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# Existence and uniqueness of elliptic systems with double phase operators and convection terms

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ABSTRACT

In this paper we study quasilinear elliptic systems driven by so-called double phase operators and nonlinear right-hand sides depending on the gradients of the solutions. Based on the surjectivity result for pseudomonotone operators we prove the existence of at least one weak solution of such systems. Furthermore, under some additional conditions on the data, the uniqueness of weak solutions is shown.


## 1. Introduction

In this paper, given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary $\partial\Omega$, we are concerned with the existence and uniqueness of solutions to the following elliptic system

$$\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p_1-2}\nabla u + \mu_1(x)|\nabla u|^{q_1-2}\nabla u\right) &= f_1(x,u,v,\nabla u,\nabla v) && \text{in } \Omega,\\
-\operatorname{div}\left(|\nabla v|^{p_2-2}\nabla v + \mu_2(x)|\nabla v|^{q_2-2}\nabla v\right) &= f_2(x,u,v,\nabla u,\nabla v) && \text{in } \Omega,\\
u = v &= 0 && \text{on } \partial\Omega,
\end{aligned} \tag{1.1}$$

where $1 < p_i < q_i < N$, $\mu_i \colon \overline{\Omega} \to [0,\infty)$ are Lipschitz continuous and $f_i \colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, $i = 1, 2$, that satisfy suitable structure conditions, see hypotheses (H) in Section 3.

Here, the operator is the so-called double-phase operator, that is

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \quad \text{for } u \in W^{1,\mathcal{H}}(\Omega),$$

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where $1 < p < q < N$ and with a suitable Sobolev Musielak-Orlicz space $W^{1,\mathcal{H}}(\Omega)$, see its definition in Section 2. Such an operator is the extension of the so-called weighted $(q,p)$-Laplacian when $\inf_{\overline{\Omega}} \mu > 0$ and of the $p$-Laplace differential operator when $\mu \equiv 0$.

The novelty of this work is an existence and uniqueness result for problems of the form (1.1) by using the surjectivity result for pseudomonotone operators, see Definition 2.2 and Theorem 2.3. To the best of our knowledge, this is the first work dealing with a double phase operator and a convection term (that is, the right-hand side depends on the gradient of the solution) in the context of elliptic systems.

Zhikov [39] was the first who studied so-called double phase operators in order to describe models of strongly anisotropic materials by studying the functional

$$u \mapsto \int \left(|\nabla u|^p + \mu(x)|\nabla u|^q\right)\,dx, \tag{1.2}$$

where $1 < p < q < N$, see also Zhikov [40], [41] and the monograph of Zhikov-Kozlov-Oleinik [42]. Functionals of the expression (1.2) have been studied by several authors with respect to regularity results and nonstandard growth, see for example, Baroni-Colombo-Mingione [4], [5], [6], Baroni-Kuusi-Mingione [7], Cupini-Marcellini-Mascolo [16], Colombo-Mingione [14], [15], Marcellini [25], [26] and the references therein.

The motivation of this work was on the one hand the work of Gasiński-Winkert [20] who proved existence and uniqueness for the problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &= f(x,u,\nabla u) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

following the paper of Averna-Motreanu-Tornatore [1]. On the other side, we were also motivated by the paper of Motreanu-Vetro-Vetro [28] who treated elliptic systems for $(p_i, q_i)$-Laplace operators of the form

$$\begin{aligned} -\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1 &= f_1(x,u_1,u_2,\nabla u_1,\nabla u_2) && \text{in } \Omega,\\ -\Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2 &= f_2(x,u_1,u_2,\nabla u_1,\nabla u_2) && \text{in } \Omega,\\ u_1 = u_2 &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

The idea in the current paper is to combine both problems (1.3) and (1.4) which gives our model problem (1.1). Such new class of problems brings lots of difficulties to be overcome like the Orlicz space in order to deal with the double phase operator, the gradient dependence of the right-hand side which implies that we cannot use variational tools and the fact that we treat this for elliptic systems. Our results extend those in Gasiński-Winkert [20] and Motreanu-Vetro-Vetro [28].

In the case of single-valued equations like (1.3) without convection we refer to the works of Colasuonno-Squassina [13], Gasiński-Papageorgiou [18], Gasiński-Winkert [19], Liu-Dai [22], Perera-Squassina [35] concerning existence and multiplicity results.

Elliptic systems with the shape as in (1.4) have been considered by a number of authors. Existence results can be found, for examples, in Boccardo-de Figueiredo [8], Carl-Motreanu [10], [11], Drábek-Stavrakakis-Zographopoulos [17], Motreanu-Vetro-Vetro [29] and the references therein.

Works which are closely related to our paper dealing with certain types of double phase problems, convection terms or elliptic systems can be found in Bahrouni-Rădulescu-Repovš [2], Bahrouni-Rădulescu-Winkert [3], Cencelj-Rădulescu-Repovš [12], Marano-Marino-Moussaoui [23], Marano-Winkert [24], Marino-Winkert [27], Motreanu-Winkert [30], Papageorgiou-Rădulescu-Repovš [31], [32], [33], Rădulescu [36], Zhang-Rădulescu [38], Zheng-Gasiński-Winkert-Bai [37] and the references therein.

The paper is organized as follows. In Section 2 we recall the definition of the Musielak-Orlicz spaces $L^{\mathcal{H}}(\Omega)$ and its corresponding Sobolev spaces $W^{1,\mathcal{H}}(\Omega)$ and we recall the surjectivity result for pseudomonotone operators. In Section 3 we present the full assumptions on the data of problem (1.1), give the definition of the weak solution and state and prove our main existence result, see Theorem 3.1. In the last part, namely Section 4, we state some conditions on $f_i$, $i = 1, 2$, in order to prove the uniqueness of weak solutions of (1.1), see Theorem 4.1.

## 2. Preliminaries

For every $1 \leq r < \infty$ we consider the usual Lebesgue spaces $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ equipped with the norm $\|\cdot\|_r$. When $1 < r < \infty$ we denote by $W^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ the corresponding Sobolev spaces equipped with the norms $\|\cdot\|_{1,r}$ and $\|\cdot\|_{1,r,0}$, respectively. By $r'$, we denote the conjugate of $r \in (1,\infty)$, that is, $\frac{1}{r}+\frac{1}{r'}=1$.

For $i = 1, 2$ we define functions $\mathcal{H}_i \colon \Omega \times [0,\infty) \to [0,\infty)$ by

$$\mathcal{H}_i(x,t) = t^{p_i} + \mu_i(x)t^{q_i},$$

where $1 < p_i < q_i < N$ and

$$\frac{q_i}{p_i} < 1 + \frac{1}{N}, \qquad \mu_i \colon \overline{\Omega} \to [0,\infty) \text{ is Lipschitz continuous.} \tag{2.1}$$

**Remark 2.1.** From the condition above we easily see that

$$q_i < p_i^*, \quad i = 1, 2,$$

where $p_i^*$ is the critical Sobolev exponent of $p_i$ given by

$$p_i^* := \frac{Np_i}{N-p_i}.$$

Indeed, for fixed $i \in \{1, 2\}$, we have to show that $q_i < \dfrac{Np_i}{N-p_i}$, that is $Nq_i - p_iq_i < Np_i$. From condition (2.1) we have $Nq_i - p_i < Np_i$. Moreover, since $q_i > 1$, we have

$$Nq_i - p_iq_i < Nq_i - p_i < Np_i,$$

which shows the assertion.

Let $\rho_{\mathcal{H}_i}(u) := \displaystyle\int_\Omega \mathcal{H}_i(x,|u|)dx$. Then the Musielak-Orlicz space $L^{\mathcal{H}_i}(\Omega)$ is given by

$$L^{\mathcal{H}_i}(\Omega) = \left\{u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \rho_{\mathcal{H}_i}(u) < +\infty\right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}_i} := \inf\left\{\tau > 0 : \rho_{\mathcal{H}_i}\left(\frac{u}{\tau}\right) \leq 1\right\}.$$

Then $L^{\mathcal{H}_i}(\Omega)$ becomes uniformly convex, and so a reflexive Banach space. Moreover we define the space

$$L^{q_i}_{\mu_i}(\Omega) = \left\{u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \mu_i(x)|u|^{q_i}\,dx < +\infty\right\}$$

endowed with the seminorm

$$\|u\|_{q_i,\mu_i} := \left(\int_\Omega \mu_i(x)|u|^{q_i}\,dx\right)^{\frac{1}{q_i}}.$$

From Colasuonno-Squassina [13, Proposition 2.15] we have the following continuous embeddings

$$L^{q_i}(\Omega) \hookrightarrow L^{\mathcal{H}_i}(\Omega) \hookrightarrow L^{p_i}(\Omega) \cap L^{q_i}_{\mu_i}(\Omega).$$

For $u \neq 0$ we have $\rho_{\mathcal{H}_i}\left(\frac{u}{\|u\|_{\mathcal{H}_i}}\right) = 1$, so it is easy to see that

$$\min\left\{\|u\|_{\mathcal{H}_i}^{p_i}, \|u\|_{\mathcal{H}_i}^{q_i}\right\} \le \|u\|_{p_i}^{p_i} + \|u\|_{q_i,\mu_i}^{q_i} \le \max\left\{\|u\|_{\mathcal{H}_i}^{p_i}, \|u\|_{\mathcal{H}_i}^{q_i}\right\} \tag{2.2}$$

for every $u \in L^{\mathcal{H}_i}(\Omega)$. Then we can introduce the corresponding Sobolev space $W^{1,\mathcal{H}_i}(\Omega)$ defined by

$$W^{1,\mathcal{H}_i}(\Omega) := \left\{u \in L^{\mathcal{H}_i}(\Omega) : |\nabla u| \in L^{\mathcal{H}_i}(\Omega)\right\}$$

with the norm

$$\|u\|_{1,\mathcal{H}_i} := \|\nabla u\|_{\mathcal{H}_i} + \|u\|_{\mathcal{H}_i},$$

where $\|\nabla u\|_{\mathcal{H}_i} = \|\,|\nabla u|\,\|_{\mathcal{H}_i}$.

Moreover, we write $W^{1,\mathcal{H}_i}_0(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{H}_i}}$ being the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}_i}(\Omega)$. Taking (2.1) into account we can refer to Colasuonno-Squassina [13, Proposition 2.18] in order to consider an equivalent norm on $W^{1,\mathcal{H}_i}_0(\Omega)$ given by

$$\|u\|_{1,\mathcal{H}_i,0} = \|\nabla u\|_{\mathcal{H}_i}.$$

Note that $W^{1,\mathcal{H}_i}(\Omega)$ as well as $W^{1,\mathcal{H}_i}_0(\Omega)$ are uniformly convex, and so reflexive Banach spaces.

Since $1 < p_i < N$, we know that the embedding

$$W^{1,\mathcal{H}_i}_0(\Omega) \hookrightarrow L^{r_i}(\Omega) \tag{2.3}$$

is compact whenever $r_i < p_i^*$, see Colasuonno-Squassina [13, Proposition 2.15].

From equation (2.2) we directly obtain

$$\min\left\{\|u\|_{1,\mathcal{H}_i,0}^{p_i}, \|u\|_{1,\mathcal{H}_i,0}^{q_i}\right\} \le \|\nabla u\|_{p_i}^{p_i} + \|\nabla u\|_{q_i,\mu_i}^{q_i} \le \max\left\{\|u\|_{1,\mathcal{H}_i,0}^{p_i}, \|u\|_{1,\mathcal{H}_i,0}^{q_i}\right\} \tag{2.4}$$

for every $u \in W^{1,\mathcal{H}_i}_0(\Omega)$.

For $1 < r < \infty$ we consider now the eigenvalue problem for the $r$-Laplacian with homogeneous Dirichlet boundary condition given by

$$\begin{aligned} -\Delta_r u &= \lambda |u|^{r-2}u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

Let us denote by $\lambda_{1,r}$ the first eigenvalue of (2.5). It is well known that $\lambda_{1,r}$ is positive, simple, and isolated, see Lê [21]. Moreover, we have the following variational characterization

$$\lambda_{1,r} = \inf_{u \in W_0^{1,r}(\Omega)} \left\{ \int_\Omega |\nabla u|^r dx : \int_\Omega |u|^r dx = 1 \right\}. \tag{2.6}$$

We now recall some definitions that we will use in the sequel.

**Definition 2.2.** Let $X$ be a reflexive Banach space, $X^*$ its dual space and denote by $\langle \cdot\,,\cdot\rangle$ its duality pairing. Let $A\colon X \to X^*$, then $A$ is called

(a) to satisfy the $(\mathrm{S}_+)$-property if $u_n \rightharpoonup u$ in $X$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$ imply $u_n \to u$ in $X$;
(b) pseudomonotone if $u_n \rightharpoonup u$ in $X$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$ imply $Au_n \rightharpoonup u$ and $\langle Au_n, u_n\rangle \to \langle Au, u\rangle$;
(c) coercive if

$$\lim_{\|u\|_X\to\infty} \frac{\langle Au, u\rangle}{\|u\|_X} = \infty. \tag{2.7}$$

Our existence result is based on the following surjectivity result for pseudomonotone operators, see, e.g., Carl-Le-Motreanu [9, Theorem 2.99] or Papageorgiou-Winkert [34, Theorem 6.1.57].

**Theorem 2.3.** *Let $X$ be a real, reflexive Banach space, let $A\colon X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and $b \in X^*$. Then, a solution of the equation $Au = b$ exists.*

We consider the space $\mathcal{W} := W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ endowed with the norm

$$\|(u,v)\|_{\mathcal{W}} := \|u\|_{1,\mathcal{H}_1,0} + \|v\|_{1,\mathcal{H}_2,0},$$

for every $(u,v) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$.

Then we consider the operator $A\colon W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega) \to (W_0^{1,\mathcal{H}_1}(\Omega))^* \times (W_0^{1,\mathcal{H}_2}(\Omega))^*$ defined by

$$\begin{aligned}\langle A(u,v),(\varphi,\psi)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} :=& \int_\Omega \left(|\nabla u|^{p_1-2}\nabla u + \mu_1(x)|\nabla u|^{q_1-2}\nabla u\right)\cdot\nabla\varphi dx\\ &+ \int_\Omega \left(|\nabla v|^{p_2-2}\nabla v + \mu_2(x)|\nabla v|^{q_2-2}\nabla v\right)\cdot\nabla\psi dx,\end{aligned} \tag{2.8}$$

where $\langle\cdot\,,\cdot\rangle_{\mathcal{H}_1\times\mathcal{H}_2}$ is the duality pairing between $W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ and its dual space $(W_0^{1,\mathcal{H}_1}(\Omega))^* \times (W_0^{1,\mathcal{H}_2}(\Omega))^*$. The next result summarizes the properties of the operator $A$.

**Lemma 2.4.** *Let $A\colon W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega) \to (W_0^{1,\mathcal{H}_1}(\Omega))^* \times (W_0^{1,\mathcal{H}_2}(\Omega))^*$ be the operator defined by (2.8). Then, $A$ is bounded, continuous, monotone (hence maximal monotone), and of type $(S_+)$.*

**Proof.** The proof is similar to the one in Liu-Dai [22, Proposition 3.1]. □

## 3. Main result

We assume the following hypotheses on the nonlinearities $f_1, f_2$.

(H) $f_1, f_2\colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that

(i) There exist $\alpha_i \in L^{\frac{r_i}{r_i-1}}(\Omega)$ $(i=1,2)$ such that

$$\begin{aligned}|f_1(x,s,t,\xi,\zeta)| &\leq A_1|s|^{a_1} + A_2|t|^{a_2} + A_3|s|^{a_3}|t|^{a_4} + A_4|\xi|^{a_5}\\ &\quad + A_5|\zeta|^{a_6} + A_6|\xi|^{a_7}|\zeta|^{a_8} + |\alpha_1(x)|,\\ |f_2(x,s,t,\xi,\zeta)| &\leq B_1|s|^{b_1} + B_2|t|^{b_2} + B_3|s|^{b_3}|t|^{b_4} + B_4|\xi|^{b_5}\\ &\quad + B_5|\zeta|^{b_6} + B_6|\xi|^{b_7}|\zeta|^{b_8} + |\alpha_2(x)|,\end{aligned}$$

for a.a. $x \in \Omega$, for all $s,t \in \mathbb{R}$ and for all $\xi,\zeta \in \mathbb{R}^N$, where $A_j, B_j$, $j=1,\dots 6$, are nonnegative constants and with $1 < r_i < p_i^*$, $i=1,2$. Moreover, the exponents $a_\ell, b_\ell$, $\ell = 1,\dots,8$, are nonnegative and satisfy the following conditions

$$\begin{array}{llll}
\text{(E1)} & a_1 \leq r_1 - 1, & \text{(E2)} & a_2 \leq \dfrac{r_1-1}{r_1} r_2,\\[2mm]
\text{(E3)} & \dfrac{a_3}{r_1} + \dfrac{a_4}{r_2} \leq \dfrac{r_1-1}{r_1}, & \text{(E4)} & a_5 \leq \dfrac{r_1-1}{r_1} p_1,\\[2mm]
\text{(E5)} & a_6 \leq \dfrac{r_1-1}{r_1} p_2, & \text{(E6)} & \dfrac{a_7}{p_1} + \dfrac{a_8}{p_2} \leq \dfrac{r_1-1}{r_1},\\[2mm]
\text{(E7)} & b_1 \leq \dfrac{r_2-1}{r_2} r_1, & \text{(E8)} & b_2 \leq r_2 - 1,\\[2mm]
\text{(E9)} & \dfrac{b_3}{r_1} + \dfrac{b_4}{r_2} \leq \dfrac{r_2-1}{r_2}, & \text{(E10)} & b_5 \leq \dfrac{r_2-1}{r_2} p_1,\\[2mm]
\text{(E11)} & b_6 \leq \dfrac{r_2-1}{r_2} p_2, & \text{(E12)} & \dfrac{b_7}{p_1} + \dfrac{b_8}{p_2} \leq \dfrac{r_2-1}{r_2}.
\end{array}$$

(ii) There exist $\omega \in L^1(\Omega)$ and $\Lambda, \Gamma \geq 0$ such that

$$f_1(x,s,t,\xi,\zeta)s + f_2(x,s,t,\xi,\zeta)t \leq \Lambda\left(|\xi|^{p_1} + |\zeta|^{p_2}\right) + \Gamma\left(|s|^{p_1} + |t|^{p_2}\right) + \omega(x), \tag{3.1}$$

for a.a. $x \in \Omega$, for all $s,t \in \mathbb{R}$ and for all $\xi,\zeta \in \mathbb{R}^N$ and with

$$\Lambda + \Gamma \max\left\{\lambda_{1,p_1}^{-1}, \lambda_{1,p_2}^{-1}\right\} < 1, \tag{3.2}$$

where $\lambda_{1,p_i}$ is the first eigenvalue of the $p_i$-Laplacian, see (2.5).

We say that $(u,v) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned}\int_\Omega \left(|\nabla u|^{p_1-2}\nabla u + \mu_1(x)|\nabla u|^{q_1-2}\nabla u\right)\cdot \nabla\varphi\,dx &= \int_\Omega f_1(x,u,v,\nabla u,\nabla v)\varphi\,dx,\\ \int_\Omega \left(|\nabla v|^{p_2-2}\nabla v + \mu_2(x)|\nabla v|^{q_2-2}\nabla v\right)\cdot \nabla\psi\,dx &= \int_\Omega f_2(x,u,v,\nabla u,\nabla v)\psi\,dx,\end{aligned} \tag{3.3}$$

is satisfied for all test functions $(\varphi,\psi) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$. Taking the embedding (2.3) into account, along with the growth conditions in (H)(i), we see that the definition of a weak solution is well-defined. Indeed, if we estimate the integral concerning the function $f_1 \colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by using condition (H)(i) we obtain several mixed terms. Let us consider, for example, the third term on the right-hand side of the growth of $f_1$. Applying Hölder's inequality we get

$$A_3 \int\limits_\Omega |u|^{a_3}|v|^{a_4}\varphi\,dx \le A_3 \left(\int\limits_\Omega |u|^{a_3 s_1}\,dx\right)^{\frac{1}{s_1}} \left(\int\limits_\Omega |v|^{a_4 s_2}\,dx\right)^{\frac{1}{s_2}} \left(\int\limits_\Omega |\varphi|^{s_3}\,dx\right)^{\frac{1}{s_3}},$$

where $(u,v) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$, $\varphi \in W_0^{1,\mathcal{H}_1}(\Omega)$ and

$$\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 1.$$

Taking $s_3 = r_1$ with $1 < r_1 < p_1^*$ and using $s_1 \le \frac{r_1}{a_3}$ as well as $s_2 \le \frac{r_2}{a_4}$ leads to

$$\frac{a_3}{r_1} + \frac{a_4}{r_2} \le \frac{r_1 - 1}{r_1},$$

which is exactly condition (E3). Note that the conditions in (H)(i) are chosen in order to prove our main results by applying the compact embedding (2.3). Of course, for the finiteness of the integrals in the weak formulation (3.3), we can also allow critical growth to have a well-defined weak formulation.

Now we are ready to formulate and prove our main result in this section.

**Theorem 3.1.** *Let $1 < p_i < q_i < N$, $i = 1,2$, and let hypotheses (2.1) and (H) be satisfied. Then, there exists a weak solution $(u,v) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ of problem (1.1).*

**Proof.** Let

$$\hat{N}_{f_i} \colon W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega) \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to L^{r_1'}(\Omega) \times L^{r_2'}(\Omega)$$

be the Nemytskij operator associated to $f_i$. Moreover, let

$$j_i^* \colon L^{r_1'}(\Omega) \times L^{r_2'}(\Omega) \to (W_0^{1,\mathcal{H}_1}(\Omega))^* \times (W_0^{1,\mathcal{H}_2}(\Omega))^*$$

be the adjoint operator for the embedding

$$j_i \colon W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega) \to L^{r_1}(\Omega) \times L^{r_2}(\Omega).$$

We then define

$$N_{f_i} := j_i^* \circ \hat{N}_{f_i} \colon W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega) \to (W_0^{1,\mathcal{H}_1}(\Omega))^* \times (W_0^{1,\mathcal{H}_2}(\Omega))^*,$$

which is well-defined by hypotheses (H)(i). We set

$$\mathcal{A}(u,v) := A(u,v) - N_{f_1}(u,v) - N_{f_2}(u,v). \tag{3.4}$$

Our aim is to apply Theorem 2.3. So, we need to show that $\mathcal{A}$ is bounded, pseudomonotone and coercive.

**Claim 1:** $\mathcal{A}$ is bounded.

The boundedness of $\mathcal{A}$ follows directly from the boundedness of $A$ and the growth conditions on $f_1$ and $f_2$ stated in (H)(i).

**Claim 2:** $\mathcal{A}$ is pseudomonotone.

To this end, let $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ be a sequence such that

$$(u_n,v_n) \rightharpoonup (u,v) \quad \text{in } W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega) \tag{3.5}$$

and

$$\limsup_{n\to\infty}\langle \mathcal{A}(u_n,v_n),(u_n-u,v_n-v)\rangle_{\mathcal{H}_1\times\mathcal{H}_2}\le 0. \tag{3.6}$$

Taking the compact embedding (2.3) into account yields

$$u_n\to u \quad \text{in } L^{r_1}(\Omega) \quad \text{and} \quad v_n\to v \quad \text{in } L^{r_2}(\Omega), \tag{3.7}$$

since $r_1<p_1^*$ and $r_2<p_2^*$, respectively. We want to show that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega f_1(x,u_n,v_n,\nabla u_n,\nabla v_n)(u_n-u)\,dx=0,\\
&\lim_{n\to\infty}\int_\Omega f_2(x,u_n,v_n,\nabla u_n,\nabla v_n)(v_n-v)\,dx=0.
\end{aligned} \tag{3.8}$$

Let us consider the first expression in (3.8). By the growth condition (H)(i) it follows

$$\begin{aligned}
&\int_\Omega f_1(x,u_n,v_n,\nabla u_n,\nabla v_n)(u_n-u)\,dx\\
&\le\int_\Omega\Big(A_1|u_n|^{a_1}+A_2|v_n|^{a_2}+A_3|u_n|^{a_3}|v_n|^{a_4}\\
&\quad+A_4|\nabla u_n|^{a_5}+A_5|\nabla v_n|^{a_6}+A_6|\nabla u_n|^{a_7}|\nabla v_n|^{a_8}+|\alpha_1(x)|\Big)|u_n-u|\,dx.
\end{aligned} \tag{3.9}$$

Applying Hölder's inequality, (3.7) and conditions (E1) and (E2), respectively, we obtain

$$\begin{aligned}
A_1\int_\Omega |u_n|^{a_1}|u_n-u|\,dx&\le A_1\left(\int_\Omega |u_n|^{a_1r_1'}\,dx\right)^{\frac{1}{r_1'}}\|u_n-u\|_{r_1}\\
&\le C_1\left(1+\|u_n\|_{r_1}^{r_1-1}\right)\|u_n-u\|_{r_1}\to 0
\end{aligned}$$

and

$$\begin{aligned}
A_2\int_\Omega |v_n|^{a_2}|u_n-u|\,dx&\le A_2\left(\int_\Omega v_n^{a_2r_1'}\,dx\right)^{\frac{1}{r_1'}}\|u_n-u\|_{r_1}\\
&\le C_2\left(1+\|v_n\|_{r_2}^{\frac{r_2}{r_1'}}\right)\|u_n-u\|_{r_1}\to 0
\end{aligned}$$

for some $C_1,C_2>0$. Moreover, Hölder's inequality with exponents $x_1,y_1,z_1>1$ such that

$$x_1a_3\le r_1,\quad y_1a_4\le r_2,\quad z_1=r_1,\quad \frac{1}{x_1}+\frac{1}{y_1}+\frac{1}{z_1}=1$$

gives, by hypothesis (E3),

$$A_3\int_\Omega |u_n|^{a_3}|v_n|^{a_4}|u_n-u|\,dx\le A_3\|u_n\|_{a_3x_1}^{a_3}\|v_n\|_{a_4y_1}^{a_4}\|u_n-u\|_{r_1}\to 0.$$

Next we apply Hölder's inequality with exponents $r_1, r_1'$ and use (E4) and (E5) to get

$$\begin{aligned} A_4 \int\limits_\Omega |\nabla u_n|^{a_5}|u_n - u|\,dx &\le A_4 \left(\int\limits_\Omega |\nabla u_n|^{a_5 r_1'}\,dx\right)^{\frac{1}{r_1'}} \|u_n - u\|_{r_1} \\ &\le C_3 \left(1 + \|\nabla u_n\|_{p_1}^{\frac{p_1}{r_1'}}\right) \|u_n - u\|_{r_1} \to 0 \end{aligned}$$

and

$$\begin{aligned} A_5 \int\limits_\Omega |\nabla v_n|^{a_6}|u_n - u|\,dx &\le A_5 \left(\int\limits_\Omega |\nabla v_n|^{a_6 r_1'}\,dx\right)^{\frac{1}{r_1'}} \|u_n - u\|_{r_1} \\ &\le C_4 \left(1 + \|\nabla v_n\|_{p_2}^{\frac{p_2}{r_1'}}\right) \|u_n - u\|_{r_1} \to 0 \end{aligned}$$

for some $C_3, C_4 > 0$. Furthermore, condition (E6) allows us to apply Hölder's inequality with exponents $x_2, y_2, z_2 > 1$ such that

$$x_2 a_7 \le p_1, \quad y_2 a_8 \le p_2, \quad z_2 = r_1, \quad \frac{1}{x_2} + \frac{1}{y_2} + \frac{1}{z_2} = 1$$

in order to have

$$A_6 \int\limits_\Omega |\nabla u_n|^{a_7}|\nabla v_n|^{a_8}(u_n - u)\,dx \le A_6 \|\nabla u_n\|_{a_7 x_2}^{a_7} \|\nabla v_n\|_{a_8 y_2}^{a_8} \|u_n - u\|_{r_1} \to 0,$$

since both $\|\nabla u_n\|_{a_7 x_2}$ and $\|\nabla v_n\|_{a_8 y_2}$ are bounded. Finally, for the last term in (3.9) we have

$$\int\limits_\Omega |\alpha_1(x)|(u_n - u)\,dx \le \|\alpha_1\|_{r_1'} \|u_n - u\|_{r_1} \to 0.$$

Combining all the calculations above give

$$\lim_{n\to\infty} \int\limits_\Omega f_1\left(x, u_n, v_n, \nabla u_n, \nabla v_n\right)(u_n - u)\,dx = 0.$$

Applying similar arguments proves that

$$\lim_{n\to\infty} \int\limits_\Omega f_2\left(x, u_n, v_n, \nabla u_n, \nabla v_n\right)(v_n - v)\,dx = 0.$$

Hence, (3.8) is fulfilled. We now take the weak formulation (3.3), replace $u$ by $u_n$, $v$ by $v_n$, $\varphi$ by $u_n - u$ and $\psi$ by $v_n - v$ and use (3.6) as well as (3.8) in order to have

$$\limsup_{n\to\infty} \langle A(u_n, v_n), (u_n - u, v_n - v)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} = \limsup_{n\to\infty} \langle \mathcal{A}(u_n, v_n), (u_n - u, v_n - v)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} \le 0. \tag{3.10}$$

Since $A$ satisfies the $(S_+)$-property, see Lemma 2.4, we derive from (3.5) and (3.10) that

$$(u_n, v_n) \to (u, v) \quad \text{in } W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega).$$

Since $\mathcal{A}$ is continuous we have $\mathcal{A}(u_n, v_n) \to \mathcal{A}(u, v)$ in $(W_0^{1,\mathcal{H}_1}(\Omega))^* \times (W_0^{1,\mathcal{H}_2}(\Omega))^*$, which proves that $\mathcal{A}$ is pseudomonotone.

**Claim 3:** $\mathcal{A}$ is coercive.

First of all, taking into account the representation (2.6) and replacing $r$ by $p_1$ and $p_2$, respectively, we have

$$\|u\|_{p_1}^{p_1} \leq \lambda_{1,p_1}^{-1} \|\nabla u\|_{p_1}^{p_1} \quad \text{and} \quad \|v\|_{p_2}^{p_2} \leq \lambda_{1,p_2}^{-1} \|\nabla v\|_{p_2}^{p_2}, \tag{3.11}$$

for all $(u, v) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$. Note that $W_0^{1,\mathcal{H}_1}(\Omega) \subseteq W_0^{1,p_1}(\Omega)$ and $W_0^{1,\mathcal{H}_2}(\Omega) \subseteq W_0^{1,p_2}(\Omega)$. Applying these facts along with (3.11), (3.1), and (2.4) leads to

$$\begin{aligned}
\langle \mathcal{A}(u, v), (u, v)\rangle_{\mathcal{H}_1\times\mathcal{H}_2} &= \int_\Omega \left(|\nabla u|^{p_1-2}\nabla u + \mu_1(x)|\nabla u|^{q_1-2}\nabla u\right) \cdot \nabla u \, dx \\
&\quad + \int_\Omega \left(|\nabla v|^{p_2-2}\nabla v + \mu_2(x)|\nabla v|^{q_2-2}\nabla v\right) \cdot \nabla v \, dx \\
&\quad - \int_\Omega f_1(x, u, v, \nabla u, \nabla v) u \, dx - \int_\Omega f_2(x, u, v, \nabla u, \nabla v) v \, dx \\
&\geq \|\nabla u\|_{p_1}^{p_1} + \|\nabla u\|_{q_1,\mu_1}^{q_1} + \|\nabla v\|_{p_2}^{p_2} + \|\nabla v\|_{q_2,\mu_2}^{q_2} \\
&\quad - \Lambda \left(\|\nabla u\|_{p_1}^{p_1} + \|\nabla v\|_{p_2}^{p_2}\right) - \Gamma \left(\|u\|_{p_1}^{p_1} + \|v\|_{p_2}^{p_2}\right) - \|\omega\|_1 \\
&\geq \left(1 - \Lambda - \Gamma \lambda_{1,p_1}^{-1}\right) \|\nabla u\|_{p_1}^{p_1} + \|\nabla u\|_{q_1,\mu_1}^{q_1} \\
&\quad + \left(1 - \Lambda - \Gamma \lambda_{1,p_2}^{-1}\right) \|\nabla v\|_{p_2}^{p_2} + \|\nabla v\|_{q_2,\mu_2}^{q_2} - \|\omega\|_1 \\
&\geq \left(1 - \Lambda - \Gamma \max\left\{\lambda_{1,p_1}^{-1}, \lambda_{1,p_2}^{-1}\right\}\right) \left(\min\left\{\|u\|_{1,\mathcal{H}_1,0}^{p_1}, \|u\|_{1,\mathcal{H}_1,0}^{q_1}\right\}\right. \\
&\quad \left. + \min\left\{\|v\|_{1,\mathcal{H}_2,0}^{p_2}, \|v\|_{1,\mathcal{H}_2,0}^{q_2}\right\}\right) - \|\omega\|_{L^1(\Omega)}.
\end{aligned}$$

Since $1 < p_i < q_i$ and condition (3.2) holds, it follows that (2.7) is satisfied, and hence $\mathcal{A}$ is coercive.

From the Claims 1–3 we see that $\mathcal{A}$ is bounded, pseudomonotone and coercive. Therefore, by Theorem 2.3, there exists $(u, v) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ such that $\mathcal{A}(u, v) = 0$. Taking into account the definition of $\mathcal{A}$, see equation (3.4), it follows that $(u, v)$ is a weak solution of problem (1.1). That finishes the proof. $\square$

## 4. A uniqueness result

Now we consider the uniqueness of solutions of (1.1). To this end, let $\mathbf{f}\colon \Omega \times \mathbb{R}^2 \times (\mathbb{R}^N)^2 \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{f}(x, s, \xi) = (f_1(x, s, \xi), f_2(x, s, \xi))$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}^2$ and for all $\xi \in (\mathbb{R}^N)^2$. We suppose the following conditions on $f$:

(U1) There exists $c_1 \geq 0$ such that

$$(\mathbf{f}(x, s, \xi) - \mathbf{f}(x, t, \xi)) \cdot (s - t) \leq c_1 |s - t|^2$$

for a.a. $x \in \Omega$, for all $s, t \in \mathbb{R}^2$ and for all $\xi \in (\mathbb{R}^N)^2$.

(U2) There exist $\rho = (\rho_1, \rho_2)$ with $\rho_i \in L^{s_i}(\Omega)$, $1 < s_i < p_i^*$ and $c_2 \geq 0$ such that $\mathbf{f}(x, s, \cdot) - \rho(x)$ is linear on $(\mathbb{R}^N)^2$ for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}^2$ and

$$|\mathbf{f}(x, s, \xi) - \rho(x)| \leq c_2 |\xi|$$

for a.a. $x \in \Omega$, for all $s \in \mathbb{R}^2$ and for all $\xi \in (\mathbb{R}^N)^2$.

Our main result in this section reads as follows.

**Theorem 4.1.** *Let (2.1), (H), (U1), and (U2) be satisfied. If $2 = p_i < q_i < N$ for $i = 1, 2$ and*

$$c_1 \lambda_{1,2}^{-1} + c_2 \left(2\lambda_{1,2}^{-1}\right)^{\frac{1}{2}} < 1, \tag{4.1}$$

*then there exists a unique weak solution of problem (1.1).*

**Proof.** Let $u = (u_1, u_2), v = (v_1, v_2) \in W_0^{1,\mathcal{H}_1}(\Omega) \times W_0^{1,\mathcal{H}_2}(\Omega)$ be two weak solutions of (1.1). Considering the weak formulation for $u$ and $v$, choosing $\varphi = u_1 - v_1$ as well as $\psi = u_2 - v_2$ and subtracting the related equations gives

$$\begin{aligned}
&\int_\Omega |\nabla(u_1 - v_1)|^2 \, dx + \int_\Omega |\nabla(u_2 - v_2)|^2 \, dx \\
&\quad + \int_\Omega \mu_1(x) \left(|\nabla u_1|^{q_1-2} \nabla u_1 - |\nabla v_1|^{q_1-2} \nabla v_1\right) \cdot \nabla(u_1 - v_1) \, dx \\
&\quad + \int_\Omega \mu_2(x) \left(|\nabla u_2|^{q_2-2} \nabla u_2 - |\nabla v_2|^{q_2-2} \nabla v_2\right) \cdot \nabla(u_2 - v_2) \, dx \\
&= \int_\Omega (\mathbf{f}(x, u, \nabla u) - \mathbf{f}(x, v, \nabla u)) \cdot (u - v) \, dx \\
&\quad + \int_\Omega (\mathbf{f}(x, v, \nabla u) - \rho(x) - \mathbf{f}(x, v, \nabla v) + \rho(x)) \cdot (u - v) \, dx.
\end{aligned} \tag{4.2}$$

By the monotonicity of $\xi \mapsto |\xi|^{q_i-2}\xi$ we see that the third and the fourth integral on the left-hand side of (4.2) are nonnegative, that is,

$$\begin{aligned}
&\int_\Omega |\nabla(u_1 - v_1)|^2 \, dx + \int_\Omega |\nabla(u_2 - v_2)|^2 \, dx \\
&\quad + \int_\Omega \mu_1(x) \left(|\nabla u_1|^{q_1-2} \nabla u_1 - |\nabla v_1|^{q_1-2} \nabla v_1\right) \cdot \nabla(u_1 - v_1) \, dx \\
&\quad + \int_\Omega \mu_2(x) \left(|\nabla u_2|^{q_2-2} \nabla u_2 - |\nabla v_2|^{q_2-2} \nabla v_2\right) \cdot \nabla(u_2 - v_2) \, dx \\
&\geq \int_\Omega |\nabla(u_1 - v_1)|^2 \, dx + \int_\Omega |\nabla(u_2 - v_2)|^2 \, dx \\
&= \|\nabla(u_1 - v_1)\|_2^2 + \|\nabla(u_2 - v_2)\|_2^2.
\end{aligned} \tag{4.3}$$

On the other side, by applying (U1) to the first integral on the right-hand side of (4.2) and (U2) to the second we obtain along with Hölder's inequality

$$
\begin{aligned}
&\int_\Omega (\mathbf{f}(x,u,\nabla u) - \mathbf{f}(x,v,\nabla u)) \cdot (u-v)\,dx \\
&\quad + \int_\Omega (\mathbf{f}(x,v,\nabla u) - \rho(x) - \mathbf{f}(x,v,\nabla v) + \rho(x)) \cdot (u-v)\,dx \\
&\le c_1 \left(\|u_1 - v_1\|_2^2 + \|u_2 - v_2\|_2^2\right) \\
&\quad + \int_\Omega (f_1(x,v_1,v_2,(u_1-v_1)\nabla(u_1-v_1),(u_1-v_1)\nabla(u_2-v_2)) - \rho_1(x))\,dx \\
&\quad + \int_\Omega (f_2(x,v_1,v_2,(u_2-v_2)\nabla(u_1-v_1),(u_2-v_2)\nabla(u_2-v_2)) - \rho_2(x))\,dx \\
&\le c_1 \lambda_{1,2}^{-1} \left(\|\nabla(u_1-v_1)\|_2^2 + \|\nabla(u_2-v_2)\|_2^2\right) \\
&\quad + c_2 \int_\Omega (|u_1-v_1| + |u_2-v_2|) \left(|\nabla(u_1-v_2)|^2 + |\nabla(u_2-v_2)|^2\right)^{\frac{1}{2}}\,dx \\
&\le \left(c_1\lambda_{1,2}^{-1} + c_2\left(2\lambda_{1,2}^{-1}\right)^{\frac{1}{2}}\right)\left(\|\nabla(u_1-v_1)\|_2^2 + \|\nabla(u_2-v_2)\|_2^2\right).
\end{aligned}
\tag{4.4}
$$

Combining (4.2), (4.3) and (4.4) gives

$$
\begin{aligned}
&\|\nabla(u_1-v_1)\|_2^2 + \|\nabla(u_2-v_2)\|_2^2 \\
&\le \left(c_1\lambda_{1,2}^{-1} + c_2\left(2\lambda_{1,2}^{-1}\right)^{\frac{1}{2}}\right)\left(\|\nabla(u_1-v_1)\|_2^2 + \|\nabla(u_2-v_2)\|_2^2\right).
\end{aligned}
\tag{4.5}
$$

Taking (4.1) into account, we see from (4.5) that $u_1 = v_1$ and $u_2 = v_2$ and so the solution of (1.1) is unique. □

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