frac{1}{q_1-p}} \text{ with } c_{12} = \left[\frac{(p+\kappa-1)}{c_9^{q_1}(q_1+\kappa-1)}\right]^{\frac{1}{q_1-p}} > 0.
$$

Letting $\lambda \to 0$ yields a contradiction since $1 < p < q_1$. Hence, we find $\lambda^* \in (0, \tilde{\lambda}]$ such that $\Theta_\lambda|_{\mathcal{N}_\lambda^-} > 0$ for all $\lambda \in (0, \lambda^*]$. $\qquad \square$

Next, we will show that the functional Θ_λ achieves its global minimum restricted to the set $\mathcal{N}_\lambda^-$.

Proposition 3.8. *If hypotheses (H) hold and $\lambda \in (0, \lambda^*]$, then there exists $v_\lambda \in \mathcal{N}_\lambda^-$ with $v_\lambda \geq 0$ such that*

$$
m_\lambda^- = \inf_{\mathcal{N}_\lambda^-} \Theta_\lambda = \Theta_\lambda(v_\lambda) > 0.
$$

Proof. First note that for $v \in \mathcal{N}_\lambda^-$ by using the embedding $W^{1,p}(\Omega) \hookrightarrow L^{q_1}(\Omega)$, we obtain

$$
\lambda(q_1 + \kappa - 1)\|v\|_{q_1}^{q_1} > (p + \kappa - 1)\|v\|_{1,p}^p \geq (p + \kappa - 1)\frac{1}{c_9^p}\|v\|_{q_1}^p
$$

for $c_9 > 0$; see the proof of Proposition 3.7. Therefore,

$$
\|v\|_{q_1} \geq \left[\frac{p+\kappa-1}{\lambda c_9^p(q_1+\kappa-1)}\right]^{\frac{1}{q_1-p}}. \tag{3.46}
$$

Let $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$ be a minimizing sequence. Then, since $\mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, we know from Proposition 3.1 that $\{v_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\Omega)$ is bounded. We may assume that

$$
v_n \rightharpoonup v_\lambda \text{ in } W^{1,\mathcal{H}}(\Omega), \ v_n \to v_\lambda \text{ in } L^{q_1}(\Omega) \text{ and } v_n \rightharpoonup v_\lambda \text{ in } L^{p_*}(\partial\Omega).
$$

From (3.46), we see that $v_\lambda \neq 0$. Now we will use the point $t_{v_\lambda}^2 > 0$ (see 3.19) for which we have

$$
\eta_{v_\lambda}(t_{v_\lambda}^2) = \lambda \|v_\lambda\|_{q_1}^{q_1} \text{ and } \eta_{v_\lambda}'(t_{v_\lambda}^2) < 0.
$$

In the proof of Proposition 3.4, we derived that $t_{v_\lambda}^2 v_\lambda \in \mathcal{N}_\lambda^-$.

Let us now show that $\lim_{n\to+\infty} \rho(v_n) = \rho(v_\lambda)$ for a subsequence (still denoted by v_n). As in the proof of Proposition 3.4, we can prove Claims 1–3 via contradiction. Indeed, then we have in each case for a subsequence

$$
\Theta_\lambda(t_{v_\lambda}^2 v_\lambda) < \lim_{n\to\infty} \Theta_\lambda(t_{v_\lambda}^2 v_n).
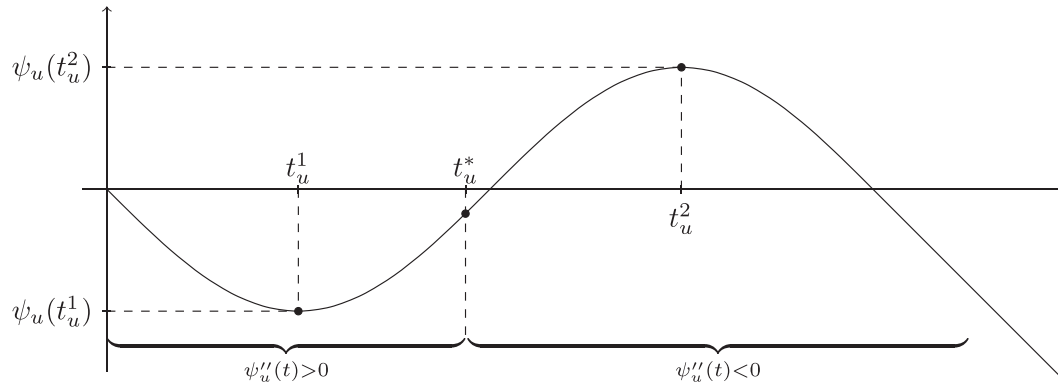$$

FIGURE 1 The graph of $\psi_u(t)$

Since $\Theta_\lambda(t^2_{v_\lambda} v_n) \le \Theta_\lambda(v_n)$ (note that it is the global maximum since $\psi''_{v_n}(1) < 0$; see Figure 1) and $t^2_{v_\lambda} v_\lambda \in \mathcal{N}^-_\lambda$, we obtain

$$m^-_\lambda \le \Theta_\lambda(t^2_{v_\lambda} v_\lambda) < m^-_\lambda,$$

a contradiction. Therefore, for a subsequence, we have $\lim_{n\to+\infty} \rho(v_n) = \rho(v_\lambda)$ and since the integrand corresponding to the modular function $\rho(\cdot)$ is uniformly convex, this implies that $v_n \to v_\lambda$ in $W^{1,\mathcal{H}}(\Omega)$ and the continuity of Θ_λ gives $\Theta_\lambda(v_n) \to \Theta_\lambda(v_\lambda)$ and so $\Theta_\lambda(v_\lambda) = m^-_\lambda$.

Since $v_n \in \mathcal{N}^-_\lambda$ for all $n \in \mathbb{N}$, we have

$$(p + \kappa - 1)\|v_n\|^p_{1,p} + (q + \kappa - 1)\|\nabla v_n\|^q_{q,\mu} + (p_* + \kappa - 1)\|v_n\|^{p_*}_{p_*,\beta,\partial\Omega}$$
$$- \lambda(q_1 + \kappa - 1)\|v_n\|^{q_1}_{q_1} < 0. \tag{3.47}$$

Now we pass to the limit in (3.47) as $n \to +\infty$ in order to get

$$(p + \kappa - 1)\|v_\lambda\|^p_{1,p} + (q + \kappa - 1)\|\nabla v_\lambda\|^q_{q,\mu} + (p_* + \kappa - 1)\|v_\lambda\|^{p_*}_{p_*,\beta,\partial\Omega} - \lambda(q_1 + \kappa - 1)\|v_\lambda\|^{q_1}_{q_1} \le 0. \tag{3.48}$$

From Proposition 3.3, we know that equality in (3.48) cannot happen, so we have a strict inequality. Therefore, $v_\lambda \in \mathcal{N}^-_\lambda$. Since the treatment also works for $|v_\lambda|$ instead of v_λ, we may assume that $v_\lambda(x) \ge 0$ for a. a. $x \in \Omega$ such that $v_\lambda \ne 0$. Proposition 3.7 finally shows that $m^-_\lambda > 0$. □

Finally, we reach a second weak solution of problem (1.1).

Proposition 3.9. *If hypotheses (H) hold and $\lambda \in (0, \lambda^*)$, then v_λ is a weak solution of problem (1.1) such that $\Theta_\lambda(v_\lambda) > 0$.*

Proof. Following the proof of Proposition 3.5 replacing u_λ by v_λ in the definition of Ξ_h, we can show for every $t \in [0, \delta_0]$ there exists $\chi(t) > 0$ such that

$$\chi(t)(v_\lambda + th) \in \mathcal{N}^-_\lambda \quad \text{and} \quad \chi(t) \to 1 \text{ as } t \to 0^+.$$

From Proposition 3.8, we derive that

$$m^-_\lambda = \Theta_\lambda(v_\lambda) \le \Theta_\lambda(\chi(t)(v_\lambda + th)) \quad \text{for all } t \in [0, \delta_0]. \tag{3.49}$$

Claim: $v_\lambda > 0$ for a. a. $x \in \Omega$

Let us suppose there is a set B with positive measure such that $v_\lambda = 0$ in B. Let $h \in W^{1,\mathcal{H}}(\Omega)$ with $h > 0$ and let $t \in (0, \delta_0)$ (see 3.49); then $(\chi(t)(v_\lambda + th))^{1-\kappa} > (\chi(t)v_\lambda)^{1-\kappa}$ a. e. in $\Omega \setminus B$. From (3.49) and since $\psi_{v_\lambda}(1)$ is the global maximum (see Figure 1) which implies $\psi_{v_\lambda}(1) \ge \psi_{v_\lambda}(\chi(t))$, we then obtain

$$0 \leq \frac{\Theta_\lambda(\chi(t)(v_\lambda + th)) - \Theta_\lambda(v_\lambda)}{t}$$

$$\leq \frac{\Theta_\lambda(\chi(t)(v_\lambda + th)) - \Theta_\lambda(\chi(t)v_\lambda)}{t}$$

$$= \frac{1}{p} \frac{\|\chi(t)(v_\lambda + th)\|_{1,p}^p - \|\chi(t)v_\lambda\|_{1,p}^p}{t} + \frac{1}{q} \frac{\|\nabla(\chi(t)(v_\lambda + th))\|_{q,\mu}^q - \|\nabla(\chi(t)v_\lambda)\|_{q,\mu}^q}{t}$$

$$+ \frac{1}{p_*} \frac{\|\chi(t)(v_\lambda + th)\|_{p_*,\beta,\partial\Omega}^{p_*} - \|\chi(t)v_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}}{t} - \frac{\chi(t)^{1-\kappa}}{(1-\kappa)t^\kappa} \int_B \zeta(x)h^{1-\kappa} dx$$

$$- \frac{1}{1-\kappa} \int_{\Omega\setminus B} \zeta(x) \frac{(\chi(t)(v_\lambda + th))^{1-\kappa} - (\chi(t)v_\lambda)^{1-\kappa}}{t} dx - \frac{\lambda}{q_1} \frac{\|\chi(t)(v_\lambda + th)\|_{q_1}^{q_1} - \|\chi(t)v_\lambda\|_{q_1}^{q_1}}{t}$$

$$< \frac{1}{p} \frac{\|\chi(t)(v_\lambda + th)\|_{1,p}^p - \|\chi(t)v_\lambda\|_{1,p}^p}{t} + \frac{1}{q} \frac{\|\nabla(\chi(t)(v_\lambda + th))\|_{q,\mu}^q - \|\nabla(\chi(t)v_\lambda)\|_{q,\mu}^q}{t}$$

$$+ \frac{1}{p_*} \frac{\|\chi(t)(v_\lambda + th)\|_{p_*,\beta,\partial\Omega}^{p_*} - \|\chi(t)v_\lambda\|_{p_*,\beta,\partial\Omega}^{p_*}}{t} - \frac{\chi(t)^{1-\kappa}}{(1-\kappa)t^\kappa} \int_B \zeta(x)h^{1-\kappa} dx$$

$$- \frac{\lambda}{q_1} \frac{\|\chi(t)(v_\lambda + th)\|_{q_1}^{q_1} - \|\chi(t)v_\lambda\|_{q_1}^{q_1}}{t}.$$

From the considerations above, we see that

$$0 \leq \frac{\Theta_\lambda(\chi(t)(v_\lambda + th)) - \Theta_\lambda(\chi(t)v_\lambda)}{t} \to -\infty \text{ as } t \to 0^+,$$

which is a contradiction. This shows the claim and so $v_\lambda > 0$ a. e. in Ω.

The rest of the proof works in the same way as the proof of Proposition 3.6. Indeed, (3.39) and (3.40) can be shown similarly using again (3.49) and the inequality $\psi_{v_\lambda}(1) \geq \psi_{v_\lambda}(\chi(t))$ along with $v_\lambda > 0$. The last part of Proposition 3.6 is the same replacing u_λ by v_λ and finally we know from Proposition 3.8 that $\Theta_\lambda(v_\lambda) > 0$. □

The proof of Theorem 1.1 follows now from Propositions 3.6 and 3.9.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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