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Global a priori bounds for weak solutions of quasilinear elliptic systems with nonlinear boundary condition

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ABSTRACT

In this paper we study quasilinear elliptic systems with nonlinear boundary condition with fully coupled perturbations even on the boundary. Under very general assumptions our main result says that each weak solution of such systems belongs to $L^\infty(\overline{\Omega}) \times L^\infty(\overline{\Omega})$. The proof is based on Moser's iteration scheme. The results presented here can also be applied to elliptic systems with homogeneous Dirichlet boundary condition.

1. Introduction

In this paper we study the boundedness of weak solutions of the following quasilinear elliptic system

$$\begin{aligned}
-\operatorname{div} \mathcal{A}_1(x,u,\nabla u) &= \mathcal{B}_1(x,u,v,\nabla u,\nabla v) && \text{in } \Omega,\\
-\operatorname{div} \mathcal{A}_2(x,v,\nabla v) &= \mathcal{B}_2(x,u,v,\nabla u,\nabla v) && \text{in } \Omega,\\
\mathcal{A}_1(x,u,\nabla u)\cdot \nu &= \mathcal{C}_1(x,u,v) && \text{on } \partial\Omega,\\
\mathcal{A}_2(x,v,\nabla v)\cdot \nu &= \mathcal{C}_2(x,u,v) && \text{on } \partial\Omega,
\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ with $N > 1$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$ and the functions $\mathcal{A}_i\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}_i\colon \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$, and $\mathcal{C}_i\colon \partial\Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $i=1,2$, satisfy suitable (p,q)-structure conditions with $1 < p,q < \infty$.

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The main goal of this paper is to prove the existence of a priori bounds for weak solutions of problem (1.1) under very general conditions on the data. Indeed, the novelties of our work can be stated as follows:

(i) Problem (1.1) is fully coupled even with the gradient of the solutions and with a nonlinear boundary condition.
(ii) Critical growth is allowed even on the boundary.

The proof of our result uses a modified version of Moser's iteration technique whose arguments are essentially based on the monographs of Drábek-Kufner-Nicolosi [9] and Struwe [32]. We extend with our work recent results of the authors [19] from the case of a single equation to a system which is a difficult task to undertake. To the best of our knowledge, a priori bounds for problem (1.1) under such weak conditions have not been published before and so our results extend several works in this direction.

Let us comment on some relevant references concerning a priori bounds for elliptic systems. In 1992, Clément-de Figueiredo-Mitidieri [5] studied the semilinear elliptic system

$$\begin{cases} -\Delta u = f(v) & \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \\ -\Delta v = g(u) & \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where f, g are smooth functions such that $\alpha, \beta \in (0, \infty)$ exist with

$$\lim_{s \to 0} \frac{f(s)}{s^p} = \alpha \quad \text{and} \quad \lim_{s \to \infty} \frac{g(s)}{s^q} = \beta,$$

where $1 \leq p, q < \infty$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N} \quad \text{if } N \geq 3. \tag{1.3}$$

Condition (1.3) is the crucial assumption in their proof of a priori bounds for weak solutions of (1.2) and it can be shown that this condition is optimal. The proof uses the methods applied in the paper of de Figueiredo-Lions-Nussbaum [11] in which condition (1.3) first appeared. Since both papers deal not only with a priori bounds but also with the existence of positive solutions, it is worth mentioning the pioneer work of Lions in [16] concerning the existence of positive solutions for semilinear elliptic equations. An extension of [5] was done by the same authors in [6] to problems of the form

$$\begin{cases} -\Delta u = f(x, u, v, Du, Dv) & \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \\ -\Delta v = g(x, u, v, Du, Dv) & \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where a priori L^∞-estimates are established for positive solutions of (1.4) via a method which combines Hardy-Sobolev-type inequalities and interpolation. In de Figueiredo-Yang [12] a priori bounds for solutions of (1.4) (without the gradient dependence on f and g) are obtained via the so-called blow up method and the results are much more general than those in [6].

In 2004, a new method for a priori estimates for solutions of semilinear elliptic systems of the form

$$\begin{cases} -\Delta u = f(x, u, v) & \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \\ -\Delta v = g(x, u, v) & \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

was presented by Quittner-Souplet [29] which is based on a bootstrap argument. In addition, we refer to this work because it gives an overview about the different techniques concerning a priori estimates, see the Introduction of [29] and also the references. Concerning a priori estimates for very weak solutions with power nonlinearities we mention the work of Quittner [28].

A priori bounds and existence of positive solutions for strongly coupled p-Laplace systems have been established by Zou [37] for systems given by

$$\begin{cases} -\Delta_m u + u^a v^b = 0 & \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \\ -\Delta_m v + u^c v^d = 0 & \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Delta_m u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$ denotes the m-Laplacian.

In 2010, Bartsch-Dancer-Wang [3] studied the local and global bifurcation structure of positive solutions of the system

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta v^2 u & \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

of nonlinear Schrödinger type equations. They developed a new Liouville type theorem for nonlinear elliptic systems which provides a priori bounds for solution branches of (1.5). Singular quasilinear elliptic systems in $\mathbb{R}^N$ have been recently studied by Marano-Marino-Moussaoui [17] for (p_1, p_2)-Laplace systems given by

$$\begin{cases} -\Delta_{p_1} u = a_1(x) f(u, v) & \text{in } \mathbb{R}^N, \\ -\Delta_{p_2} v = a_2(x) g(u, v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

where a version of Moser's iterations is applied in order to obtain L^∞-bounds for solutions of (1.6), see also Marino [18].

Finally, we refer to other works which are related to a priori bounds and existence of weak solutions of elliptic systems of type (1.1), see, for example, Angenent-Van der Vorst [1], Bahri-Lions [2], Choi [4], Damascelli-Pardo [8], D'Ambrosio-Mitidieri [7], Ghergu-Rădulescu [10], Hai [13], Kelemen-Quittner [14], Kosírová-Quittner [15], Mavinga-Pardo [20], Mingione [21], Mitidieri [22], Motreanu [23], Motreanu-Moussaoui [24], [25], Papageorgiou-Rădulescu-Repovš [26], Peletier-Van der Vorst [27], Ramos [30], Souto [31], Troy [33], Zhang [35], Zhou-Zhang-Liu [36], Zou [38] and the references therein.

The paper is organized as follows. In Section 2 we state the main preliminaries which will be used in the paper. Section 3 contains the main results of our work. First, we prove that any weak solution of (1.1) belongs to $L^r(\overline{\Omega}) \times L^r(\overline{\Omega})$ for any finite r, see Theorem 3.1 and then, in the second part, we are able to show that each weak solution of (1.1) is essentially bounded, that is, it belongs to $L^\infty(\overline{\Omega}) \times L^\infty(\overline{\Omega})$, see Theorem 3.2. Furthermore, we will mention that our results can also be applied to problems with homogeneous Dirichlet condition, see Theorem 3.4.

2. Preliminaries

Throughout the paper we denote by $|\cdot|$ the norm of $\mathbb{R}^N$ and $\cdot$ stands for the inner product in $\mathbb{R}^N$. For $r \in [1, \infty)$ we denote by $L^r(\Omega), L^r(\Omega; \mathbb{R}^N)$ and $W^{1,r}(\Omega)$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_r$ and $\|\cdot\|_{1,r}$ given by

$$\|u\|_r = \left(\int_\Omega |u|^r \, dx \right)^{\frac{1}{r}}, \quad \|\nabla u\|_r = \left(\int_\Omega |\nabla u|^r \, dx \right)^{\frac{1}{r}},$$

$$\|u\|_{1,r} = \left(\int_\Omega |\nabla u|^r \, dx \right)^{\frac{1}{r}} + \left(\int_\Omega |u|^r \, dx \right)^{\frac{1}{r}}.$$

For $r = \infty$, the norm of $L^\infty(\Omega)$ is given by

$$\|u\|_\infty = \operatorname*{esssup}_\Omega |u|.$$

By σ we denote the $(N-1)$-dimensional Hausdorff (surface) measure and $L^s(\partial\Omega)$, $1 \le s \le \infty$, stands for the Lebesgue space on the boundary with the norms

$$\|u\|_{s,\partial\Omega} = \left(\int_{\partial\Omega} |u|^s \, d\sigma \right)^{\frac{1}{s}} \quad (1 \le s < \infty), \qquad \|u\|_{\infty,\partial\Omega} = \operatorname*{esssup}_{\partial\Omega} |u|.$$

It is well known that the linear trace mapping $\gamma \colon W^{1,r}(\Omega) \to L^{r_2}(\partial\Omega)$ is compact for every $r_2 \in [1, r_*)$ and continuous for $r_2 = r_*$, where r_* is the critical exponent of r on the boundary given by

$$r_* = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N, \\ \text{any } m \in (1, \infty) & \text{if } r \ge N. \end{cases} \tag{2.1}$$

For simplification we will drop the usage of γ. Moreover, by the Sobolev embedding theorem, we know that there exists a linear map $i \colon W^{1,r}(\Omega) \to L^{r_1}(\Omega)$ which is compact for every $r_1 \in [1, r^*)$ and continuous for $r_1 = r^*$ where the critical exponent is given by

$$r^* = \begin{cases} \frac{Nr}{N-r} & \text{if } r < N, \\ \text{any } m \in (1, \infty) & \text{if } r \ge N. \end{cases} \tag{2.2}$$

For $a \in \mathbb{R}$, we set $a^\pm := \max\{\pm a, 0\}$ and for $u \in W^{1,r}(\Omega)$ we define $u^\pm(\cdot) := u(\cdot)^\pm$. It is clear that

$$u^\pm \in W^{1,r}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-. \tag{2.3}$$

Moreover, $|\cdot|$ stands for the Lebesgue measure on $\mathbb{R}^N$ and also for the Hausdorff surface measure and it will be clear from the context which one is used. If $s > 1$, then $s' := \frac{s}{s-1}$ denotes its conjugate.

The following propositions are needed in the proofs of our main results.

Proposition 2.1. *([34, Proposition 2.1]) Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$, let $1 < p < \infty$, and let $\hat{q}$ be such that $p \le \hat{q} < p_*$ with the critical exponent stated in (2.1) with $r = p$. Then, for every $\varepsilon > 0$, there exist constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$ such that*

$$\|u\|_{\hat{q},\partial\Omega}^p \le \varepsilon \|u\|_{1,p}^p + \tilde{c}_1 \varepsilon^{-\tilde{c}_2} \|u\|_p^p \qquad \textit{for all } u \in W^{1,p}(\Omega).$$

Proposition 2.2. *([19, Proposition 2.2]) Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $u \in L^p(\Omega)$ with $u \ge 0$ and $1 < p < \infty$ such that*

$$\|u\|_{\alpha_n} \le C$$

with a constant $C > 0$ and a sequence $(\alpha_n) \subseteq \mathbb{R}_+$ with $\alpha_n \to \infty$ as $n \to \infty$. Then, $u \in L^\infty(\Omega)$.

Proposition 2.3. *([19, Proposition 2.4]) Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $1 < p < \infty$. If $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $u \in L^\infty(\partial\Omega)$.*

In the following we will use the abbreviation

$$L^\infty(\overline{\Omega}) := L^\infty(\Omega) \cap L^\infty(\partial\Omega).$$

3. Main results

We now give the structure conditions on the nonlinearities in problem (1.1).

(H) The functions $\mathcal{A}_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{C}_i : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are Carathéodory functions such that the following holds:

(H1) $|\mathcal{A}_1(x, s, \xi)| \leq A_1|\xi|^{p-1} + A_2|s|^{r_1 \frac{p-1}{p}} + A_3$,

(H2) $|\mathcal{A}_2(x, t, \zeta)| \leq \tilde{A}_1|\zeta|^{q-1} + \tilde{A}_2|t|^{r_2 \frac{q-1}{q}} + \tilde{A}_3$,

(H3) $\mathcal{A}_1(x, s, \xi) \cdot \xi \geq A_4|\xi|^p - A_5|s|^{r_1} - A_6$,

(H4) $\mathcal{A}_2(x, t, \zeta) \cdot \zeta \geq \tilde{A}_4|\zeta|^q - \tilde{A}_5|t|^{r_2} - \tilde{A}_6$,

(H5) $|\mathcal{B}_1(x, s, t, \xi, \zeta)| \leq B_1|s|^{b_1} + B_2|t|^{b_2} + B_3|s|^{b_3}|t|^{b_4} + B_4|\xi|^{b_5} + B_5|\zeta|^{b_6} + B_6|\xi|^{b_7}|\zeta|^{b_8} + B_7$,

(H6) $|\mathcal{B}_2(x, s, t, \xi, \zeta)| \leq \tilde{B}_1|s|^{\tilde{b}_1} + \tilde{B}_2|t|^{\tilde{b}_2} + \tilde{B}_3|s|^{\tilde{b}_3}|t|^{\tilde{b}_4} + \tilde{B}_4|\xi|^{\tilde{b}_5} + \tilde{B}_5|\zeta|^{\tilde{b}_6} + \tilde{B}_6|\xi|^{\tilde{b}_7}|\zeta|^{\tilde{b}_8} + \tilde{B}_7$,

(H7) $|\mathcal{C}_1(x, s, t)| \leq C_1|s|^{c_1} + C_2|t|^{c_2} + C_3|s|^{c_3}|t|^{c_4} + C_4$,

(H8) $|\mathcal{C}_2(x, s, t)| \leq \tilde{C}_1|s|^{\tilde{c}_1} + \tilde{C}_2|t|^{\tilde{c}_2} + \tilde{C}_3|s|^{\tilde{c}_3}|t|^{\tilde{c}_4} + \tilde{C}_4$,

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, for all $s, t \in \mathbb{R}$, for all $\xi, \zeta \in \mathbb{R}^N$, with nonnegative constants $A_i, \tilde{A}_i, B_j, \tilde{B}_j, C_k, \tilde{C}_k$ $(i \in \{1, \dots, 6\},\ j \in \{1, \dots, 7\},\ k \in \{1, \dots, 4\})$ and with $1 < p, q < \infty$. Moreover, the exponents $b_i, \tilde{b}_i, c_j, \tilde{c}_j, r_1, r_2$ $(i \in \{1, \dots, 8\}, j \in \{1, \dots, 4\})$ are nonnegative and satisfy the following assumptions

(E1) $r_1 \leq p^*$	(E2) $r_2 \leq q^*$	
(E3) $b_1 \leq p^* - 1$	(E4) $b_2 < \dfrac{q^*}{p^*}(p^* - p)$	(E5) $\dfrac{b_3}{p^*} + \dfrac{b_4}{q^*} < \dfrac{p^* - p}{p^*}$
(E6) $b_5 \leq p - 1$	(E7) $b_6 < \dfrac{q}{p^*}(p^* - p)$	(E8) $\dfrac{b_7}{p} + \dfrac{b_8}{q} < \dfrac{p^* - p}{p^*}$
(E9) $\tilde{b}_1 < \dfrac{p^*}{q^*}(q^* - q)$	(E10) $\tilde{b}_2 \leq q^* - 1$	(E11) $\dfrac{\tilde{b}_3}{p^*} + \dfrac{\tilde{b}_4}{q^*} < \dfrac{q^* - q}{q^*}$
(E12) $\tilde{b}_5 < \dfrac{p}{q^*}(q^* - q)$	(E13) $\tilde{b}_6 \leq q - 1$	(E14) $\dfrac{\tilde{b}_7}{p} + \dfrac{\tilde{b}_8}{q} < \dfrac{q^* - q}{q^*}$
(E15) $c_1 \leq p_* - 1$	(E16) $c_2 < \dfrac{q_*}{p_*}(p_* - p)$	(E17) $\dfrac{c_3}{p_*} + \dfrac{c_4}{q_*} < \dfrac{p_* - p}{p_*}$
(E18) $\tilde{c}_1 < \dfrac{p_*}{q_*}(q_* - q)$	(E19) $\tilde{c}_6 \leq q_* - 1$	(E20) $\dfrac{\tilde{c}_3}{p_*} + \dfrac{\tilde{c}_4}{q_*} < \dfrac{q_* - q}{q_*}$,

where the numbers p^*, p_*, q^*, q_* are defined by (2.2) and (2.1).

A couple $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ is said to be a weak solution of problem (1.1) if

$$\begin{aligned} \int_\Omega \mathcal{A}_1(x,u,\nabla u)\cdot\nabla\varphi\,dx &= \int_\Omega \mathcal{B}_1(x,u,v,\nabla u,\nabla v)\varphi\,dx + \int_{\partial\Omega} \mathcal{C}_1(x,u,v)\varphi\,d\sigma \\ \int_\Omega \mathcal{A}_2(x,v,\nabla v)\cdot\nabla\psi\,dx &= \int_\Omega \mathcal{B}_2(x,u,v,\nabla u,\nabla v)\psi\,dx + \int_{\partial\Omega} \mathcal{C}_2(x,u,v)\psi\,d\sigma \end{aligned} \tag{3.1}$$

holds for all $(\varphi,\psi) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$. By hypotheses (H) and the Sobolev embedding along with the continuity of the trace operator it is clear that this definition of a weak solution is well-defined. Indeed, if we estimate the integral concerning the function $\mathcal{B}_1 \colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ using condition (H5) we obtain several mixed terms. Let us consider, for example, the third term on the right-hand side of (H5). Applying Hölder's inequality we get

$$\begin{aligned} &B_3 \int_\Omega |u|^{b_3}|v|^{b_4}\varphi\,dx \\ &\leq B_3 \left(\int_\Omega |u|^{b_3 s_1}\,dx\right)^{\frac{1}{s_1}} \left(\int_\Omega |v|^{b_4 s_2}\,dx\right)^{\frac{1}{s_2}} \left(\int_\Omega |\varphi|^{s_3}\,dx\right)^{\frac{1}{s_3}}, \end{aligned} \tag{3.2}$$

where $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, $\varphi \in W^{1,p}(\Omega)$ and

$$\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 1.$$

Taking $s_3 = p^*$ and using $s_1 \leq \frac{p^*}{b_3}$ as well as $s_2 \leq \frac{q^*}{b_4}$ leads to

$$\frac{b_3}{p^*} + \frac{b_4}{q^*} \leq \frac{p^*-1}{p^*}. \tag{3.3}$$

This condition is necessary for the finiteness of the integrals of the right-hand side of (3.2), see also Remark 3.3. Since we need some stronger conditions in order to apply Moser's iteration, we suppose condition (E5) which implies (3.3). In the same way we can prove the finiteness of all integrals in the definition of (3.1).

Our first result shows that any weak solution of problem (1.1) belongs to the space $L^r(\overline{\Omega}) \times L^r(\overline{\Omega})$ for any finite r.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N, N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let hypotheses (H) be satisfied. Then, every weak solution $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ of problem (1.1) belongs to $L^r(\overline{\Omega}) \times L^r(\overline{\Omega})$ for every $r \in (1,\infty)$.*

Proof. Let $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ be a weak solution of (1.1) in the sense of (3.1). We only show that $u \in L^r(\overline{\Omega})$, the proof for v can be done in the same way. Moreover, taking (2.3) into account, without any loss of generality, we can assume that $u, v \geq 0$ (otherwise we prove the result for u^+, v^+ and u^-, v^-, respectively). Moreover, throughout the proof we will denote by M_i, $i = 1, 2, \dots,$ constants which may depend on some natural norms of u and v.

For every $h \geq 0$ we set $u_h := \min\{u, h\}$ and choose $\varphi = uu_h^{\kappa p} \in W^{1,p}(\Omega)$ for $\kappa > 0$ as test function in the first equation of (3.1). Since $\nabla \varphi = u_h^{\kappa p} \nabla u + \kappa p u u_h^{\kappa p - 1} \nabla u_h$ this results in

$$\begin{aligned} &\int_\Omega (\mathcal{A}_1(x,u,\nabla u) \cdot \nabla u) u_h^{\kappa p}\, dx + \kappa p \int_\Omega (\mathcal{A}_1(x,u,\nabla u)\cdot \nabla u_h) u u_h^{\kappa p-1}\, dx \\ &= \int_\Omega \mathcal{B}_1(x,u,v,\nabla u,\nabla v) u u_h^{\kappa p}\, dx + \int_{\partial\Omega} \mathcal{C}_1(x,u,v) u u_h^{\kappa p}\, d\sigma. \end{aligned} \tag{3.4}$$

Now we apply (H3) to the first term on the left-hand side of (3.4) which gives

$$\begin{aligned} &\int_\Omega (\mathcal{A}_1(x,u,\nabla u)\cdot \nabla u) u_h^{\kappa p}\, dx \\ &\geq \int_\Omega \left(A_4 |\nabla u|^p - A_5 u^{r_1} - A_6\right) u_h^{\kappa p}\, dx \\ &\geq A_4 \int_\Omega |\nabla u|^p u_h^{\kappa p}\, dx - (A_5 + A_6) \int_\Omega u^{p^*} u_h^{\kappa p}\, dx - (A_5 + A_6)|\Omega|. \end{aligned}$$

In the same way we use (H3) to the second term on the left-hand side. This shows

$$\begin{aligned} &\kappa p \int_\Omega (\mathcal{A}_1(x,u,\nabla u)\cdot \nabla u_h) u u_h^{\kappa p-1}\, dx \\ &= \kappa p \int_{\{x\in\Omega:\, u(x)\leq h\}} (\mathcal{A}_1(x,u,\nabla u)\cdot\nabla u) u_h^{\kappa p}\, dx \\ &\geq \kappa p \int_{\{x\in\Omega:\, u(x)\leq h\}} \left(A_4|\nabla u|^p - A_5 u^{r_1} - A_6\right) u_h^{\kappa p}\, dx \\ &\geq A_4 \kappa p \int_{\{x\in\Omega:\, u(x)\leq h\}} |\nabla u|^p u_h^{\kappa p}\, dx \\ &\quad - \kappa p (A_5 + A_6) \int_\Omega u^{p^*} u_h^{\kappa p}\, dx - \kappa p (A_5 + A_6)|\Omega|. \end{aligned}$$

Taking (H5) into account we get for the first term on the right-hand side of (3.4) the following estimate

$$\begin{aligned} &\int_\Omega \mathcal{B}_1(x,u,v,\nabla u,\nabla v) u u_h^{\kappa p}\, dx \\ &\leq \int_\Omega \big(B_1 u^{b_1} + B_2 v^{b_2} + B_3 u^{b_3} v^{b_4} + B_4 |\nabla u|^{b_5} \\ &\quad + B_5 |\nabla v|^{b_6} + B_6 |\nabla u|^{b_7} |\nabla v|^{b_8} + B_7\big) u u_h^{\kappa p}\, dx. \end{aligned} \tag{3.5}$$

We are going to estimate each term of the inequality above separately. First, taking into account assumption (E3), we have

$$B_1 \int_\Omega u^{b_1} u u_h^{\kappa p}\, dx \leq B_1 \int_\Omega u^{p^*} u_h^{\kappa p}\, dx + B_1 |\Omega|.$$

Moreover, thanks to Hölder's inequality with $s_1 > 1$ such that $b_2 s_1 = q^*$, which is possible by (E4), we have

$$\begin{aligned} B_2 \int_\Omega v^{b_2} u u_h^{\kappa p}\, dx &\le B_2 \left(\int_\Omega v^{b_2 s_1}\, dx \right)^{1/s_1} \left(\int_\Omega (u u_h^{\kappa p})^{s_1'}\, dx \right)^{1/s_1'} \\ &\le M_1 \left(1 + \|u u_h^\kappa\|_{p s_1'}^p \right). \end{aligned}$$

Applying again Hölder's inequality with exponents $x_1, y_1, z_1 > 1$ such that

$$b_3 x_1 = p^*, \qquad b_4 y_1 = q^*, \qquad \frac{1}{z_1} = 1 - \frac{1}{x_1} - \frac{1}{y_1} \tag{3.6}$$

leads to

$$\begin{aligned} &B_3 \int_\Omega u^{b_3} v^{b_4} u u_h^{\kappa p}\, dx \\ &\le B_3 \left(\int_\Omega u^{b_3 x_1}\, dx \right)^{1/x_1} \left(\int_\Omega v^{b_4 y_1}\, dx \right)^{1/y_1} \left(\int_\Omega (u u_h^{\kappa p})^{z_1}\, dx \right)^{1/z_1} \\ &\le M_2 \left(1 + \|u u_h^\kappa\|_{p z_1}^p \right). \end{aligned}$$

Note that from (E5) it follows that $b_3 < p^*$ as well as $b_4 < q^*$ and so the choice in (3.6) is possible. Thanks to Young's inequality with $\frac{p}{b_5} > 1$ we have

$$\begin{aligned} B_4 \int_\Omega |\nabla u|^{b_5} u u_h^{\kappa p}\, dx &= B_4 \int_\Omega \left(\left(\frac{A_4}{2B_4} \right)^{\frac{b_5}{p}} |\nabla u|^{b_5} u_h^{\kappa b_5} \right) \left(\left(\frac{A_4}{2B_4} \right)^{-\frac{b_5}{p}} u u_h^{\kappa(p-b_5)} \right) dx \\ &\le \frac{A_4}{2} \int_\Omega |\nabla u|^p u_h^{\kappa p}\, dx + B_4 \left(\frac{A_4}{2B_4} \right)^{-\frac{b_5}{p-b_5}} \int_\Omega u^{\frac{p}{p-b_5}} u_h^{\kappa p}\, dx \\ &\le \frac{A_4}{2} \int_\Omega |\nabla u|^p u_h^{\kappa p}\, dx + M_3 \left(1 + \int_\Omega u^{p^*} u_h^{\kappa p}\, dx \right). \end{aligned}$$

We apply Hölder's inequality with $s_2 > 1$ such that $b_6 s_2 = q$ in order to get

$$\begin{aligned} B_5 \int_\Omega |\nabla v|^{b_6} u u_h^{\kappa p}\, dx &\le B_5 \left(\int_\Omega |\nabla v|^{b_6 s_2}\, dx \right)^{1/s_2} \left(\int_\Omega (u u_h^{\kappa p})^{s_2'}\, dx \right)^{1/s_2'} \\ &\le M_4 \left(1 + \|u u_h^\kappa\|_{p s_2'}^p \right). \end{aligned}$$

As before, by Hölder's inequality with $x_2, y_2, z_2 > 1$ such that

$$b_7 x_2 = p, \qquad b_8 y_2 = q, \qquad \frac{1}{z_2} = 1 - \frac{1}{x_2} - \frac{1}{y_2} \tag{3.7}$$

we obtain

$$
\begin{aligned}
&B_6 \int_\Omega |\nabla u|^{b_7} |\nabla v|^{b_8} u u_h^{\kappa p}\, dx \\
&\leq B_6 \left(\int_\Omega |\nabla u|^{b_7 x_2}\, dx \right)^{1/x_2} \left(\int_\Omega |\nabla v|^{b_8 y_2}\, dx \right)^{1/y_2} \left(\int_\Omega (u u_h^\kappa)^{z_2}\, dx \right)^{1/z_2} \\
&\leq M_5 \left(1 + \|u u_h^\kappa\|_{p z_2}^p \right),
\end{aligned}
$$

which is possible because of (E8). Finally, for the last term on the right-hand side of (3.5) we have

$$
B_7 \int_\Omega u u_h^{\kappa p}\, dx \leq B_7 \int_\Omega u^{p^*} u_h^{\kappa p}\, dx + B_7 |\Omega|.
$$

Hypothesis (H7) gives the following estimate for the boundary term of (3.4)

$$
\int_{\partial\Omega} \mathcal{C}_1(x,u,v) u u_h^{\kappa p}\, d\sigma \leq \int_{\partial\Omega} \left(C_1 u^{c_1} + C_2 v^{c_2} + C_3 u^{c_3} v^{c_4} + C_4 \right) u u_h^{\kappa p}\, d\sigma. \tag{3.8}
$$

Exploiting the condition on c_1 in the first term of (3.8) and applying Hölder's inequality with $t_1 > 1$ such that $c_2 t_1 = q_*$ to the second one we have

$$
C_1 \int_{\partial\Omega} u^{c_1+1} u_h^{\kappa p}\, d\sigma \leq C_1 \int_{\partial\Omega} u^{p_*} u_h^{\kappa p}\, d\sigma + C_1 |\partial\Omega|
$$

and

$$
\begin{aligned}
C_2 \int_{\partial\Omega} v^{c_2} u u_h^{\kappa p}\, d\sigma &\leq C_2 \left(\int_{\partial\Omega} v^{c_2 t_1}\, d\sigma \right)^{1/t_1} \left(\int_{\partial\Omega} (u u_h^{\kappa p})^{t_1'}\, d\sigma \right)^{1/t_1'} \\
&\leq M_6 \left(1 + \|u u_h^\kappa\|_{p t_1', \partial\Omega}^p \right),
\end{aligned}
$$

respectively. For the third term of (3.8) we apply Hölder's inequality with exponents $x_3, y_3, z_3 > 1$ such that

$$
c_3 x_3 = p_*, \qquad c_4 y_3 = q_*, \qquad \frac{1}{z_3} = 1 - \frac{1}{x_3} - \frac{1}{y_3} \tag{3.9}
$$

in order to get

$$
\begin{aligned}
&C_3 \int_{\partial\Omega} u^{c_3} v^{c_4} u u_h^{\kappa p}\, d\sigma \\
&\leq C_3 \left(\int_{\partial\Omega} u^{c_3 x_3}\, d\sigma \right)^{1/x_3} \left(\int_{\partial\Omega} v^{c_4 y_3}\, d\sigma \right)^{1/y_3} \left(\int_{\partial\Omega} (u u_h^{\kappa p})^{z_3}\, d\sigma \right)^{1/z_3} \\
&\leq M_7 \left(1 + \|u u_h^\kappa\|_{p z_3, \partial\Omega}^p \right).
\end{aligned}
$$

Finally, for the last term of (3.8) we have

$$
C_4 \int_{\partial\Omega} u u_h^{\kappa p}\, d\sigma \leq C_4 \int_{\partial\Omega} u^{p_*} u_h^{\kappa p}\, d\sigma + C_4 |\partial\Omega|.
$$

Note that from the choice of s_1, s_2 and t_1 in combination with (E4), (E7) and (E16) we have

$$s_1', s_2' < \frac{p^*}{p} \quad \text{and} \quad t_1' < \frac{p_*}{p}.$$

Furthermore, by (3.6), (3.7), (3.9) and the conditions (E5), (E8) and (E17) we see that

$$z_1, z_2 < \frac{p^*}{p} \quad \text{and} \quad z_3 < \frac{p_*}{p}.$$

Now we combine all the calculations above and set

$$s := \max\{s_1', s_2', z_1, z_2\} \in \left(1, \frac{p^*}{p}\right) \tag{3.10}$$

as well as

$$t := \max\{t_1', z_3\} \in \left(1, \frac{p_*}{p}\right) \tag{3.11}$$

which finally gives

$$\begin{aligned}
&A_4\left(\frac{1}{2}\int_\Omega |\nabla u|^p u_h^{\kappa p}\,dx + \kappa p \int_{\{x\in\Omega:\, u(x)\le h\}} |\nabla u|^p u_h^{\kappa p}\,dx\right)\\
&\le [(\kappa p+1)(A_5+A_6)+B_1+B_7+M_3]\int_\Omega u^{p^*}u_h^{\kappa p}\,dx + (C_1+C_4)\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}\,d\sigma\\
&\quad + M_8\|uu_h^\kappa\|_{ps}^p + M_9\|uu_h^\kappa\|_{pt,\partial\Omega}^p + M_{10}(\kappa+1).
\end{aligned}$$

Simplifying the inequality above leads to

$$\begin{aligned}
&\frac{A_4}{2}\frac{\kappa p+1}{(\kappa+1)^p}\int_\Omega |\nabla uu_h^\kappa|^p\,dx\\
&\le M_{11}(\kappa p+1)\int_\Omega u^{p^*}u_h^{\kappa p}\,dx + M_{12}\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}\,d\sigma + M_8\|uu_h^\kappa\|_{ps}^p\\
&\quad + M_9\|uu_h^\kappa\|_{pt,\partial\Omega}^p + M_{10}(\kappa+1),
\end{aligned} \tag{3.12}$$

see Marino-Winkert [19, Inequality after (3.7)]. Dividing by $\frac{A_4}{2}$, summarizing the constants and adding on both sides of (3.12) the nonnegative term $\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_p^p$ gives

$$\begin{aligned}
&\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p\\
&\le \frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_p^p + M_{13}(\kappa p+1)\int_\Omega u^{p^*}u_h^{\kappa p}\,dx + M_{14}\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}\,d\sigma\\
&\quad + M_{15}\|uu_h^\kappa\|_{ps}^p + M_{16}\|uu_h^\kappa\|_{pt,\partial\Omega}^p + M_{17}(\kappa+1)\\
&\le M_{18}\left(\frac{\kappa p+1}{(\kappa+1)^p}+1\right)\|uu_h^\kappa\|_{ps}^p + M_{13}(\kappa p+1)\int_\Omega u^{p^*}u_h^{\kappa p}\,dx\\
&\quad + M_{14}\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}\,d\sigma + M_{16}\|uu_h^\kappa\|_{pt,\partial\Omega}^p + M_{17}(\kappa+1),
\end{aligned} \tag{3.13}$$

where we applied Hölder's inequality in the last passage.

Now, let $L, G > 0$ and set $a := u^{p^*-p}$ and $b := u^{p_*-p}$. By using Hölder's inequality and the continuous embeddings $i\colon W^{1,p}(\Omega) \to L^{p^*}(\Omega)$ and $\gamma\colon W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$ we obtain

$$\begin{aligned}
&\int_\Omega u^{p^*} u_h^{\kappa p}\,dx\\
&= \int_{\{x\in\Omega\colon a(x)\le L\}} a(uu_h^\kappa)^p\,dx + \int_{\{x\in\Omega\colon a(x)>L\}} a(uu_h^\kappa)^p\,dx\\
&\le L\int_\Omega (uu_h^\kappa)^p\,dx\\
&\quad + \left(\int_{\{x\in\Omega\colon a(x)>L\}} a^{\frac{p^*}{p^*-p}}\,dx\right)^{\frac{p^*-p}{p}} \left(\int_\Omega (uu_h^\kappa)^{p^*}\,dx\right)^{\frac{p}{p^*}}\\
&\le L|\Omega|^{1/s'}\|uu_h^\kappa\|_{ps}^p + \left(\int_{\{x\in\Omega\colon a(x)>L\}} a^{\frac{p^*}{p^*-p}}\,dx\right)^{\frac{p^*-p}{p^*}} c_\Omega^p\|uu_h^\kappa\|_{1,p}^p
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\int_{\partial\Omega} u^{p_*} u_h^{\kappa p}\,d\sigma\\
&= \int_{\{x\in\partial\Omega\colon b(x)\le G\}} b(uu_h^\kappa)^p\,d\sigma + \int_{\{x\in\partial\Omega\colon b(x)>G\}} b(uu_h^\kappa)^p\,d\sigma\\
&\le G\int_{\partial\Omega} (uu_h^\kappa)^p\,d\sigma\\
&\quad + \left(\int_{\{x\in\partial\Omega\colon b(x)>G\}} b^{\frac{p_*}{p_*-p}}\,d\sigma\right)^{\frac{p_*-p}{p_*}} \left(\int_{\partial\Omega} (uu_h^\kappa)^{p_*}\,d\sigma\right)^{\frac{p}{p_*}}\\
&\le G|\partial\Omega|^{1/t'}\|uu_h^\kappa\|_{pt,\partial\Omega} + \left(\int_{\{x\in\partial\Omega\colon b(x)>G\}} b^{\frac{p_*}{p_*-p}}\,d\sigma\right)^{\frac{p_*-p}{p_*}} c_{\partial\Omega}^p\|uu_h^\kappa\|_{1,p}^p
\end{aligned} \tag{3.15}$$

with the embedding constants c_Ω and $c_{\partial\Omega}$. We point out that

$$\begin{aligned}
H(L) &:= \left(\int_{\{x\in\Omega\colon a(x)>L\}} a^{\frac{p^*}{p^*-p}}\,dx\right)^{\frac{p^*-p}{p^*}} \to 0 \quad \text{as } L\to\infty,\\
K(G) &:= \left(\int_{\{x\in\partial\Omega\colon b(x)>G\}} b^{\frac{p_*}{p_*-p}}\,d\sigma\right)^{\frac{p_*-p}{p_*}} \to 0 \text{ as } G\to\infty.
\end{aligned} \tag{3.16}$$

Combining (3.13), (3.14), (3.15) and (3.16) yields

$$\begin{aligned}
&\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p\\
&\leq M_{19}\left(\frac{\kappa p+1}{(\kappa+1)^p}+1+(\kappa p+1)L|\Omega|^{1/s'}\right)\|uu_h^\kappa\|_{ps}^p\\
&\quad+M_{13}(\kappa p+1)H(L)c_\Omega^p\|uu_h^\kappa\|_{1,p}^p+(M_{16}+M_{14}G|\partial\Omega|^{1/t'})\|uu_h^\kappa\|_{pt,\partial\Omega}^p\\
&\quad+M_{14}K(G)c_{\partial\Omega}^p\|uu_h^\kappa\|_{1,p}^p+M_{17}(\kappa+1).
\end{aligned}\tag{3.17}$$

Taking (3.16) into account we choose $L=L(\kappa,u)>0$ and $G=G(\kappa,u)>0$ such that

$$M_{13}(\kappa p+1)H(L)c_\Omega^p=\frac{\kappa p+1}{4(\kappa+1)^p}\quad\text{and}\quad M_{14}K(G)c_{\partial\Omega}^p=\frac{\kappa p+1}{4(\kappa+1)^p}.$$

Therefore, inequality (3.17) can be written as

$$\begin{aligned}
&\frac{\kappa p+1}{2(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p\\
&\leq M_{19}\left(\frac{\kappa p+1}{(\kappa+1)^p}+1+(\kappa p+1)L(\kappa,u)|\Omega|^{1/s'}\right)\|uu_h^\kappa\|_{ps}^p\\
&\quad+(M_{16}+M_{14}G(\kappa,u)|\partial\Omega|^{1/t'})\|uu_h^\kappa\|_{pt,\partial\Omega}^p+M_{17}(\kappa+1).
\end{aligned}\tag{3.18}$$

Taking into account (3.11) we have $pt<p_*$. Thus, we can apply Proposition 2.1 to estimate the boundary term in (3.18). This gives

$$\begin{aligned}
\|uu_h^\kappa\|_{pt,\partial\Omega}^p&\leq\varepsilon_1\|uu_h^\kappa\|_{1,p}^p+\tilde{c}_1\varepsilon_1^{-\tilde{c}_2}\|uu_h^\kappa\|_p^p\\
&\leq\varepsilon_1\|uu_h^\kappa\|_{1,p}^p+\tilde{c}_1\varepsilon_1^{-\tilde{c}_2}|\Omega|^{1/s'}\|uu_h^\kappa\|_{ps}^p
\end{aligned}\tag{3.19}$$

by Hölder's inequality. Now we choose ε_1 such that

$$\varepsilon_1\left(M_{16}+M_{14}G(\kappa,u)|\partial\Omega|^{1/t'}\right)=\frac{\kappa p+1}{4(\kappa+1)^p}.$$

Applying (3.19) to (3.18) and summarizing the constants results in

$$\|uu_h^\kappa\|_{1,p}^p\leq M_{20}(\kappa,u,v)[\|uu_h^\kappa\|_{ps}^p+1]\tag{3.20}$$

with a constant $M_{20}(\kappa,u,v)$ depending on κ and on the solution pair (u,v), see the calculations above.

Now we are in the position to use the Sobolev embedding theorem on the left-hand side of (3.20). We have

$$\|uu_h^\kappa\|_{p^*}\leq c_\Omega\|uu_h^\kappa\|_{1,p}\leq M_{21}(\kappa,u,v)\left[\|uu_h^\kappa\|_{ps}^p+1\right]^{\frac{1}{p}}.\tag{3.21}$$

Since, due to (3.10), $ps<p^*$, we can start with the bootstrap arguments. Choosing κ_1 such that $(\kappa_1+1)ps=p^*$, (3.21) becomes

$$\begin{aligned}
\|uu_h^{\kappa_1}\|_{p^*}&\leq M_{21}(\kappa_1,u,v)\left[\|uu_h^{\kappa_1}\|_{ps}^p+1\right]^{\frac{1}{p}}\\
&\leq M_{21}(\kappa_1,u,v)\left[\|u^{\kappa_1+1}\|_{ps}^p+1\right]^{\frac{1}{p}}\\
&=M_{21}(\kappa_1,u,v)\left[\|u\|_{p^*}^{(\kappa_1+1)p}+1\right]^{\frac{1}{p}}<\infty,
\end{aligned}\tag{3.22}$$

where we have used the estimate $u_h(x) \leq u(x)$ for a.e. $x \in \Omega$. The usage of Fatou's Lemma as $h \to \infty$ in (3.22) gives

$$\|u\|_{(\kappa_1+1)p^*} = \|u^{\kappa_1+1}\|_{p^*}^{\frac{1}{\kappa_1+1}} \leq M_{22}(\kappa_1, u, v) \left[\|u\|_{p^*}^{(\kappa_1+1)p} + 1\right]^{\frac{1}{(\kappa_1+1)p}} < \infty. \tag{3.23}$$

Hence, $u \in L^{(\kappa_1+1)p^*}(\Omega)$. Repeating the steps from (3.21)-(3.23) for each κ, we choose a sequence with the following properties

$$\begin{aligned} \kappa_2 &: (\kappa_2+1)ps = (\kappa_1+1)p^*, \\ \kappa_3 &: (\kappa_3+1)ps = (\kappa_2+1)p^*, \\ &\vdots \qquad\qquad \vdots \quad . \end{aligned}$$

Observe that the sequence (κ_n) is constructed in such a way that $\kappa_n + 1 = (\frac{p^*}{ps})^n$ for every $n \in \mathbb{N}$, with $\frac{p^*}{ps} > 1$, taking into account (3.10). This implies that

$$\|u\|_{(\kappa+1)p^*} \leq M_{23}(\kappa, u, v) \tag{3.24}$$

for any finite $\kappa > 0$ with $M_{23}(\kappa, u, v)$ being a positive constant which depends both on κ and on the solution pair (u, v) itself. Therefore, $u \in L^r(\Omega)$ for any $r < \infty$.

Now we are going to prove that $u \in L^r(\partial\Omega)$ for any finite r. To this end, let us consider again inequality (3.18), that is,

$$\begin{aligned} &\frac{\kappa p+1}{2(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p \\ &\leq M_{19}\left(\frac{\kappa p+1}{(\kappa+1)^p} + 1 + (\kappa p+1)L(\kappa, u)|\Omega|^{1/s'}\right)\|uu_h^\kappa\|_{ps}^p \\ &\quad + (M_{16} + M_{14}G(\kappa, u)|\partial\Omega|^{1/t'})\|uu_h^\kappa\|_{pt,\partial\Omega}^p + M_{17}(\kappa+1). \end{aligned} \tag{3.25}$$

Exploiting (3.24), inequality (3.25) can be written in the simple form

$$\|uu_h^\kappa\|_{1,p} \leq M_{24}(\kappa, u, v)\left[\|uu_h^\kappa\|_{pt,\partial\Omega}^p + 1\right]^{\frac{1}{p}}. \tag{3.26}$$

Applying the embedding $\gamma\colon W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$ to the right-hand side of (3.26) gives

$$\|uu_h^\kappa\|_{p_*,\partial\Omega} \leq c_{\partial\Omega}\|uu_h^\kappa\|_{1,p} \leq M_{25}(\kappa, u, v)\left[\|uu_h^\kappa\|_{pt,\partial\Omega}^p + 1\right]^{\frac{1}{p}}.$$

Since $pt < p_*$, we can proceed as before with a bootstrap argument, thus obtaining

$$\|u\|_{(\kappa+1)p_*,\partial\Omega} \leq M_{26}(\kappa, u, v)$$

for any finite number κ with $M_{26}(\kappa, u, v)$ being a positive constant depending on κ and on the solution pair (u, v). Hence, $u \in L^r(\partial\Omega)$ for every $r < \infty$. Combining this with the first part of the proof shows that $u \in L^r(\overline{\Omega})$ for every finite r. The same arguments can be applied for the function v starting with the second equation in (3.1). This completes the proof. □

The next result states the L^∞-boundedness of weak solutions of problem (1.1).

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^N, N > 1$, be a bounded domain with a Lipschitz boundary $\partial\Omega$ and let the hypotheses (H) be satisfied. Then, for any weak solution $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ it holds $(u,v) \in L^\infty(\overline{\Omega}) \times L^\infty(\overline{\Omega})$.*

Proof. Let $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ be a weak solution of problem (1.1). As in the proof of Theorem 3.1 we will suppose that $u, v \geq 0$ and we only prove that $u \in L^\infty(\overline{\Omega})$, since the proof that $v \in L^\infty(\overline{\Omega})$ works in a similar way. We repeat the proof of Theorem 3.1 until inequality (3.13), that is

$$\begin{aligned}
&\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p \\
&\leq M_{27}\left(\frac{\kappa p+1}{(\kappa+1)^p}+1\right)\|uu_h^\kappa\|_{ps}^p + M_{28}(\kappa p+1)\int_\Omega u^{p^*}u_h^{\kappa p}\,dx \\
&\qquad + M_{29}\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}\,d\sigma + M_{30}\|uu_h^\kappa\|_{pt,\partial\Omega}^p + M_{31}(\kappa+1).
\end{aligned} \tag{3.27}$$

Recall that $ps < p^*$ and $pt < p_*$. Hence, we can fix numbers $p_1 \in (ps, p^*)$ and $p_2 \in (pt, p_*)$. Then, by Hölder's inequality and the $L^r(\overline{\Omega})$-boundedness of u for any finite r, see Theorem 3.1, we have for the terms on the right-hand side of (3.27) the following

$$\begin{aligned}
\|uu_h^\kappa\|_{ps}^p &\leq |\Omega|^{\frac{p_1-ps}{p_1 s}}\left(\int_\Omega (uu_h^\kappa)^{p_1}\,dx\right)^{\frac{p}{p_1}} \leq M_{32}\|uu_h^\kappa\|_{p_1}^p, \\
\int_\Omega u^{p^*}u_h^{\kappa p}\,dx &= \int_\Omega u^{p^*-p}(uu_h^\kappa)^p\,dx \\
&\leq \left(\int_\Omega u^{\frac{p^*-p}{p_1-p}p_1}\,dx\right)^{\frac{p_1-p}{p_1}}\left(\int_\Omega (uu_h^\kappa)^{p_1}\,dx\right)^{\frac{p}{p_1}} \\
&\leq M_{33}\|uu_h^\kappa\|_{p_1}^p, \\
\int_{\partial\Omega} u^{p_*}u_h^{\kappa p}\,d\sigma &= \int_{\partial\Omega} u^{p_*-p}(uu_h^\kappa)^p\,d\sigma \\
&\leq \left(\int_{\partial\Omega} u^{\frac{p_*-p}{p_2-p}p_2}\,d\sigma\right)^{\frac{p_2-p}{p_2}}\left(\int_{\partial\Omega} (uu_h^\kappa)^{p_2}\,d\sigma\right)^{\frac{p}{p_2}} \\
&\leq M_{34}\|uu_h^\kappa\|_{p_2,\partial\Omega}^p, \\
\|uu_h^\kappa\|_{pt,\partial\Omega}^p &\leq |\partial\Omega|^{\frac{p_2-pt}{p_2}}\left(\int_{\partial\Omega} (uu_h^\kappa)^{p_2}\,d\sigma\right)^{\frac{p}{p_2}} \leq M_{35}\|uu_h^\kappa\|_{p_2,\partial\Omega}^p.
\end{aligned} \tag{3.28}$$

Observe that M_{33}, M_{34} are finite thanks to Theorem 3.1. More precisely, they are such that

$$M_{33} = M_{33}\left(\|u\|_{\frac{p^*-p}{p_1-p}p_1}\right), \quad M_{34} = M_{34}\left(\|u\|_{\frac{p_*-p}{p_2-p}p_2,\partial\Omega}\right).$$

Then (3.27) becomes

$$\begin{aligned}\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p &\leq M_{36}\left(\frac{\kappa p+1}{(\kappa+1)^p}+\kappa p+2\right)\|uu_h^\kappa\|_{p_1}^p\\ &\quad+M_{37}\|uu_h^\kappa\|_{p_2,\partial\Omega}^p+M_{31}(\kappa+1),\end{aligned} \tag{3.29}$$

where we used the estimates in (3.28). Now we are going to apply again Proposition 2.1 to the boundary term. This gives, after using Hölder's inequality,

$$\begin{aligned}\|uu_h^\kappa\|_{p_2,\partial\Omega}^p &\leq \varepsilon_2\|uu_h^\kappa\|_{1,p}^p+\bar{c}_1\varepsilon_2^{-\bar{c}_2}\|uu_h^\kappa\|_p^p\\ &\leq \varepsilon_2\|uu_h^\kappa\|_{1,p}^p+\bar{c}_1\varepsilon_2^{-\bar{c}_2}M_{38}\|uu_h^\kappa\|_{p_1}^p.\end{aligned} \tag{3.30}$$

Choosing ε_2 such that $M_{37}\varepsilon_2=\frac{\kappa p+1}{2(\kappa+1)^p}$ and applying (3.30) to (3.29) yields

$$\frac{\kappa p+1}{2(\kappa+1)^p}\|uu_h^\kappa\|_{1,p}^p \leq \left[M_{39}(\kappa p+2)+M_{40}\bar{c}_1\varepsilon_2^{-\bar{c}_2}\right]\|uu_h^\kappa\|_{p_1}^p+M_{31}(\kappa+1). \tag{3.31}$$

Inequality (3.31) can be written in the form

$$\|uu_h^\kappa\|_{1,p}^p \leq M_{41}((\kappa+1)^p)^{M_{42}}\left[\|uu_h^\kappa\|_{p_1}^p+1\right].$$

By the Sobolev embedding and the $L^r(\Omega)$-boundedness of u we obtain

$$\begin{aligned}\|uu_h^\kappa\|_{p^*} \leq c_\Omega\|uu_h^\kappa\|_{1,p} &\leq M_{43}(\kappa+1)^{M_{44}}\left[\|uu_h^\kappa\|_{p_1}^p+1\right]^{\frac{1}{p}}\\ &\leq M_{43}(\kappa+1)^{M_{44}}\left[\|u^{\kappa+1}\|_{p_1}^p+1\right]^{\frac{1}{p}}<\infty.\end{aligned} \tag{3.32}$$

Applying Fatou's Lemma to (3.32) then gives

$$\|u\|_{(\kappa+1)p^*}=\|u^{\kappa+1}\|_{p^*}^{\frac{1}{\kappa+1}} \leq M_{43}^{\frac{1}{\kappa+1}}\left((\kappa+1)^{M_{44}}\right)^{\frac{1}{\kappa+1}}\left[\|u^{\kappa+1}\|_{p_1}^p+1\right]^{\frac{1}{(\kappa+1)p}}. \tag{3.33}$$

Since

$$\left((\kappa+1)^{M_{44}}\right)^{\frac{1}{\sqrt{\kappa+1}}} \geq 1 \quad \text{and} \quad \lim_{\kappa\to\infty}\left((\kappa+1)^{M_{44}}\right)^{\frac{1}{\sqrt{\kappa+1}}}=1,$$

there exists $M_{45}>1$ such that

$$\left((\kappa+1)^{M_{44}}\right)^{\frac{1}{\kappa+1}} \leq M_{45}^{\frac{1}{\sqrt{\kappa+1}}}. \tag{3.34}$$

From (3.33), taking (3.34) into account, we have

$$\|u\|_{(\kappa+1)p^*} \leq M_{43}^{\frac{1}{\kappa+1}}M_{45}^{\frac{1}{\sqrt{\kappa+1}}}\left[\|u^{\kappa+1}\|_{p_1}^p+1\right]^{\frac{1}{(\kappa+1)p}}. \tag{3.35}$$

Suppose now there exists a sequence $\kappa_n\to\infty$ such that

$$\|u^{\kappa_n+1}\|_{p_1}^p \leq 1,$$

that is

$$\|u\|_{(\kappa_n+1)p_1} \leq 1.$$

Then, Proposition 2.2 implies that $\|u\|_\infty<\infty$. On the contrary, suppose that there exists $\kappa_0>0$ such that

$$\|u^{\kappa+1}\|_{p_1}^p > 1 \quad \text{for every } \kappa \geq \kappa_0.$$

Then, (3.35) becomes

$$\|u\|_{(\kappa+1)p^*} \leq M_{43}^{\frac{1}{\kappa+1}} M_{45}^{\frac{1}{\sqrt{\kappa+1}}} \left[2\|u^{\kappa+1}\|_{p_1}^p\right]^{\frac{1}{(\kappa+1)p}} \leq M_{46}^{\frac{1}{\kappa+1}} M_{45}^{\frac{1}{\sqrt{\kappa+1}}} \|u\|_{(\kappa+1)p_1}$$

for every $\kappa \geq \kappa_0$.

Now we choose κ in the following way

$$\begin{aligned}
\kappa_1 &: (\kappa_1+1)p_1 = (\kappa_0+1)p^*,\\
\kappa_2 &: (\kappa_2+1)p_1 = (\kappa_1+1)p^*,\\
\kappa_3 &: (\kappa_3+1)p_1 = (\kappa_2+1)p^*,\\
&\quad\vdots \qquad\qquad\qquad \vdots \quad .
\end{aligned}$$

This leads to

$$\|u\|_{(\kappa_n+1)p^*} \leq M_{46}^{\frac{1}{\kappa_n+1}} M_{45}^{\frac{1}{\sqrt{\kappa_n+1}}} \|u\|_{(\kappa_{n-1}+1)p^*}$$

for every $n \in \mathbb{N}$ with (κ_n) given by $(\kappa_n+1) = (\kappa_0+1)\left(\frac{p^*}{p_1}\right)^n$. It follows

$$\|u\|_{(\kappa_n+1)p^*} \leq M_{46}^{\sum_{i=1}^{n} \frac{1}{\kappa_i+1}} M_{45}^{\sum_{i=1}^{n} \frac{1}{\sqrt{\kappa_i+1}}} \|u\|_{(\kappa_0+1)p^*}.$$

Since

$$\frac{1}{\kappa_i+1} = \frac{1}{\kappa_0+1}\left(\frac{p_1}{p^*}\right)^i \quad \text{and} \quad \frac{p_1}{p^*} < 1,$$

there exists $M_{47} > 0$ such that

$$\|u\|_{(\kappa_n+1)p^*} \leq M_{47}\|u\|_{(\kappa_0+1)p^*} < \infty,$$

where the right-hand side is finite thanks to Theorem 3.1. Now we may apply again Proposition 2.2. This ensures that $u \in L^\infty(\Omega)$. Moreover, Proposition 2.3 gives $u \in L^\infty(\partial\Omega)$ and so, $u \in L^\infty(\overline{\Omega})$. □

Remark 3.3. The conditions on the exponents in hypotheses (H) are not the natural ones. Precisely, in order to have a well-defined weak solution it is enough to require the following assumptions

(E1) $r_1 \leq p^*$	(E2) $r_2 \leq q^*$	
(E3) $b_1 \leq p^*-1$	(E4') $b_2 \leq \frac{q^*}{p^*}(p^*-1)$	(E5') $\frac{b_3}{p^*} + \frac{b_4}{q^*} \leq \frac{p^*-1}{p^*}$
(E6) $b_5 \leq p-1$	(E7') $b_6 \leq \frac{q}{p^*}(p^*-1)$	(E8') $\frac{b_7}{p} + \frac{b_8}{q} \leq \frac{p^*-1}{p^*}$
(E9') $\tilde{b}_1 \leq \frac{p^*}{q^*}(q^*-1)$	(E10) $\tilde{b}_2 \leq q^*-1$	(E11') $\frac{\tilde{b}_3}{p^*} + \frac{\tilde{b}_4}{q^*} \leq \frac{q^*-1}{q^*}$
(E12') $\tilde{b}_5 \leq \frac{p}{q^*}(q^*-1)$	(E13) $\tilde{b}_6 \leq q-1$	(E14') $\frac{\tilde{b}_7}{p} + \frac{\tilde{b}_8}{q} \leq \frac{q^*-1}{q^*}$

(E15) $c_1 \leq p_* - 1$ (E16') $c_2 \leq \frac{q_*}{p_*}(p_* - 1)$ (E17') $\frac{c_3}{p_*} + \frac{c_4}{q_*} \leq \frac{p_* - 1}{p_*}$

(E18') $\tilde{c}_1 \leq \frac{p_*}{q_*}(q_* - 1)$ (E19) $\tilde{c}_6 \leq q_* - 1$ (E20') $\frac{\tilde{c}_3}{p_*} + \frac{\tilde{c}_4}{q_*} \leq \frac{q_* - 1}{q_*}$.

In order to apply Moser's iteration we needed to strengthen the hypotheses for (E4'), (E5'), (E7'), (E8'), (E9'), (E11'), (E12'), (E14'), (E16'), (E17'), (E18'), and (E20'). We also point out that hypotheses (H1) and (H2) are not explicitly needed in the proofs of Theorems 3.1 and 3.2, but they are necessary to have a well-defined weak solution as defined in (3.1).

Furthermore, the bounds obtained in Theorem 3.1 and 3.2 depend on the data in hypotheses (H) and also on the solution pair (u, v). In particular, the bound for u also depends on v and vice-versa.

In the last part we want to mention that the results obtained in Theorems 3.1 and 3.2 can be easily applied to problems of the form (1.1) with a homogeneous Dirichlet condition. Indeed, consider the problem

$$\begin{aligned} -\operatorname{div} \mathcal{A}_1(x, u, \nabla u) &= \mathcal{B}_1(x, u, v, \nabla u, \nabla v) && \text{in } \Omega, \\ -\operatorname{div} \mathcal{A}_2(x, v, \nabla v) &= \mathcal{B}_2(x, u, v, \nabla u, \nabla v) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.36}$$

We suppose the following assumptions on the data in problem (3.36).

($\tilde{\text{H}}$) The functions $\mathcal{A}_i \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathcal{B}_i \colon \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, 2$, are Carathéodory functions such that

($\tilde{\text{H}}$1) $|\mathcal{A}_1(x, s, \xi)| \leq A_1|\xi|^{p-1} + A_2|s|^{r_1 \frac{p-1}{p}} + A_3$,

($\tilde{\text{H}}$2) $|\mathcal{A}_2(x, t, \zeta)| \leq \tilde{A}_1|\zeta|^{q-1} + \tilde{A}_2|t|^{r_2 \frac{q-1}{q}} + \tilde{A}_3$,

($\tilde{\text{H}}$3) $\mathcal{A}_1(x, s, \xi) \cdot \xi \geq A_4|\xi|^p - A_5|s|^{r_1} - A_6$,

($\tilde{\text{H}}$4) $\mathcal{A}_2(x, t, \zeta) \cdot \zeta \geq \tilde{A}_4|\zeta|^q - \tilde{A}_5|t|^{r_2} - \tilde{A}_6$,

($\tilde{\text{H}}$5) $|\mathcal{B}_1(x, s, t, \xi, \zeta)| \leq B_1|s|^{b_1} + B_2|t|^{b_2} + B_3|s|^{b_3}|t|^{b_4} + B_4|\xi|^{b_5} + B_5|\zeta|^{b_6} + B_6|\xi|^{b_7}|\zeta|^{b_8} + B_7$,

($\tilde{\text{H}}$6) $|\mathcal{B}_2(x, s, t, \xi, \zeta)| \leq \tilde{B}_1|s|^{\tilde{b}_1} + \tilde{B}_2|t|^{\tilde{b}_2} + \tilde{B}_3|s|^{\tilde{b}_3}|t|^{\tilde{b}_4} + \tilde{B}_4|\xi|^{\tilde{b}_5} + \tilde{B}_5|\zeta|^{\tilde{b}_6} + \tilde{B}_6|\xi|^{\tilde{b}_7}|\zeta|^{\tilde{b}_8} + \tilde{B}_7$,

for a.e. $x \in \Omega$, for all $s, t \in \mathbb{R}$, and for all $\xi, \zeta \in \mathbb{R}^N$, with nonnegative constants $A_i, \tilde{A}_i, B_j, \tilde{B}_j$ ($i \in \{1, \ldots, 6\}, j \in \{1, \ldots, 7\}$) and with $1 < p, q < \infty$. Moreover, the exponents $b_i, \tilde{b}_i, r_1, r_2$ ($i \in \{1, \ldots, 8\}$) are nonnegative and satisfy the following assumptions

(E1) $r_1 \leq p^*$ (E2) $r_2 \leq q^*$

(E3) $b_1 \leq p^* - 1$ (E4) $b_2 < \frac{q^*}{p^*}(p^* - p)$ (E5) $\frac{b_3}{p^*} + \frac{b_4}{q^*} < \frac{p^* - p}{p^*}$

(E6) $b_5 \leq p - 1$ (E7) $b_6 < \frac{q}{p^*}(p^* - p)$ (E8) $\frac{b_7}{p} + \frac{b_8}{q} < \frac{p^* - p}{p^*}$

(E9) $\tilde{b}_1 < \frac{p^*}{q^*}(q^* - q)$ (E10) $\tilde{b}_2 \leq q^* - 1$ (E11) $\frac{\tilde{b}_3}{p^*} + \frac{\tilde{b}_4}{q^*} < \frac{q^* - q}{q^*}$

(E12) $\tilde{b}_5 < \frac{p}{q^*}(q^* - q)$ (E13) $\tilde{b}_6 \leq q - 1$ (E14) $\frac{\tilde{b}_7}{p} + \frac{\tilde{b}_8}{q} < \frac{q^* - q}{q^*}$,

where the numbers p^*, p_*, q^*, q_* are defined by (2.1) and (2.2).

A couple $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is said to be a weak solution of problem (3.36) if

$$\int_\Omega \mathcal{A}_1(x, u, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega \mathcal{B}_1(x, u, v, \nabla u, \nabla v) \varphi \, dx$$
$$\int_\Omega \mathcal{A}_2(x, v, \nabla v) \cdot \nabla \psi \, dx = \int_\Omega \mathcal{B}_2(x, u, v, \nabla u, \nabla v) \psi \, dx$$

holds for all $(\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

We can state the following result for problem (3.36).

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let hypotheses $(\tilde{H})$ be satisfied. Then, every weak solution $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of problem (3.36) belongs to $L^\infty(\overline{\Omega}) \times L^\infty(\overline{\Omega})$.*

The proof of Theorem 3.4 works exactly in the same way as the proofs of Theorems 3.1 and 3.2.

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