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Critical logarithmic double phase equations with sign-changing potentials in $\mathbb{R}^N$

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A B S T R A C T

This paper is concerned with the existence of solutions to the following double phase equation with logarithmic nonlinearity

$$
\begin{aligned}
&-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)+|u|^{p-2}u+\mu(x)|u|^{q-2}u\\
&=K_1(x)|u|^{p^*-2}u+\lambda K_2(x)|u|^{r-2}u\log(|u|)+\gamma K_3(x)|u|^{\beta-2}u \quad \text{in } \mathbb{R}^N,
\end{aligned}
$$

with dimension $N \geq 2$, parameters $\lambda, \gamma > 0$, $1 < p < q < N$, $\mu\colon \mathbb{R}^N \to [0,\infty)$ is a Lipschitz continuous function, exponents $q \leq r < p^*$ and $1 < \beta < p^*$. Here, the weight functions K_1 and K_3 are positive, while K_2 may change sign on $\mathbb{R}^N$. First, under quite general assumptions, we give basic properties of the corresponding function space and prove a compactness results. Then, we study the equation above for the two cases: the superlinear case ($q < \beta < r < p^*$) and the linear case ($\beta < r = q < p^*$). Moreover, we deal with the radial situation in the two previous cases.

1. Introduction

In this paper, we study the existence and properties of solutions for a quasilinear equation, driven by an operator of double phase type and involving a logarithmic nonlinearity as well as a critical Sobolev term. Namely, we deal with the following equation

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$$\begin{aligned}&-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right)+|u|^{p-2}u+\mu(x)|u|^{q-2}u\\&=K_1(x)|u|^{p^*-2}u+\lambda K_2(x)|u|^{r-2}u\log(|u|)+\gamma K_3(x)|u|^{\beta-2}u,\quad \text{in } \mathbb{R}^N,\end{aligned}\tag{1.1}$$

where the main operator on the left-hand side is the so-called double phase operator satisfying the structural assumption:

(H_1) *$1<p<q<N$, $q<p^*=\frac{Np}{N-p}$ and $\mu\colon \mathbb{R}^N\to\mathbb{R}_+=[0,\infty)$ is Lipschitz continuous such that $\mu(\cdot)\in L^\infty(\mathbb{R}^N)$.*

Here, we consider parameters λ, $\gamma>0$ and exponents $q\le r<p^*$, $1<\beta<p^*$. Concerning the functions K_1, K_2, $K_3\colon \mathbb{R}^N\to\mathbb{R}$, along the paper, we assume the following conditions:

(H_2) *$K_1\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $K_1(x)>0$ for all $x\in\mathbb{R}^N$ and if $\{A_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure $|A_n|\le R$ for all $n\in\mathbb{N}$ and some $R>0$, then*

$$\lim_{n\to\infty}\int\limits_{A_n\cap B^c_\rho(0)} K_1(x)\,\mathrm{d}x=0,$$

for some $\rho>0$.

(H_3) *$K_3\in L^\infty(\mathbb{R}^N,\mathbb{R}^+)$, $0<K_3<K_1$ on $\mathbb{R}^N$, $K_2\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $|K_2|\le K_1$ on $\mathbb{R}^N$ and there exist $x_0\in\mathbb{R}^N$ and $\widetilde{\rho}>0$ such that*

$$K_2(x)>0\quad\text{for } x\in B(x_0,\widetilde{\rho}).$$

The novelty of our work is the fact that we combine several different and interesting phenomena in one single equation. More precisely, problem (1.1) contains

(i) a double phase operator;
(ii) a double lack of compactness, due to the free action of translation group in $\mathbb{R}^N$ and the critical Sobolev nonlinearity;
(iii) a logarithmic nonlinearity.

To the best of our knowledge, this is the first paper proving the existence of solutions with the combined effects generated by the above features.

The double phase operator is related to the energy functional

$$u\mapsto\int\limits_\Omega\left(|\nabla u|^p+\mu(x)|\nabla u|^q\right)\mathrm{d}x,\tag{1.2}$$

where Ω is an arbitrary domain in $\mathbb{R}^N$. Clearly, the integrand has unbalanced growth. The integral functional (1.2) was first introduced by Zhikov [33–35] to provide models for strongly anisotropic materials in the framework of homogenization. This functional belongs to a class of functionals with non-standard growth conditions introduced by Marcellini in [25,26]. The main characteristic of the double phase functional (1.2) is the change of ellipticity on the set $\{x\in\Omega\colon \mu(x)=0\}$. Indeed, its energy density exhibits ellipticity in the gradient of order q in the set $\{x\in\Omega\colon \mu(x)>\varepsilon\}$ for any fixed $\varepsilon>0$ and of order p on the points x where $\mu(x)$ vanishes. Thus, the integrand in (1.2) switches between two different phases of elliptic behaviors. The analysis of non-autonomous energy functionals with energy density changing its ellipticity and growth properties according to a point has been developed in several remarkable papers, see for example, the works of Baroni-Colombo-Mingione [7], Baroni-Kuusi-Mingione [8] and Colombo-Mingione [11]. A regularity theory for local minimizers of energy functionals such as (1.2) was recently developed in the papers of De Filippis [14] and De Filippis-Mingione [15,16]. Recently, some contributions devoted to solve equations driven by the double phase operator in $\mathbb{R}^N$ have been published, we refer to Ambrosio-Essebei [2], Arora-Fiscella-Mukherjee-Winkert [3], Bahrouni-Rădulescu [5], Bahrouni-Rădulescu-Repovš [6], Ge-Pucci [17], Ge-Yuan [18], Le [20], Liu-Dai [21], Li-Liu [22], Liu-Winkert [24] and Stegliński [28]. However, all these works do not allow a logarithmic term on the right-hand side of the problem.

Problems involving nonlinearities of logarithmic type have been widely studied in literature dealing both with a local and a nonlocal structure and different types of operators. In this context we refer to the contributions by Biswas-Bahrouni-Fiscella [9], d'Avenia-Squassina-Zenari [13], Liang-Pu-Rădulescu [23], Tian [29] and Truong [30] and the references therein. In particular in [29], Tian proved that the following problem

$$\begin{cases} -\Delta u = a(x)u\log(|u|) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least two nontrivial solutions provided that $a(\cdot)$ changes sign on $\Omega \subset \mathbb{R}^N$ with Ω being bounded. In [13], d'Avenia-Squassina-Zenari considered the following fractional logarithmic Schrödinger equation

$$(-\Delta)^s u + W(x)u = u\log\left(|u|^2\right), \quad x \in \mathbb{R}^N,$$

where $W\colon \mathbb{R}^N \to \mathbb{R}^+$ is a continuous function. By employing the fractional logarithmic Sobolev inequality, the authors of [13] showed the existence of infinitely many solutions. Moreover, Truong [30] studied the following fractional p-Laplacian equations with logarithmic nonlinearity

$$(-\Delta)_p^s u + V(x)|u|^{p-2} = \lambda l(x)|u|^{p-2}u\log(|u|), \quad x \in \mathbb{R}^N,$$

where $l\colon \mathbb{R}^N \to \mathbb{R}$ is a sign-changing weight function. Using the Nehari manifold approach, the author in [30] proved the existence of at least two nontrivial solutions, see also Biswas-Bahrouni-Fiscella [9] for the case of logarithmic fractional equations with variable exponent.

Concerning a double phase situation, we can just refer to Aberqi-Benslimane-Elmassoudi-Ragusa [1] who studied the problem

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + V(x)|u|^{p-2}u \\ &= \lambda K(x)|u|^{r-2}u\log(|u|) \quad \text{in } \mathcal{D}, \quad u\big|_{\partial D} = 0, \end{aligned} \tag{1.3}$$

where $\mathcal{D} \subset \mathcal{M}$ is an open bounded subset of a smooth complete compact Riemannian N-manifold. By a variational technique on a suitable Nehari manifold, the existence of a nonnegative solution of (1.3) has been proved whenever $r \in (1,p)$.

Motivated by the above papers, in this work we are interested in finding nontrivial solutions for equation (1.1) in $\mathbb{R}^N$. First, we are going to prove a compactness result for the double phase space $W^{1,\mathcal{H}}(\mathbb{R}^N)$ into a suitable weighted Lebesgue space, in order to deal with the critical term. Then, we establish a result which provides an estimate for the logarithmic nonlinearity. As applications of these abstract tools, we distinguish two main situations for (1.1) depending on the behavior of r: the superlinear case $q < \beta < r < p^*$ and the linear case $\beta < r = q < p^*$.

As we will see, the effects of the logarithmic nonlinearity and of the double lack of compactness, due to the unboundedness of the domain and the presence of criticality, prevent us from using variational methods in a standard way. In particular, in order to deal with the superlinear logarithmic term, we strongly need the nonlinearity of β exponent in (1.1). Indeed, we want to get a mountain pass solution for (1.1) in this case. For this, we need a delicate asymptotic property of the mountain pass level, as λ goes to ∞. The proof of this asymptotic condition is obtained by a tricky combination of the superlinear logarithmic term and of the β-nonlinearity. Concerning the linear logarithmic case, we can get a solution of (1.1) by minimization, under more delicate assumptions on data, in particular considering K_3 sufficiently small.

The outline of the paper is the following: in Section 2, we collect some preliminary results and we present the variational setting in which equation (1.1) will be studied. In Section 3 we prove some abstract results as explained above. In Section 4 we deal with the two situations, that is, the superlinear and linear cases. Finally, in the last section we study the existence of radial solution for equation (1.1), see Section 5.

2. Variational setting

In this section, we first recall basic results about Musielak-Orlicz spaces in $\mathbb{R}^N$. As usual, we denote by $L^m(\mathbb{R}^N)$ the classical Lebesgue space equipped with the norm $\|\cdot\|_m$ for $1 \leq m \leq \infty$. Moreover, $W^{1,m}(\mathbb{R}^N)$ stands for the Sobolev spaces endowed with the norm $\|\nabla \cdot\|_m + \|\cdot\|_m$, for any $1 < m < \infty$.

Let us consider the nonlinear function $\mathcal{H}\colon \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{H}(x,t) := t^p + \mu(x)t^q,$$

by assuming that (H_1) holds true. Then, the Musielak-Orlicz Lebesgue space $L^{\mathcal{H}}(\mathbb{R}^N)$ is given by

$$L^{\mathcal{H}}(\mathbb{R}^N) := \left\{ u \;\middle|\; u\colon \mathbb{R}^N \to \mathbb{R} \text{ is measurable and } \varrho_{\mathcal{H}}(u) := \int_{\mathbb{R}^N} \mathcal{H}(x,|u|)\,\mathrm{d}x < \infty \right\}$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf\left\{ \tau > 0 \;\middle|\; \varrho_{\mathcal{H}}\left(\frac{u}{\tau}\right) \leq 1 \right\},$$

where the modular function is given by

$$\varrho_{\mathcal{H}}(u) := \int_{\mathbb{R}^N} \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_{\mathbb{R}^N} \left[|u|^p + \mu(x)\,|u|^q \right] \mathrm{d}x.$$

In addition, we introduce the weighted space

$$L^q_\mu(\mathbb{R}^N) := \left\{ u \;\middle|\; u\colon \mathbb{R}^N \to \mathbb{R} \text{ is measurable and } \int_{\mathbb{R}^N} \mu(x)|u|^q\,\mathrm{d}x < \infty \right\}$$

with the seminorm

$$\|u\|_{q,\mu} := \left(\int_{\mathbb{R}^N} \mu(x)|u|^q\,\mathrm{d}x \right)^{\frac{1}{q}}.$$

While, the corresponding Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\mathbb{R}^N)$ is defined by

$$W^{1,\mathcal{H}}(\mathbb{R}^N) := \left\{ u \in L^{\mathcal{H}}(\mathbb{R}^N) \;\middle|\; |\nabla u| \in L^{\mathcal{H}}(\mathbb{R}^N) \right\}$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}} := \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. Note that the norm $\|u\|_{1,\mathcal{H}}$ on $W^{1,\mathcal{H}}(\mathbb{R}^N)$ is equivalent to

$$\|u\| := \inf\left\{\tau > 0 \,\middle|\, \int_{\mathbb{R}^N} \left[\left(\frac{|\nabla u|}{\tau}\right)^p + \mu(x)\left(\frac{|\nabla u|}{\tau}\right)^q + \left|\frac{u}{\tau}\right|^p + \mu(x)\left|\frac{u}{\tau}\right|^q\right] \mathrm{d}x \le 1\right\},$$

where

$$\varrho(u) := \int_{\mathbb{R}^N} \left[|\nabla u|^p + \mu(x)\,|\nabla u|^q + |u|^p + \mu(x)|u|^q\right] \mathrm{d}x$$

is the associated modular. We know that $L^{\mathcal{H}}(\mathbb{R}^N)$ and $W^{1,\mathcal{H}}(\mathbb{R}^N)$ are separable reflexive Banach spaces, see Liu-Dai [21, Theorem 2.7]. Moreover, $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,\mathcal{H}}(\mathbb{R}^N)$, see Harjulehto-Hästö [19, Proposition 6.4.4] and Crespo-Blanco-Gasiński-Harjulehto-Winkert [12, Theorems 2.24 and 2.28].

The following relations between the norm $\|\cdot\|$ and the corresponding modular function $\varrho(\cdot)$ can be found in Liu-Dai [21, Proposition 2.6].

Lemma 2.1. *Let (H_1) be satisfied, $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ and $c > 0$. Then the following hold:*

(i) *for $u \neq 0$ we have $\|u\| = c$ if and only if $\varrho(\frac{u}{c}) = 1$;*
(ii) *$\|u\| < 1$ implies $\|u\|^q \le \varrho(u) \le \|u\|^p$;*
(iii) *$\|u\| > 1$ implies $\|u\|^p \le \varrho(u) \le \|u\|^q$.*

The following embedding result can be found in Liu-Dai [21, Theorem 2.7].

Lemma 2.2. *Let (H_1) be satisfied. Then, the embedding $W^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N)$ is continuous for any $m \in [p, p^*]$. Also, $W^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^m_{\mathrm{loc}}(\mathbb{R}^N)$ is compact for any $m \in [1, p^*)$.*

Let us denote

$$W^{1,\mathcal{H}}_{\mathrm{rad}}(\mathbb{R}^N) := \left\{u \in W^{1,\mathcal{H}}(\mathbb{R}^N) \colon u \text{ is radially symmetric}\right\}.$$

By u being radially symmetric, we mean a function $u\colon \mathbb{R}^N \to \mathbb{R}$ satisfying $u(x) = u(y)$ for any $x, y \in \mathbb{R}^N$ with $|x| = |y|$. In the last section, we look for solutions of (1.1) in $W^{1,\mathcal{H}}_{\mathrm{rad}}(\mathbb{R}^N)$. For this, we need the following compact result given in Liu-Dai [21, Theorem 2.8].

Lemma 2.3. *Let (H_1) be satisfied. Then, the embedding $W^{1,\mathcal{H}}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N)$ is compact for any $m \in (p, p^*)$.*

Now, let us recall the definition of a weak solution of equation (1.1).

Definition 2.4. We say that $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ is a weak solution of (1.1) if

$$\int_{\mathbb{R}^N} \left[|\nabla u|^{p-2}\,\nabla u \cdot \nabla v + \mu(x)\,|\nabla u|^{q-2}\,\nabla u \cdot \nabla v\right] \mathrm{d}x$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^N} \left[|u|^{p-2} uv + \mu(x) |u|^{q-2} uv \right] \mathrm{d}x \\
&= \int_{\mathbb{R}^N} K_1(x)|u|^{p^*-2}uv \, \mathrm{d}x + \lambda \int_{\mathbb{R}^N} K_2(x)|u|^{r-2}u \log(|u|)v \, \mathrm{d}x \\
&+ \gamma \int_{\mathbb{R}^N} K_3(x)|u|^{\beta-2}uv \, \mathrm{d}x,
\end{aligned}
$$

for any $v \in W^{1,\mathcal{H}}(\mathbb{R}^N) \setminus \{0\}$.

The energy functional $I_\lambda \colon W^{1,\mathcal{H}}(\mathbb{R}^N) \to \mathbb{R}$ associated to equation (1.1) is defined by

$$
\begin{aligned}
I_\lambda(u) =& \frac{1}{p}\left(\|\nabla u\|_p^p + \|u\|_p^p\right) + \frac{1}{q}\left(\|\nabla u\|_{q,\mu}^q + \|u\|_{q,\mu}^q\right) - \int_{\mathbb{R}^N} K_1(x)\frac{|u|^{p^*}}{p^*} \, \mathrm{d}x \\
&- \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r}|u|^r \log(|u|) \, \mathrm{d}x + \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r^2}|u|^r \, \mathrm{d}x - \gamma \int_{\mathbb{R}^N} \frac{K_3(x)}{\beta}|u|^\beta \, \mathrm{d}x.
\end{aligned}
$$

Of course, weak solution of (1.1) are critical points of I_λ. By Lemma 3.3, we will see that I_λ is well defined and of class $C^1(W^{1,\mathcal{H}}(\mathbb{R}^N), \mathbb{R})$. Also, for I_λ we do not specify dependence on parameter γ since in Sections 4 and 5 we will consider either $\gamma = \lambda$ or $\gamma = 1$.

A delicate property for I_λ concerns the study of compactness in $W^{1,\mathcal{H}}(\mathbb{R}^N)$. For this, we say that $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ is a Palais-Smale sequence for I_λ at level $c \in \mathbb{R}$ if

$$
I_\lambda(u_n) \to c \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{in } \left(W^{1,\mathcal{H}}(\mathbb{R}^N)\right)^* \quad \text{as } n \to \infty. \tag{2.1}
$$

We say that I_λ satisfies the Palais-Smale condition at level c ($(PS)_c$ condition for short) if any Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ at level c admits a convergent subsequence in $W^{1,\mathcal{H}}(\mathbb{R}^N)$.

3. The weighted Lebesgue space and the logarithmic term

In this section, we examine the continuous and the compact embedding of $W^{1,\mathcal{H}}(\mathbb{R}^N)$ in a suitable weighted Lebesgue spaces. Moreover, we give some new logarithmic estimations that will be useful in the sequel. For this purpose we define, for any $1 < s < \infty$, the following Lebesgue space

$$
L^s_{K_1}(\mathbb{R}^N) := \left\{ u \colon \mathbb{R}^N \to \mathbb{R} \;\middle|\; u \text{ is measurable and } \int_{\mathbb{R}^N} K_1(x)|u|^s \, \mathrm{d}x < \infty \right\},
$$

where K_1 satisfying (H_2) multiplies the critical Sobolev term in (1.1).

We can prove the following compactness result.

Proposition 3.1. *Let (H_2) be satisfied. Then, $W^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^s_{K_1}(\mathbb{R}^N)$ is compact for any $s \in (p, p^*)$.*

Proof. Let us fix $s \in (p, p^*)$ and let $\varepsilon > 0$. It is easy to see that

$$
\lim_{t\to 0} \frac{|t|^s}{|t|^p} = \lim_{t\to\infty} \frac{|t|^s}{|t|^{p^*}} = 0.
$$

Thus, there exist numbers $0 < t_0 < t_1$ and a positive constant $C > 0$ such that

$$K_1(x)|t|^s \leq \varepsilon C(|t|^p + |t|^{p^*}) + \chi_{[t_0,t_1]}(x)K_1(x)|t|^p,$$

for any $t \in \mathbb{R}$ and any $x \in \mathbb{R}^N$. We set

$$F(u) := \|u\|_p^p + \|u\|_{p^*}^{p^*}.$$

Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ be a sequence such that $u_n \rightharpoonup u$ in $W^{1,\mathcal{H}}(\mathbb{R}^N)$. By Lemma 2.2 we have that $\{F(u_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Denoting

$$A_n := \{x \in \mathbb{R}^N : t_0 < |u_n(x)| < t_1\},$$

it holds $\sup_{n\in\mathbb{N}} |A_n| < \infty$. Hence, from (H_2), there exists a positive radius $\rho > 0$ such that

$$\begin{aligned}\int_{B_\rho^c(0)} K_1(x)|u_n|^s \,\mathrm{d}x &\leq \varepsilon CF(u_n) + \int_{B_\rho^c(0)} \chi_{[t_0,t_1]}(|u_n(x)|)K_1(x)|u_n|^p \,\mathrm{d}x \\ &\leq \varepsilon CF(u_n) + t_1^p \int_{B_\rho^c(0)\cap A_n} K_1(x)\,\mathrm{d}x \\ &\leq (C' + t_1^q)\varepsilon,\end{aligned} \tag{3.1}$$

for any $n \in \mathbb{N}$ sufficiently large. On the other hand, since $u \in L^s_{K_1}(\mathbb{R}^N)$, we know that

$$\lim_{r\to\infty} \int_{B_r^c(0)} K_1(x)|u|^s \,\mathrm{d}x = 0.$$

From this, there exists $r_\varepsilon > \rho > 0$ such that

$$\int_{B_{r_\varepsilon(0)}^c} K_1(x)|u|^s \,\mathrm{d}x \leq \varepsilon \tag{3.2}$$

and by applying (3.1) we get

$$\int_{B_{r_\varepsilon}^c(0)} K_1(x)|u_n|^s \,\mathrm{d}x \leq \int_{B_\rho^c(0)} K_1(x)|u_n|^s \,\mathrm{d}x \leq (C' + t_1^q)\varepsilon. \tag{3.3}$$

Therefore, by combining (3.2) and (3.3), we deduce that

$$\begin{aligned}\int_{B_{r_\varepsilon}^c(0)} K_1(x)|u_n - u|^s \,\mathrm{d}x &\leq 2^{s-1} \int_{B_{r_\varepsilon}^c(0)} K_1(x)(|u_n|^s + |u|^s)\,\mathrm{d}x \\ &\leq 2^{s-1}(1 + C' + t_1^q)\varepsilon.\end{aligned} \tag{3.4}$$

Now, since $s \in (p, p^*)$ and $K_1 \in L^\infty(\mathbb{R}^N)$, by Lemma 2.2 we get that

$$\lim_{n\to\infty} \int_{B_{r_\varepsilon}(0)} K_1(x)|u_n - u|^s \,\mathrm{d}x = 0. \tag{3.5}$$

Combining (3.4) and (3.5), we conclude for $\varepsilon > 0$ small enough, that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K_1(x)|u_n - u|^s \,\mathrm{d}x = 0.$$

Consequently, we infer that

$$u_n \to u \quad \text{in } L^s_{K_1}(\mathbb{R}^N) \text{ for any } s \in (p, p^*).$$

This finishes the proof. □

We conclude the section proving results which allow us to handle the logarithmic nonlinearity in (1.1). First, we recall the following technical tool, whose proof can be found in Xiang-Hu-Yang [32].

Lemma 3.2.

(i) *For any $\sigma > 0$, we have*

$$\log(t) \le \frac{1}{e\sigma} t^\sigma \quad \textit{for any } t \in [1, \infty).$$

(ii) *For any $\sigma > 0$, we have*

$$t^\sigma |\log(t)| \le \frac{1}{e\sigma} \quad \textit{for any } t \in (0, 1).$$

Lemma 3.3. *Let (H_1) – (H_3) be satisfied, $u \in W^{1,\mathcal{H}}(\mathbb{R}^N) \setminus \{0\}$ and $r \in (p, p^*)$. Moreover, we assume that*

$$\|u\| = 1 \quad \textit{or} \quad \int_{\mathbb{R}^N} K_2(x)|u|^r \,\mathrm{d}x = 0. \tag{3.6}$$

Then, there is a constant $C(p, q, r, K_1, K_2) > 0$ such that

$$\int_{\mathbb{R}^N} K_2(x)|u|^r \log(|u|) \,\mathrm{d}x \le C(p, q, r, K_1, K_2)\|u\|^r.$$

Proof. It is easy to see that, due to assumption (3.6), we have

$$\int_{\mathbb{R}^N} K_2(x)|u|^r \log(|u|) \,\mathrm{d}x = \int_{\mathbb{R}^N} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x = J_1 + J_2,$$

where

$$J_1 := \int_{\{x\in\mathbb{R}^N:\, K_2(x)\ge 0\}} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x,$$

$$J_2 := \int_{\{x\in\mathbb{R}^N:\, K_2(x)< 0\}} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x.$$

Let $\sigma_1 > 0$ be such that $r + \sigma_1 \in (p, p^*)$. Therefore, by Proposition 3.1, Lemma 3.2(ii) and condition (H_3), we obtain

$$
\begin{aligned}
J_1 &= \int\limits_{\{x\in\mathbb{R}^N\colon K_2(x)>0\}} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x \\
&\le \frac{1}{e\sigma_1}\|u\|^{-\sigma_1} \int\limits_{\{x\in\mathbb{R}^N\colon K_2(x)>0\}} K_2(x)|u|^{r+\sigma_1} \mathrm{d}x \\
&\le \frac{1}{e\sigma_1}\|u\|^{-\sigma_1} \int\limits_{\{x\in\mathbb{R}^N\colon K_2(x)>0\}} K_1(x)|u|^{r+\sigma_1} \mathrm{d}x \le C\|u\|^r,
\end{aligned}
$$

for some constant $C>0$. We set

$$
\Omega^- := \left\{x\in\mathbb{R}^N\colon K_2(x)<0 \text{ and } |u(x)|\le\|u\|\right\}.
$$

Then we get

$$
\begin{aligned}
J_2 &\le \int\limits_{\Omega^-} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x \\
&= \int\limits_{\Omega^-} -K_2(x)|u|^r \log\left(\frac{\|u\|}{|u|}\right) \mathrm{d}x.
\end{aligned}
$$

Let $\sigma_2>0$ such that $r-\sigma_2\in(p,p^*)$. Then, from Proposition 3.1, Lemma 3.2(ii) and condition (H_3) it follows that

$$
\begin{aligned}
J_2 &\le \frac{1}{e\sigma_2}\|u\|^{\sigma_2} \int\limits_{\Omega^-} -K_2(x)|u|^{r-\sigma_2} \mathrm{d}x \\
&\le \frac{1}{e\sigma_2}\|u\|^{\sigma_2} \int\limits_{\Omega^-} K_1(x)|u|^{r-\sigma_2} \mathrm{d}x \\
&\le C\|u\|^r,
\end{aligned}
$$

for suitable constant $C>0$. This concludes the proof. □

Lemma 3.4. *Let (H_1) – (H_3) be satisfied, $u\in W^{1,\mathcal{H}}(\mathbb{R}^N)\setminus\{0\}$ and $r\in(p,p^*)$. Then it holds*

$$
\int\limits_{\mathbb{R}^N} K_2(x)|u|^r \log(|u|)\,\mathrm{d}x \le C\|u\|^r + \log(\|u\|)\int\limits_{\mathbb{R}^N} K_2(x)|u|^r \mathrm{d}x,
$$

where $C=C(K_2,p,r,p^*)$ is a positive constant.

Proof. We set

$$
\Omega_1 := \left\{x\in\mathbb{R}^N\colon |u(x)|\le\|u\|\right\} \quad \text{and} \quad \Omega_2 := \left\{x\in\mathbb{R}^N\colon |u(x)|\ge\|u\|\right\}.
$$

Then, we have

$$
\begin{aligned}
\int\limits_{\mathbb{R}^N} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x &= \int\limits_{\Omega_1} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x \\
&\quad + \int\limits_{\Omega_2} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x.
\end{aligned}
$$

Let us check the first integration. Taking into account Lemma 3.2(ii) with $\sigma = \varepsilon$, for a suitable $\varepsilon > 0$, joint with Lemma 2.2, we get

$$\begin{aligned} &\int_{\Omega_1} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x \\ &\leq \|u\|^r \int_{\Omega_1} |K_2(x)| \left(\frac{|u|}{\|u\|}\right)^r \left|\log\left(\frac{|u|}{\|u\|}\right)\right| \mathrm{d}x \\ &\leq \|K_2\|_\infty \|u\|^r \int_{\Omega_1} \left(\frac{|u|}{\|u\|}\right)^{r-\varepsilon} \left(\frac{|u|}{\|u\|}\right)^{\varepsilon} \left|\log\left(\frac{|u|}{\|u\|}\right)\right| \mathrm{d}x \\ &\leq \frac{1}{e\varepsilon} \|K_2\|_\infty \|u\|^r \int_{\Omega_1} \left(\frac{|u|}{\|u\|}\right)^{r-\varepsilon} \mathrm{d}x \\ &\leq C\|u\|^r, \end{aligned} \tag{3.7}$$

for a suitable constant $C > 0$.

Next, we calculate the second integral. For that, using Lemma 3.2(i) with $\sigma = p^* - r$, along with Lemma 2.2 again, we obtain

$$\begin{aligned} \int_{\Omega_2} K_2(x)|u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x &\leq \|K_2\|_\infty \int_{\Omega_2} |u|^r \log\left(\frac{|u|}{\|u\|}\right) \mathrm{d}x \\ &\leq \|K_2\|_\infty \frac{1}{e(p^*-r)} \int_{\Omega_2} |u|^r \left(\frac{|u|}{\|u\|}\right)^{p^*-r} \mathrm{d}x \\ &\leq \frac{\|K_2\|_\infty \frac{1}{e(p^*-r)}}{\|u\|^{p^*-r}} \|u\|^{p^*} \\ &= C\|u\|^r, \end{aligned} \tag{3.8}$$

for a suitable $C = C(K_2, p, r, p^*) > 0$. Combining (3.7) and (3.8), we get the desired assertion of the lemma. □

4. The existence results

In this section, we distinguish two situations depending on the behavior of r.

4.1. The superlinear case

In this part, we discuss the existence of solutions for (1.1) when the logarithmic term is superlinear, namely $r > q$. In order to handle this sign-changing nonlinearity, we need that $\gamma = \lambda$. Hence, we consider the following equation

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + |u|^{p-2}u + \mu(x)|u|^{q-2}u \\ &= K_1(x)|u|^{p^*-2}u + \lambda K_2(x)|u|^{r-2}u\log(|u|) + \lambda K_3(x)|u|^{\beta-2}u \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{4.1}$$

where $1 < p < q < \beta < r < p^*$. Moreover, we assume a further assumption for weight functions, such that

(H_4) *K_2, $K_3 \in L^1(\mathbb{R}^N, \mathbb{R}^+)$ and there exists $q < \sigma < \beta$ such that*

$$K_2(x) \le \frac{e\,(r-\beta)\,r\,(\beta-\sigma)}{\beta(r-\sigma)} K_3(x), \quad \text{for any } x \in \mathbb{R}^N.$$

Our main result is the existence of a mountain pass solution for (4.1).

Theorem 4.1. *Let (H_1) – (H_4) be satisfied and let $1 < p < q < \beta < r < p^*$. Then, there exists $\lambda^* > 0$ such that, if $\lambda \ge \lambda^*$, equation (4.1) admits at least one nontrivial weak solution.*

Hence, we first study the mountain pass geometry of the functional I_λ.

Lemma 4.2. *Let (H_1) – (H_3) be satisfied and let $\lambda > 0$. Then we have the following statements:*

(i) *there exist $\delta = \delta(\lambda) > 0$ and $\alpha = \alpha(\lambda) > 0$ such that $I_\lambda(u) \ge \alpha$ for any $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with $\|u\| = \delta$;*
(ii) *there exist $\psi \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ and $T = T(\lambda, \psi) > 0$ such that $\|T\psi\| > \delta$ and $I_\lambda(T\psi) < 0$.*

Proof. (i) Let $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with $\|u\| \le 1$ and let $s > 0$ such that $r + s \in (q, p^*)$. By Proposition 3.1, Lemmas 2.1, 3.2, Hölder's and Young's inequalities, we get

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{q}\varrho(u) - \frac{\lambda}{r} \int\limits_{\{x\in\mathbb{R}^N :\, |u(x)|>1\}} K_2(x)|u|^r \log(|u|)\,\mathrm{d}x \\ &\quad - \frac{1}{p^*}\int\limits_{\mathbb{R}^N} K_1(x)|u|^{p^*}\,\mathrm{d}x - \frac{\lambda}{\beta}\int\limits_{\mathbb{R}^N} K_3(x)|u|^{\beta}\,\mathrm{d}x \\ &\ge \frac{1}{q}\|u\|^q - \frac{C_1}{p^*}\|u\|^{p^*} - \frac{C_2\lambda}{r}\|u\|^{r+s} - \frac{\lambda C_3}{\beta}\|u\|^{\beta} \\ &\ge \frac{1}{2q}\|u\|^q - \left(\frac{C_1}{p^*} + \frac{C_2\lambda}{r} + \frac{\lambda C_3}{\beta}\right)\|u\|^{\beta}, \end{aligned} \tag{4.2}$$

where C_1, C_2, C_3 are positive constants. Now, we take $\delta > 0$ such that

$$\delta < \min\left\{1, \left[\frac{1}{4q\left(\frac{C_1}{p^*} + \frac{C_2\lambda}{r} + \frac{\lambda C_3}{\beta}\right)}\right]\right\}^{\frac{1}{\beta-q}}.$$

Then, by (4.2), we infer that $I_\lambda(u) \ge \frac{\delta^q}{4q} =: \alpha$ for any $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with $\|u\| = \delta$.

(ii) Let $\psi \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with $\psi \ge 0$ such that

$$\int\limits_{\mathbb{R}^N} K_2(x)|\psi|^r \log(|\psi|)\,\mathrm{d}x > 0. \tag{4.3}$$

For $t > 1$, we have

$$
\begin{aligned}
I_\lambda(t\psi) &= \frac{1}{p}\left(\|\nabla t\psi\|_p^p + \|t\psi\|_p^p\right) + \frac{1}{q}\left(\|\nabla t\psi\|_{q,\mu}^q + \|t\psi\|_{q,\mu}^q\right) \\
&\quad - \int_{\mathbb{R}^N} \frac{K_1(x)}{p^*}|t\psi|^{p^*}\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r}|t\psi|^r \log(|t\psi|)\,\mathrm{d}x \\
&\quad + \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r^2}|t\psi|^r\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \frac{K_3(x)}{\beta}|t\psi|^\beta\,\mathrm{d}x \\
&\le \frac{t^p}{p}\left(\|\nabla \psi\|_p^p + \|\psi\|_p^p\right) + \frac{t^q}{q}\left(\|\nabla \psi\|_{q,\mu}^q + \|\psi\|_{q,\mu}^q\right) \\
&\quad + \lambda t^r \int_{\mathbb{R}^N} \frac{K_2(x)}{r^2}|\psi|^r - t^{p^*}\int_{\mathbb{R}^N} \frac{K_1(x)}{p^*}|\psi|^{p^*}\,\mathrm{d}x.
\end{aligned}
\tag{4.4}
$$

Since $p < q < r < p^*$, we can take $v_0 = T\psi$ with $T = T(\lambda) > 1$ large enough, concluding the proof. □

Thanks to Lemma 4.2, by Willem [31] we can set the positive critical mountain pass value by

$$
c_\lambda := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_\lambda(\gamma(t))
$$

with

$$
\Gamma := \left\{\gamma \in C\left([0,1], W^{1,\mathcal{H}}(\mathbb{R}^N)\right) :\ \gamma(0) = 0,\ I_\lambda\left(\gamma(1)\right) < 0\right\}.
$$

We first prove that c_λ can be controlled by the threshold $\overline{c}$ set as

$$
\overline{c} := \left(\frac{1}{\sigma} - \frac{1}{p^*}\right) S^{\frac{p^*}{p^*-p}} > 0, \tag{4.5}
$$

where σ is given in (H_4), while $S > 0$ is the best constant of the Sobolev embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}_{K_1}(\mathbb{R}^N)$, considering $K_1 \in L^\infty(\mathbb{R}^N)$, set here as

$$
S := \inf_{u\in W^{1,p}(\mathbb{R}^N)} \frac{\|\nabla u\|_p^p + \|u\|_p^p}{\left\|[K_1]^{\frac{1}{p^*}} u\right\|_{p^*}^p}. \tag{4.6}
$$

Lemma 4.3. *Let (H_1) – (H_4) be satisfied. Then there exists $\lambda^* > 0$ such that $c_\lambda < \overline{c}$ for any $\lambda \ge \lambda^*$.*

Proof. Fix $\lambda > 0$ and let $\psi \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ as constructed in Lemma 4.2. That is, $\psi \ge 0$ and (4.3) holds true. By (4.4) we have $\lim_{t\to\infty} I_\lambda(t\psi) = -\infty$. Then, there exists $t_\lambda > 0$ satisfying $I_\lambda(t_\lambda\psi) = \sup_{t\ge 0} I_\lambda(t\psi)$. Hence, $\langle I'_\lambda(t_\lambda\psi), \psi\rangle = 0$ so that

$$
\begin{aligned}
& t_\lambda^{p-1}\left(\|\nabla \psi\|_p^p + \|\psi\|_p^p\right) + t_\lambda^{q-1}\left(\|\nabla \psi\|_{q,\mu}^q + \|\psi\|_{q,\mu}^q\right) \\
&= t_\lambda^{p^*-1}\int_{\mathbb{R}^N} K_1(x)|\psi|^{p^*}\,\mathrm{d}x + \lambda t_\lambda^{r-1}\int_{\mathbb{R}^N} K_2(x)|\psi|^r \log(|t_\lambda\psi|)\,\mathrm{d}x \\
&\quad + \lambda t_\lambda^{\beta-1}\int_{\mathbb{R}^N} K_3(x)|\psi|^\beta\,\mathrm{d}x.
\end{aligned}
\tag{4.7}
$$

We claim that $\{t_\lambda\}_{\lambda>0}$ is bounded. Indeed, denoting by $\Lambda = \{\lambda > 0\colon t_\lambda \ge 1\}$, we see that

$$t_\lambda^{q-1}\varrho(\psi) \geq t_\lambda^{p^*-1}\int\limits_{\mathbb{R}^N} K_1(x)|\psi|^{p^*}\,\mathrm{d}x \quad \text{for any } \lambda\in\Lambda$$

as $\lambda > 0$, $\psi \geq 0$ and by (4.3). Hence, we get the boundedness of $\{t_\lambda\}_{\lambda\in\Lambda}$. Clearly by the construction of Λ also $\{t_\lambda\}_{\lambda\in(\mathbb{R}\setminus\Lambda)}$ is bounded. This proofs the claim.

We fix now a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that $\lambda_n \to \infty$ as $n\to\infty$. Clearly $\{t_{\lambda_n}\}_{n\in\mathbb{N}}$ is bounded. Hence, there exist a number $t_0 \geq 0$ and a subsequence of $\{\lambda_n\}_{n\in\mathbb{N}}$, that we still denote by $\{\lambda_n\}_{n\in\mathbb{N}}$, such that $t_{\lambda_n} \to t_0$ as $n\to\infty$.

We claim that $t_0 = 0$. Indeed, if $t_0 > 0$ then by the dominated convergence theorem we have

$$\int\limits_{\mathbb{R}^N} K_2(x)|\psi|^r \log(|t_{\lambda_n}\psi|)\,\mathrm{d}x \to \int\limits_{\mathbb{R}^N} K_2(x)|\psi|^r \log(|t_0\psi|)\,\mathrm{d}x \quad \text{as } n\to\infty. \tag{4.8}$$

Also, by Lemma 3.2 with $\sigma = r-\beta$, we get

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} K_3(x)|t_0\psi|^\beta\,\mathrm{d}x + \int\limits_{\mathbb{R}^N} K_2(x)|t_0\psi|^r\log(|t_0\psi|)\,\mathrm{d}x\\
&= \int\limits_{\mathbb{R}^N} K_3(x)|t_0\psi|^\beta\,\mathrm{d}x + \int\limits_{\{x\in\mathbb{R}^N:\ |t_0\psi(x)|<1\}} K_2(x)|t_0\psi|^r\log(|t_0\psi|)\,\mathrm{d}x\\
&\quad + \int\limits_{\{x\in\mathbb{R}^N:\ |t_0\psi(x)|\geq 1\}} K_2(x)|t_0\psi|^r\log(|t_0\psi|)\,\mathrm{d}x\\
&\geq \int\limits_{\mathbb{R}^N} K_3(x)|t_0\psi|^\beta\,\mathrm{d}x - \int\limits_{\{x\in\mathbb{R}^N:\ |t_0\psi(x)|<1\}} K_2(x)|t_0\psi|^r\left|\log(|t_0\psi|)\right|\,\mathrm{d}x\\
&\geq \int\limits_{\mathbb{R}^N} \left[K_3(x) - \frac{1}{e(r-\beta)}K_2(x)\right]|t_0\psi|^\beta\,\mathrm{d}x > 0
\end{aligned} \tag{4.9}$$

where the last inequality follows from (H_4), since $\frac{r(\beta-\sigma)}{\beta(r-\sigma)} < 1$ being $r > \beta$. Thus, recalling that $\lambda_n \to \infty$, from (4.7) with $\lambda = \lambda_n$, we get

$$\begin{aligned}
&t_0^{p-1}\left(\|\nabla\psi\|_p^p + \|\psi\|_p^p\right) + t_0^{q-1}\left(\|\nabla\psi\|_{q,\mu}^q + \|\psi\|_{q,\mu}^q\right)\\
&= \lim_{n\to\infty}\left[t_{\lambda_n}^{p^*-1}\int\limits_{\mathbb{R}^N} K_1(x)|\psi|^{p^*}\,\mathrm{d}x + \lambda_n\left(t_{\lambda_n}^{r-1}\int\limits_{\mathbb{R}^N} K_2(x)|\psi|^r\log(|t_{\lambda_n}\psi|)\,\mathrm{d}x\right.\right.\\
&\qquad\left.\left. + t_{\lambda_n}^{\beta-1}\int\limits_{\mathbb{R}^N} K_3(x)|\psi|^\beta\,\mathrm{d}x\right)\right] = \infty,
\end{aligned}$$

that is the desired contradiction. Hence, $t_0 = 0$ and $t_\lambda \to 0$ as $\lambda\to\infty$, since the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is arbitrary.

Consider now the path $\gamma_0(t) = tT\psi$ for $t\in[0,1]$, with T given in Lemma 4.2. Clearly, $\gamma_0\in\Gamma$. Then, Lemma 4.2 gives

$$
\begin{aligned}
0 < c_\lambda &\leq \max_{t\in[0,1]} I_\lambda(\gamma_0(t)) \leq \sup_{t\geq 0} I_\lambda(t\psi) = I_\lambda(t_\lambda\psi) \\
&\leq \varrho(t_\lambda\psi) + \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r^2}|t_\lambda\psi|^r \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r}|t_\lambda\psi|^r \log(|t_\lambda\psi|)\,\mathrm{d}x \\
&\quad - \lambda \int_{\mathbb{R}^N} \frac{K_3(x)}{\beta}|t_\lambda\psi|^\beta \,\mathrm{d}x.
\end{aligned}
\tag{4.10}
$$

We claim that $\{\lambda t_\lambda^\beta\}_{\lambda>0}$ is bounded. Suppose not, then there exists a sequence $\{\lambda_n t_{\lambda_n}^\beta\}_{n\geq 1}$ such that $\lambda_n t_{\lambda_n}^\beta \to \infty$ as $n\to\infty$. Thus, by (4.10) along with the fact that $t_{\lambda_n}\to 0$ as $n\to\infty$, we get

$$
\begin{aligned}
0 < \frac{c_{\lambda_n}}{\lambda_n t_{\lambda_n}^\beta} &\leq \frac{\varrho(t_{\lambda_n}\psi)}{\lambda_n t_{\lambda_n}^\beta} + t_{\lambda_n}^{r-\beta} \int_{\mathbb{R}^N} \frac{K_2(x)}{r^2}|\psi|^r \,\mathrm{d}x \\
&\quad + t_{\lambda_n}^{r-\beta} \int_{\mathbb{R}^N} \frac{K_2(x)}{r}|\psi|^r |\log(|t_{\lambda_n}\psi|)|\,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{K_3(x)}{\beta}|\psi|^\beta \,\mathrm{d}x
\end{aligned}
$$

from which we conclude that, by sending $n\to\infty$ and since $\beta < r$,

$$
0 \leq -\int_{\mathbb{R}^N} K_3(x)|\psi|^\beta \,\mathrm{d}x < 0.
$$

This is a contradiction.

Thus, being $\{\lambda t_\lambda^\beta\}_{\lambda>0}$ bounded and considering $r>\beta$, by (4.10) again we get $\lim_{\lambda\to\infty} c_\lambda = 0$. Then, we can conclude the proof of the lemma. $\square$

Now, we discuss the compactness property for the functional I_λ given by the Palais-Smale condition.

Lemma 4.4. *Let (H_1) – (H_3) be satisfied and let $\lambda > 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ be a bounded $(PS)_c$ sequence with $c\in\mathbb{R}$. Then, up to a subsequence, $\nabla u_n(x)\to\nabla u(x)$ a.e. in $\mathbb{R}^N$ as $n\to\infty$.*

Proof. Since $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ is bounded, using the reflexivity of $W^{1,\mathcal{H}}(\mathbb{R}^N)$, there exists $u\in W^{1,\mathcal{H}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{1,\mathcal{H}}(\mathbb{R}^N)$.

For any $k\in\mathbb{N}$, let $T_k\colon \mathbb{R}\to\mathbb{R}$ be the truncation function defined by

$$
T_k(t) = \begin{cases} t & \text{if } |t|\leq k, \\ k\frac{t}{|t|} & \text{if } |t|>k. \end{cases}
$$

Let $\varepsilon>0$ be a constant such that $r+\varepsilon-1\in(p,p^*)$. By Proposition 3.1 and Lemma 3.2 we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N} K_2(x)|u_n|^{r-2}u_n\log(|u_n|)T_k(u_n-u)\,\mathrm{d}x\Big| \\
&\leq \int_{\mathbb{R}^N} |K_2(x)|\,|u_n|^{r-1}\,|\log(|u_n|)|\,|T_k(u_n-u)|\,\mathrm{d}x \\
&\leq \int_{\{x\in\mathbb{R}^N:\,|u_n(x)|<1\}} |K_2(x)|\,|u_n|^{r-1}\,|\log(|u_n|)|\,|T_k(u_n-u)|\,\mathrm{d}x \\
&\quad + \int_{\{x\in\mathbb{R}^N:\,|u_n(x)|>1\}} |K_2(x)|\,|u_n|^{r-1}\,|\log(|u_n|)|\,|T_k(u_n-u)|\,\mathrm{d}x \\
&\leq \frac{1}{e(r-1)}\int_{\mathbb{R}^N} |K_2(x)|\,|T_k(u_n-u)|\,\mathrm{d}x \\
&\quad + \frac{1}{e\varepsilon}\int_{\mathbb{R}^N} |K_2(x)|\,|u_n|^{r+\varepsilon-1}\,|T_k(u_n-u)|\,\mathrm{d}x \\
&\leq \frac{k}{e(r-1)}\int_{\mathbb{R}^N} |K_2(x)|\,\mathrm{d}x + \frac{C\,k}{e\varepsilon} \\
&\leq C'k,
\end{aligned}
\tag{4.11}
$$

where C, C' are two positive constants. Using (4.11) and considering a cut-off function $\varphi_R \in C_c^\infty(\mathbb{R}^N, [0,1])$ with $\varphi_R \equiv 1$ on a ball B_R with generic radius $R > 0$, the rest of the proof is similar to that of Autuori-Pucci [4, Theorem 4.4]. □

Lemma 4.5. *Let (H_1) – (H_4) be satisfied and let $\lambda > 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ be a bounded $(PS)_c$ sequence with $c < \overline{c}$ as given in (4.5). Then there exists $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u$ in $W^{1,\mathcal{H}}(\mathbb{R}^N)$ as $n \to \infty$.*

Proof. Fix $c < \overline{c}$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded $(PS)_c$ sequence in $W^{1,\mathcal{H}}(\mathbb{R}^N)$, satisfying (2.1). By Proposition 3.1 and Lemma 4.4, there exists a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, and $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ such that

$$
\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}}(\mathbb{R}^N), \qquad u_n \rightharpoonup u \text{ in } L^{p^*}(\mathbb{R}^N),\\
&\Big\|[K_1]^{\frac{1}{p^*}}(u_n-u)\Big\|_{p^*} \to l, \quad u_n \to u \text{ in } L^s_{K_1}(\mathbb{R}^N) \ \text{ for any } s\in(p,p^*),\\
&u_n(x)\to u(x) \text{ a.e. in } \mathbb{R}^N, \quad \nabla u_n(x) \to \nabla u(x) \text{ a.e. in } \mathbb{R}^N.
\end{aligned}
\tag{4.12}
$$

Since the sequences $\{|\nabla u_n|^{p-2}\nabla u_n\}_{n\in\mathbb{N}}$ and $\{\mu(x)^{\frac{1}{q'}}|\nabla u_n|^{q-2}\nabla u_n\}_{n\in\mathbb{N}}$ are bounded in $L^{p'}(\mathbb{R}^N)$ and $L^{q'}(\mathbb{R}^N)$, respectively, we deduce that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u\,\mathrm{d}x = \|\nabla u\|_p^p, \tag{4.13}
$$

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} \mu(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla u\,\mathrm{d}x = \|\nabla u\|_{q,\mu}^q. \tag{4.14}
$$

Furthermore, using Lemma 4.4 and the Lemma of Brézis-Lieb [10], we obtain

$$
\begin{aligned}
&\|\nabla u_n\|_p^p - \|\nabla u_n - \nabla u\|_p^p = \|\nabla u\|_p^p + o(1),\\
&\|u_n\|_p^p - \|u_n - u\|_p^p = \|u\|_p^p + o(1),\\
&\|\nabla u_n\|_{q,\mu}^q - \|\nabla u_n - \nabla u\|_{q,\mu}^q = \|\nabla u\|_{q,\mu}^q + o(1),\\
&\|u_n\|_{q,\mu}^q - \|u_n - u\|_{q,\mu}^q = \|u\|_{q,\mu}^q + o(1),\\
&\left\|[K_1]^{\frac{1}{\beta}} u_n\right\|_\beta^\beta - \left\|[K_1]^{\frac{1}{\beta}} (u_n - u)\right\|_\beta^\beta = \left\|[K_1]^{\frac{1}{\beta}} u\right\|_\beta^\beta + o(1)\\
&\left\|[K_1]^{\frac{1}{p^*}} u_n\right\|_{p^*}^{p^*} - \left\|[K_1]^{\frac{1}{p^*}} (u_n - u)\right\|_{p^*}^{p^*} = \left\|[K_1]^{\frac{1}{p^*}} u\right\|_{p^*}^{p^*} + o(1),
\end{aligned} \tag{4.15}
$$

as $n \to \infty$.

Now, let $s \in (p, r)$ and let $\varepsilon > 0$ be such that $r + \varepsilon \in (p, p^*)$. In light of Proposition 3.1 and Lemma 3.2, condition (H_3) and Hölder's inequality, we infer that

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^N} K_2(x)|u_n|^{r-2}u_n \log(|u_n|)(u_n - u)\,\mathrm{d}x \right|\\
&\le \int_{\mathbb{R}^N} |K_2(x)|\,|u_n|^{r-1}\,|\log(|u_n|)|\,|u_n - u|\,\mathrm{d}x\\
&= \int_{\{x\in\mathbb{R}^N:\ |u_n(x)|<1\}} |K_2(x)|\,|u_n|^{r-1}\,|\log(|u_n|)|\,|u_n - u|\,\mathrm{d}x\\
&\quad + \int_{\{x\in\mathbb{R}^N:\ |u_n(x)|>1\}} |K_2(x)|\,|u_n|^{r-1}\,|\log(|u_n|)|\,|u_n - u|\,\mathrm{d}x\\
&\le \frac{1}{e(r-s)} \int_{\mathbb{R}^N} |K_2(x)|\,|u_n|^{s-1}\,|u_n - u|\,\mathrm{d}x\\
&\quad + \frac{1}{e(r+\varepsilon)} \int_{\mathbb{R}^N} |K_2(x)|\,|u_n|^{r+\varepsilon-1}\,|u_n - u|\,\mathrm{d}x\\
&\le \frac{1}{e(r-s)} \left\|[K_1]^{\frac{1}{s}} u_n\right\|_s^{s-1} \left\|[K_1]^{\frac{1}{s}} |u_n - u|\right\|_s^s\\
&\quad + \frac{1}{e(r+\varepsilon)} \left\|[K_1]^{\frac{1}{r+\varepsilon}} u_n\right\|_{r+\varepsilon}^{r+\varepsilon-1} \left\|[K_1]^{\frac{1}{r+\varepsilon}} |u_n - u|\right\|_{r+\varepsilon}^{r+\varepsilon}\\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned} \tag{4.16}
$$

for a suitable $C > 0$. Thus, by (2.1) and (4.12) – (4.16), we get

$$
\begin{aligned}
o(1) &= \langle I'_\lambda(u_n), u_n - u\rangle\\
&= \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n + \mu(x)|\nabla u_n|^{q-2}\nabla u_n)\cdot(\nabla u_n - \nabla u)\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n + \mu(x)|u_n|^{q-2}u_n)(u_n - u)\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*-2}u_n(u_n - u)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda \int_{\mathbb{R}^N} K_2(x)|u_n|^{r-2}u_n \log(|u_n|)(u_n - u)\,\mathrm{d}x \\
&- \lambda \int_{\mathbb{R}^N} K_3(x)|u_n|^{\beta-2}u_n(u_n - u)\,\mathrm{d}x \\
= {}& \|\nabla u_n\|_p^p - \|\nabla u\|_p^p + \|u_n\|_p^p - \|u\|_p^p + \|\nabla u_n\|_{q,\mu}^q - \|\nabla u\|_{q,\mu}^q \\
&+ \|u_n\|_{q,\mu}^q - \|u\|_{q,\mu}^q - \left\|[K_1]^{\frac{1}{p^*}} u_n\right\|_{p^*}^{p^*} - \left\|[K_1]^{\frac{1}{p^*}} u\right\|_{p^*}^{p^*} + o(1),
\end{aligned}
$$

as $n \to \infty$. Hence by (4.12) and (4.15), we conclude that

$$
\begin{aligned}
&\|\nabla u_n - \nabla u\|_p^p + \|u_n - u\|_p^p + \|\nabla u_n - \nabla u\|_{q,\mu}^q + \|u_n - u\|_{q,\mu}^q \\
&= \left\|[K_1]^{\frac{1}{p^*}}(u_n - u)\right\|_{p^*}^{p^*} + o(1) = l^{p^*} + o(1).
\end{aligned} \tag{4.17}
$$

We claim that $l = 0$. Assume instead $l > 0$. Then, by (4.6) and (4.17), we can see that

$$
l \geq S^{\frac{1}{p^*-p}}. \tag{4.18}
$$

On the other hand, by Lemma 3.2 and condition (H_4), we have

$$
\begin{aligned}
&I_\lambda(u_n) - \frac{1}{\sigma}\langle I_\lambda'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{p} - \frac{1}{\sigma}\right)\varrho(u_n) + \left(\frac{1}{\sigma} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x \\
&\quad + \lambda\left(\frac{1}{\sigma} - \frac{1}{\beta}\right)\int_{\mathbb{R}^N} K_3(x)|u_n|^{\beta}\,\mathrm{d}x + \lambda\left(\frac{1}{\sigma} - \frac{1}{r}\right)\int_{\mathbb{R}^N} K_2(x)|u_n|^{r}\log(|u_n|)\,\mathrm{d}x \\
&\geq \left(\frac{1}{\sigma} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x + \lambda\left(\frac{1}{\sigma} - \frac{1}{\beta}\right)\int_{\{x\in\mathbb{R}^N:\ |u_n(x)|<1\}} K_3(x)|u_n|^{\beta}\,\mathrm{d}x \\
&\quad + \lambda\left(\frac{1}{\sigma} - \frac{1}{r}\right)\int_{\{x\in\mathbb{R}^N:\ |u_n(x)|\leq 1\}} K_2(x)|u_n|^{r}\log(|u_n|)\,\mathrm{d}x \\
&\geq \left(\frac{1}{\sigma} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x + \lambda\left(\frac{1}{\sigma} - \frac{1}{\beta}\right)\int_{\{x\in\mathbb{R}^N:\ |u_n(x)|<1\}} K_3(x)|u_n|^{\beta}\,\mathrm{d}x \\
&\quad - \left(\frac{\lambda}{e(r-\beta)}\right)\left(\frac{1}{\sigma} - \frac{1}{r}\right)\int_{\{x\in\mathbb{R}^N:\ |u_n(x)|\leq 1\}} K_2(x)|u_n|^{\beta}\,\mathrm{d}x \\
&\geq \left(\frac{1}{\sigma} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x \\
&\quad + \lambda\int_{\{x\in\mathbb{R}^N:\ |u_n(x)|\leq 1\}} \left[\left(\frac{1}{\sigma} - \frac{1}{\beta}\right)K_3(x) - \left(\frac{1}{e(r-\beta)}\right)\left(\frac{1}{\sigma} - \frac{1}{r}\right)K_2(x)\right]|u_n|^{\beta}\,\mathrm{d}x \\
&\geq \left(\frac{1}{\sigma} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x,
\end{aligned}
$$

which implies, by using (2.1), (4.12) and (4.15), that

$$c \geq \left(\frac{1}{\sigma} - \frac{1}{p^*}\right)\left(l^{p^*} + \left\|[K_1]^{\frac{1}{p^*}} u\right\|_{p^*}^{p^*}\right).$$

Therefore, from (4.5) and (4.18) we conclude that

$$\overline{c} > c \geq \left(\frac{1}{\sigma} - \frac{1}{p^*}\right) S^{\frac{p^*}{p^*-p}}.$$

Thus, we get a contradiction which yields that $l = 0$. Hence, by (4.17), we prove our desired result. □

Proof of Theorem 4.1 completed. We show the statement via the mountain pass theorem. Indeed, by Lemma 4.2 together with the mountain pass theorem without (PS) condition, see Willem [31, Theorems 1.15 and 2.8], there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ of I_λ.

Now, we show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}}(\mathbb{R}^N)$ arguing by contradiction. Then, going to a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, we have $\lim_{n\to\infty} \|u_n\| = \infty$ and $\|u_n\| \geq 1$ for any $n \in \mathbb{N}$. Let $\sigma > 0$ be such that $1 < p < q < \sigma < \beta < r < p^*$. Thus, invoking Lemmas 2.1 and 3.2 along with Young's inequality, we get

$$\begin{aligned}
&o(1) + c + C\|u_n\| \\
&= I(u_n) - \frac{1}{\sigma}\langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{p} - \frac{1}{\sigma}\right)\left(\|\nabla u_n\|_p^p + \|u_n\|_p^p\right) + \left(\frac{1}{q} - \frac{1}{\sigma}\right)\left(\|\nabla u_n\|_{q,\mu}^q + \|u_n\|_{q,\mu}^q\right) \\
&\quad - \left(\frac{1}{p^*} - \frac{1}{\sigma}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x - \lambda\left(\frac{1}{r} - \frac{1}{\sigma}\right)\int_{\mathbb{R}^N} K_2(x)|u_n|^r \log(|u_n|)\,\mathrm{d}x \\
&\geq \left(\frac{1}{q} - \frac{1}{\sigma}\right)\|u_n\|^p - \lambda\left(\frac{1}{r} - \frac{1}{\sigma}\right)\int_{\{x\in\mathbb{R}^N:\,|u_n(x)|>1|\}} K_2(x)|u_n|^r \log(|u_n|)\,\mathrm{d}x \\
&\quad - \lambda\left(\frac{1}{r} - \frac{1}{\sigma}\right)\int_{\{x\in\mathbb{R}^N:\,|u_n|<1|\}} K_2(x)|u_n|^r \log(|u_n|)\,\mathrm{d}x \\
&\geq \left(\frac{1}{q} - \frac{1}{\sigma}\right)\|u_n\|^p + C_\sigma\left(\frac{1}{r} - \frac{1}{\sigma}\right)\int_{\mathbb{R}^N} K_2(x)\,\mathrm{d}x,
\end{aligned}$$

where C is a positive constant. This leads to a contradiction. Hence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}}(\mathbb{R}^N)$. By Lemma 4.3 we can apply Lemma 4.5 to $\{u_n\}_{n\in\mathbb{N}}$, so that there exists a weak solution $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ of (4.1) such that $I_\lambda(u) = c_\lambda > 0$.

4.2. The linear case

In this subsection, we study equation (1.1) when the logarithmic term is linear, namely $r = q$. Here, we assume that $\gamma = 1$. More precisely, we consider the equation

$$\begin{aligned}
&-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + |u|^{p-2}u + \mu(x)|u|^{q-2}u \\
&= K_1(x)|u|^{p^*-2}u + \lambda K_2(x)|u|^{q-2}u\log(|u|) + K_3(x)|u|^{\beta-2}u \quad \text{in } \mathbb{R}^N,
\end{aligned} \tag{4.19}$$

where $1 < \beta < p < q < p^*$. For this, we assume a new structural assumption

$(\widetilde{H_3})$ $K_2 \in L^1(\mathbb{R}^N, \mathbb{R}^+) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ *with* $0 < K_2 < K_1$ *in* $\mathbb{R}^N$*, while* $K_3 \in L^1(\mathbb{R}^N, \mathbb{R}^+) \cap L^{\frac{q}{q-\beta}}(\mathbb{R}^N, \mathbb{R}^+)$ *with* $0 < K_3 < K_1$ *on* $\mathbb{R}^N$*.*

The main result of this part is the following theorem.

Theorem 4.6. *Let* (H_1) – (H_2) *and* $(\widetilde{H_3})$ *be satisfied and let* $1 < p < \beta < q < p^*$*. Then, for any* $\lambda > 0$*, there exists* $k_\lambda > 0$ *such that if*

$$\max\left\{\|K_3\|_1, \|K_3\|_{\frac{q}{q-\beta}}\right\} < k_\lambda,$$

equation (4.19) *admits at least one nontrivial weak solution.*

Lemma 4.7. *Let* (H_1) – (H_2) *and* $(\widetilde{H_3})$ *be satisfied and let* $\lambda > 0$*. Then, there exist* δ_λ*,* α_λ*,* $m_\lambda > 0$ *such that* $I_\lambda(u) \geq \alpha_\lambda > 0$ *for any* $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ *with* $\|u\| = \delta_\lambda$*, whenever*

$$\|K_3\|_{\frac{q}{q-\beta}} < m_\lambda.$$

Proof. Let $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with $\|u\| \leq 1$ and let $\lambda > 0$. Let us choose $s > 0$ such that $q + s \in (q, p^*)$. By Lemmas 2.1 and 3.2, Hölder and Young inequalities, we get

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{q}\varrho(u) - \frac{\lambda}{q} \int\limits_{\{x\in\mathbb{R}^N:\ |u(x)|>1\}} K_2(x)|u|^q \log(|u|)\,\mathrm{d}x \\ &\quad - \frac{1}{p^*}\int\limits_{\mathbb{R}^N} K_1(x)|u|^{p^*}\,\mathrm{d}x - \int\limits_{\mathbb{R}^N} K_3(x)|u|^\beta\,\mathrm{d}x \\ &\geq \frac{1}{q}\|u\|^q - \left(\frac{C_1}{p^*} + \lambda\frac{C_2}{q}\right)\|u\|^{q+s} - C_3\|K_3\|_{\frac{q}{q-\beta}}\|u\|^\beta \\ &\geq \frac{1}{2q}\|u\|^q - \left(\frac{C_1}{p^*} + \lambda\frac{C_2}{q}\right)\|u\|^{q+s} - \frac{(q-\beta)\,(2\beta C_3)^{\frac{\beta}{q-\beta}}}{q}\|K_3\|_{\frac{q}{q-\beta}}^{\frac{q}{q-\beta}} \end{aligned} \tag{4.20}$$

where C_3 is a positive constant. Set $\delta_\lambda > 0$ such that

$$\delta_\lambda < \min\left(1, \left[\frac{1}{4q\left(\frac{C_1}{p^*} + \frac{\lambda C_2}{r}\right)}\right]^{\frac{1}{s}}\right).$$

Then, by (4.20), we infer that

$$I(u) \geq \frac{\delta_\lambda^q}{4q} - \frac{(q-\beta)\,(2\beta C_3)^{\frac{\beta}{q-\beta}}}{q}\|K_3\|_{\frac{q}{q-\beta}}^{\frac{q}{q-\beta}}, \quad \text{for} \quad \|u\| = \delta_\lambda.$$

Let

$$m_\lambda := \frac{\delta_\lambda^{q-\beta}}{\left[8(q-\beta)\,(2\beta C_3)^{\frac{\beta}{q-\beta}}\right]^{\frac{q-\beta}{q}}}. \tag{4.21}$$

Then, if $\|K_3\|_{\frac{q}{q-\beta}} < m_\lambda$, we obtain

$$I(u) \geq \frac{\varrho_\lambda^q}{8q} =: \alpha_\lambda$$

for any $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ with $\|u\| = \delta_\lambda$. The proof is completed. □

Consider the minimizer

$$M_\lambda := \inf_{u \in B(0,\delta_\lambda)} I_\lambda(u),$$

where δ_λ is defined in Lemma 4.7. For this, we have the following control.

Lemma 4.8. *Let* (H_1) – (H_2) *and* $(\widetilde{H_3})$ *be satisfied and let* $\lambda > 0$. *Then* $-\infty < M_\lambda < 0$.

Proof. Exploiting the proof of Lemma 4.7, we prove that $M_\lambda > -\infty$. Using $(\widetilde{H_3})$, there is a function $\psi \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ such that

$$\begin{aligned} I_\lambda(t\psi) = t^\beta \Bigg[&\frac{t^{p-\beta}}{p}\left(\|\nabla\psi\|_p^p + \|\psi\|_p^p\right) + \frac{t^{q-\beta}}{q}\Bigg(\|\nabla\psi\|_{q,\mu}^q + \|\nabla\psi\|_{q,\mu}^q \\ &- \lambda \int_{\mathbb{R}^N} K_2(x)|\psi|^q \log(|t\psi|)\,\mathrm{d}x + \frac{\lambda}{q}\int_{\mathbb{R}^N} K_2(x)|\psi|^q\,\mathrm{d}x\Bigg) \\ &- \frac{t^{p^*-\beta}}{p^*}\int_{\mathbb{R}^N} K_1(x)|\psi|^{p^*}\,\mathrm{d}x - \int_{\mathbb{R}^N}\frac{K_3(x)}{\beta}|\psi|^\beta\,\mathrm{d}x\Bigg] < 0, \end{aligned}$$

for $t > 0$ small enough. In the previous inequality we used the fact that $\beta < p$. Thus we conclude the proof. □

Now, we get the compactness property for I_λ, under the threshold c_K given by

$$c_K := \left(\frac{p^*-q}{2p^*q}\right) S^{\frac{p^*}{p^*-p}} - \frac{p^*-\beta}{\beta}\left(\frac{q-\beta}{q\beta}\right)^{\frac{p^*}{p^*-\beta}}\left(\frac{2q\beta}{p^*-q}\right)^{\frac{\beta}{p^*-\beta}}\|K_3\|_1 \tag{4.22}$$

with S set as in (4.6).

Lemma 4.9. *Let* (H_1) – (H_2) *and* $(\widetilde{H_3})$ *be satisfied and let* $\lambda > 0$. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}}(\mathbb{R}^N)$ *be a bounded* $(PS)_c$ *sequence with* $c < c_K$. *Then there exists* $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ *such that, up to a subsequence,* $u_n \to u$ *in* $W^{1,\mathcal{H}}(\mathbb{R}^N)$ *as* $n \to \infty$.

Proof. Fix $c < c_K$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded $(PS)_c$ sequence in $W^{1,\mathcal{H}}(\mathbb{R}^N)$ satisfying (2.1). Arguing as in the proof of Lemma 4.5, we can find that

$$l \geq S^{\frac{1}{p^*-p}}, \tag{4.23}$$

with l given in (4.12).

Next, by Lemma 3.2 and Young's inequality, considering also assumption $(\widetilde{H_3})$, we have

$$\begin{aligned} & I_\lambda(u_n) - \frac{1}{q}\langle I'_\lambda(u_n), u_n\rangle \\ & \geq \left(\frac{1}{q} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x + \left(\frac{1}{q} - \frac{1}{\beta}\right)\int_{\mathbb{R}^N} K_3(x)|u_n|^{\beta}\,\mathrm{d}x \\ & \geq \left(\frac{p^*-q}{2p^*q}\right)\int_{\mathbb{R}^N} K_1(x)|u_n|^{p^*}\,\mathrm{d}x - \frac{p^*-\beta}{\beta}\left(\frac{q-\beta}{q\beta}\right)^{\frac{p^*}{p^*-\beta}}\left(\frac{2q\beta}{p^*-q}\right)^{\frac{\beta}{p^*-\beta}}\|K_3\|_1, \end{aligned}$$

which implies, by using (2.1), (4.12) and (4.15), that

$$\begin{aligned} c \geq & \left(\frac{p^*-q}{2p^*q}\right)\left(l^{p^*} + \left\|[K_1]^{\frac{1}{p^*}}u\right\|_{p^*}^{p^*}\right) \\ & - \frac{p^*-\beta}{\beta}\left(\frac{q-\beta}{q\beta}\right)^{\frac{p^*}{p^*-\beta}}\left(\frac{2q\beta}{p^*-q}\right)^{\frac{\beta}{p^*-\beta}}\|K_3\|_1. \end{aligned}$$

Therefore, from (4.23) and again $(\widetilde{H_3})$, we conclude that

$$\begin{aligned} c > & \left(\frac{p^*-q}{2p^*q}\right)S^{\frac{p^*}{p^*-p}} \\ & - \frac{p^*-\beta}{\beta}\left(\frac{q-\beta}{q\beta}\right)^{\frac{p^*}{p^*-\beta}}\left(\frac{2q\beta}{p^*-q}\right)^{\frac{\beta}{p^*-\beta}}\|K_3\|_1. \end{aligned}$$

Thus, we get a contradiction from (4.22) and we know that $l = 0$. Hence, by (4.17), our desired result follows. □

Proof of Theorem 4.6 completed. Let $\lambda > 0$ and let us set

$$m := \left(\frac{p^*-r}{2p^*r}\right)S^{p^{\frac{p^*}{p^*-p}}}\left[\frac{p^*-\beta}{\beta}\left(\frac{r-\beta}{r\beta}\right)^{\frac{p^*}{p^*-\beta}}\left(\frac{2r\beta}{p^*-r}\right)^{\frac{\beta}{p^*-\beta}}\right]^{-1}.$$

Then, we define $\kappa_\lambda := \min\{m_\lambda, m\}$, with m_λ given in (4.21).

Now, let us assume that

$$\max\left\{\|K_3\|_1, \|K_3\|_{\frac{q}{q-\beta}}\right\} < \kappa_\lambda.$$

By Lemmas 4.7, 4.8 and applying the Ekeland's variational principle for the complete metric space $\overline{B}(0,\delta_\lambda)$, then there exists a $(PS)_{M_\lambda}$ sequence $\{v_n\}_n \subset \overline{B}(0,\delta_\lambda)$. Again, from Lemma 4.8 we have $M_\lambda < 0$. Thus, since when $\|K_3\|_1 < m$ we have that $c_K > 0$, with c_K given in (4.22), we can apply Lemma 4.9 to $\{v_n\}_n$, and this time we get $v \in W^{1,\mathcal{H}}(\mathbb{R}^N)$ a weak solution of (4.19) such that $I_\lambda(v) = M_\lambda < 0$.

5. Radial solution for the double phase equation

In this section, we study the existence of radial solution for equation (1.1) when $K_1 = 1$. In addition, we need the following structural assumption

(H_5) *μ, K_2 and K_3 are three radial functions.*

The main result of this section reads as follows.

Theorem 5.1. *Let (H_1) – (H_2) and (H_5) be satisfied.*

- (i) *Suppose that (H_3) – (H_4) hold true. If $1 < p < q < \beta < r < p^*$ and $\gamma = \lambda$, then for any $\lambda \geq \lambda^*$, with λ^* as given in Lemma 4.3, equation (1.1) admits at least one nontrivial radial weak solution.*
- (ii) *Suppose that $(\widetilde{H_3})$ holds true. If $1 < \beta < p < q = r < p^*$ and $\gamma = 1$, then for any $\lambda > 0$ equation (1.1) admits at least one nontrivial radial weak solution.*

Proof. (i) Using Theorem 2.3 and replacing $W^{1,\mathcal{H}}(\mathbb{R}^N)$ by $W^{1,\mathcal{H}}_{\text{rad}}(\mathbb{R}^N)$ in combination with the argument employed in the proof of Theorem 4.1, I_λ admits a nontrivial critical point $u_0 \in W^{1,\mathcal{H}}_{\text{rad}}(\mathbb{R}^N)$. Next, we will show that u is a critical point of I_λ in the space $W^{1,\mathcal{H}}(\mathbb{R}^N)$. To this end, we will apply the Principle of Symmetric Criticality, see Palais [27]. Thus, let $O(N)$ denotes the group of rotations in $\mathbb{R}^N$ and the action $O(N) \times W^{1,\mathcal{H}}(\mathbb{R}^N) \to W^{1,\mathcal{H}}(\mathbb{R}^N)$ is given by

$$(gu)(x) = u(g(x)) \quad \text{for any } x \in \mathbb{R}^N,$$

which is isometric. Furthermore, since μ is a radial function, the functional I_λ is invariant under the action of g, since for any $g \in O(N)$ and any $u \in W^{1,\mathcal{H}}(\mathbb{R}^N)$, we have

$$\begin{aligned} I_\lambda(gu) &= \varrho(gu) - \int_{\mathbb{R}^N} \frac{|gu|^{p^*}}{p^*}\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r}|gu|^r \log(|gu|)\,\mathrm{d}x \\ &\quad + \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r^2}|gu|^r\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \frac{K_3(x)}{\beta}|gu|^\beta\,\mathrm{d}x \\ &= \varrho(u) - \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{p^*}\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r}|u|^r \log(|gu|)\,\mathrm{d}x \\ &\quad + \lambda \int_{\mathbb{R}^N} \frac{K_2(x)}{r^2}|u|^r\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \frac{K_3(x)}{\beta}|u|^\beta\,\mathrm{d}x. \end{aligned}$$

From the above information, it is easy to check that

$$\operatorname{Fix}(O(N)) = \left\{u \in W^{1,\mathcal{H}}(\mathbb{R}^N) \colon gu = u \text{ for any } g \in O(N)\right\} = W^{1,\mathcal{H}}_{\text{rad}}(\mathbb{R}^N).$$

Hence by the Principe of Symmetric Criticality of Palais, u_0 is a nontrivial critical point of I_λ in $W^{1,\mathcal{H}}(\mathbb{R}^N)$.

(ii) The second assertion follows by combining the above argument with the existence result of Theorem 4.6. □

CRediT authorship contribution statement

The authors contributed equally to this work.

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