Research Article

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Existence and multiplicity of solutions for fractional Schrödinger-p-Kirchhoff equations in $\mathbb{R}^N$

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Abstract: This paper concerns the existence and multiplicity of solutions for a nonlinear Schrödinger–Kirchhoff-type equation involving the fractional p-Laplace operator in $\mathbb{R}^N$. Precisely, we study the Kirchhoff-type problem

$$\left(a + b \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, \mathrm{d}x\, \mathrm{d}y\right)(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x, u) \quad \text{in } \mathbb{R}^N,$$

where $a, b > 0$, $(-\Delta)_p^s$ is the fractional p-Laplacian with $0 < s < 1 < p < \frac{N}{s}$, $V\colon \mathbb{R}^N \to \mathbb{R}$ and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions while V can have negative values and f fulfills suitable growth assumptions. According to the interaction between the attenuation of the potential at infinity and the behavior of the nonlinear term at the origin, using a penalization argument along with L^∞-estimates and variational methods, we prove the existence of a positive solution. In addition, we also establish the existence of infinitely many solutions provided the nonlinear term is odd.

Keywords: Fractional p-Laplacian, L^∞-estimates, Kirchhoff-type equation, multiple solutions, penalization technique, unbounded domain

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1 Introduction and main results

In this article, we consider the following fractional p-Laplacian Kirchhoff-type elliptic problem:

$$\begin{cases} \left(a + b \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, \mathrm{d}x\, \mathrm{d}y\right)(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x, u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $a, b > 0$, $p \in (1, \infty)$, $s \in (0, 1)$, $N > sp$, V is a continuous function which may vanishing at infinity and f is a continuous function verifying suitable growth assumptions. Here, $(-\Delta)_p^s$ is the fractional p-Laplacian operator,

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which (up to normalization factors) is defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\delta\to 0^+} \int\limits_{\mathbb{R}^N\setminus B_\delta(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,\mathrm{d}y, \quad x\in\mathbb{R}^N,$$

for any $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\delta(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius δ.

In recent years, there has been a surge of interest in the study of partial differential equations involving nonlocal fractional Laplace operators. This type of nonlocal operator comes up naturally in the real world in many different applications, such as phase transitions, game theory, finance, image processing, Lévy processes, and optimization; see, for example the works of Applebaum [16], Di Nezza-Palatucci-Valdinoci [26] and their references for more details.

In the case $a = 1$, $b = 0$ and $p = 2$, (1.1) becomes the fractional Laplacian equation of the type

$$(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which can be seen as the fractional form of the following classical stationary Schrödinger equation

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

During the last years, equations (1.2) and (1.3) have been widely considered. Indeed, by using appropriate techniques and assuming different conditions of the potential V and the nonlinearity f, several existence, multiplicity, and concentration results of equations (1.2) and (1.3) have been established. We refer to Alves and Miyagaki [3], Ambrosio [9–11], Figueiredo and Siciliano [28], Li, Sun and Tersian [33] and Willem [48], see also the references therein.

In the case $s = 1$ and $p = 2$, (1.1) turns into the classical Kirchhoff-type equation of the form

$$-\Bigl(a + b\int\limits_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x\Bigr)\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

which was proposed by Kirchhoff [32] as a generalization of the well-known d'Alembert's wave equation

$$\rho u_{tt} - \Bigl(\frac{p_0}{h} + \frac{E}{2L}\int_0^L |u_x|^2\,\mathrm{d}x\Bigr)u_{xx} = f(x,u),$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and p_0 is the initial tension. In [1], it was pointed out that problem (1.4) models several physical systems, in which u described a process dependent on its own average value. Nonlocal effects also have applications in biological systems. In fact, the parabolic version of the equation can be used to describe the growth and movement of specific species. The motion modeled by the integral term is assumed to depend on the energy of the whole system, where u is its population density. Some interesting results concerning (1.4) can be found in [19] or [23]. Since Lions' work [36], problem (1.4) began to attract the attention of several mathematicians, we refer to the papers of Chen and Li [22], He and Zou [31], Perera and Zhang [41], Sun, Li, Cencelj and Gabrovšek [46] and the references therein.

On the other hand, the study of fractional p-Kirchhoff problems has attracted considerable attention. Pucci, Xiang and Zhang [42] dealt with a nonhomogeneous fractional p-Laplacian Kirchhoff–Schrödinger equation given by

$$M\Bigl(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\Bigr)(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) + g(x) \quad \text{in } \mathbb{R}^N,$$

where the potential V satisfies a Bartsch–Wang-type condition. In [51], Xiang, Zhang and Ferrara studied the existence of two solutions for a nonhomogeneous fractional p-Kirchhoff problem, where the nonlinearity is convex-concave. In [21], Caponi and Pucci dealt with the existence, multiplicity, and asymptotic behavior of entire solutions for a series of stationary Kirchhoff fractional p-Laplacian equations. Subsequently, Liang and Rădulescu applied Kajikiya's new version of the symmetric mountain pass lemma to study a class of fractional p-Kirchhoff-type Schrödinger–Choquard equations in [35]. In [29], Fiscella and Pucci obtained the existence and the asymptotic behavior of nontrivial solutions for stationary fractional p-Laplacian Kirchhoff equations involving a Hardy potential and different critical nonlinearities. In [52], Xiang, Zhang and Rădulescu obtained a multiplicity result for a fractional p-Kirchhoff system driven by a nonlocal integro-differential operator with zero Dirichlet boundary data. Moreover, Liang, Molica Bisci and Zhang [34] studied the multiplicity of solutions of a class of noncooperative critical fractional p-Laplacian elliptic system with homogeneous Dirichlet boundary conditions by using the Limit Index Theory and the fractional version of the concentration compactness principle. The existence and multiplicity of solutions for a critical fractional p-Kirchhoff-type problem involving discontinuous nonlinearity has been obtained by Xiang and Zhang [50] while Ambrosio, Isernia and Rădulescu [14] discussed the concentration of positive solutions for fractional p-Kirchhoff-type problems given in the form

$$\begin{cases} (\varepsilon^{sp} a + \varepsilon^{2sp-3} b[u]^p_{W^{s,p}(\mathbb{R}^3)})(-\Delta)^s_p u + V(x)u^{p-1} = f(u) & \text{in } \mathbb{R}^3, \\ u \in W^{s,p}(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where ε is a small positive parameter and $a, b > 0$. Another interesting work has been done by Thin, Xiang and Zhang [47] who studied the existence of solutions for critical Kirchhoff–Schrödinger-type fractional p-Laplacian problems with potential vanishing at infinity defined by

$$M\bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}^N} V(x)|u(x)|^p\,\mathrm{d}x\bigg)((-\Delta)^s_p u(x) + V(x)|u|^{p-2}u) = K(x)(\lambda f(x, u) + |u|^{p^*_s-2}u),$$

where $p^*_s = \frac{Np}{N-ps}$, M, K, V are nonnegative continuous functions satisfying suitable conditions and $\lambda > 0$ is a real parameter. Very recently, Lv and Zheng [37, 38], studied critical fractional p-Kirchhoff equations involving competitive nonlinearities or logarithmic nonlinearity while Xiong, Chen, Chen and Sun [54] considered concave-convex fractional p-Kirchhoff-type elliptic equation with steep well potential. Finally, other interesting results in this direction can be found in the papers of Ambrosio [12], Ambrosio and Isernia [13], Ambrosio and Servadei [15], Arora, Fiscella, Mukherjee and Winkert [17, 18], Fiscella and Pucci [30], Nyamoradi and Zaidan [40], Pucci, Xiang and Zhang [43], Song and Shi [45], Xiang, Molica Bisci, Tian and Zhang [39], Xiang, Zhang and Rădulescu [49] and Xiang, Zhang and Rădulescu [53]. However, in the above works, the potential V is always nonnegative, that is,

$$\inf_{x\in\mathbb{R}^N} V(x) \geq V(x_0) \geq 0,$$

where $V(x_0) \geq 0$ is a constant.

In the past two decades, many studies have focused on the potential that can vanish at infinity, that is, $V(x) \to 0$ as $|x| \to \infty$, or briefly, $V_\infty = 0$. We refer the reader to Alves, Figueiredo and Yang [2], Alves and Souto [4], Ambrosetti, Felli and Malchiodi [5], Ambrosetti, Malchiodi and Ruiz [6], Ambrosetti and Wang [8], de B. Silva and Soares [24], and references therein. It is worth mentioning that a penalization technique and corresponding L^∞-estimates have been applied. It should also be emphasized that the existence result shows the interplay between the behavior of the nonlinear term at the origin and the decay of the potential at infinity. A key factor in establishing this relationship is the result of the L^∞-estimates for the penalized problem, which does not depend on the behavior of the nonlinear term near the origin.

Motivated by the papers of Alves and Souto [4], de B. Silva and Soares [24] and Ambrosio, Isernia and Rădulescu [14] as well as due to the large interest shared by the mathematical community on fractional p-Laplacian problems, we study the existence and multiplicity of solutions to problem (1.1) where the potential V may assume negative values. Along the paper, we always assume f and V satisfy the following assumptions:

(f_1) $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists $\vartheta > p$ such that

$$\limsup_{z\to 0} \left|\frac{zf(x, z)}{z^\vartheta}\right| < +\infty \quad \text{uniformly for all } x \in \mathbb{R}^N.$$

(f_2) There exist $a_1, a_2 > 0$ and $q \in (p, p^*_s)$ with $p^*_s = \frac{Np}{N-sp}$ such that

$$|f(x, z)| \leq a_1|z|^{q-1} + a_2 \quad \text{for all } (x, z) \in \mathbb{R}^N \times \mathbb{R}.$$

(f_3) There exist $\theta > 2p$ and $S_0 \geq 0$ such that

$$zf(x, z) \geq \theta F(x, z) > 0 \quad \text{for all } |z| \geq S_0 \text{ and for all } x \in \mathbb{R}^N,$$

where $F(x, z) := \int_0^z f(x, t)\,\mathrm{d}t$.

(V_1) $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function and either $V \geq 0$ in $\mathbb{R}^N$ and satisfies

$$V(x) \leq V_\infty \quad \text{for all } x \in B_{r_0}(x_0) \tag{1.5}$$

for some $V_\infty, r_0 > 0$ and $x_0 \in \mathbb{R}^N$ or

$$\Omega := \{x \in \mathbb{R}^N : V(x) < 0\}$$

is a nonempty bounded set and

$$\inf_\Omega V > -\frac{\mathcal{S}}{|\Omega|^{\frac{sp}{N}}},$$

where $\mathcal{S} > 0$ is the best constant for the embedding $W^{s,p}(\mathbb{R}^N)$ into $L^{p_s^*}(\mathbb{R}^N)$.

(V_2) There are constants $\Lambda > 0$ and $R > 0$ ($R > |x_0| + r_0$, for $r_0 > 0$ and $x_0 \in \mathbb{R}^N$ given by (1.5), if $V \geq 0$) such that

$$\inf_{|x|\geq R} |x|^{\frac{(N-sp)(\vartheta-p)}{p-1}} V(x) \geq \Lambda$$

with $\vartheta > p$ given by (f_1).

Now, we state our first main result of this work:

Theorem 1.1. *Suppose hypotheses* (V_1)–(V_2) *and* (f_1)–(f_3) *hold. Then there exists a constant* $\Lambda^* > 0$ *such that* (1.1) *has a positive solution for every* $\Lambda \geq \Lambda^*$.

Note that Λ^* given in Theorem 1.1 depends on the radius $R > 0$ given in condition (V_2). In particular, when condition (f_3) holds with $S_0 = 0$, and V satisfies the following version of (V_2):

(V_3) There are constants $\Lambda > 0$ and $R > 0$ ($R > |x_0| + r_0$, for $r_0 > 0$ and $x_0 \in \mathbb{R}^N$ given by (1.5), if $V \geq 0$) such that

$$\frac{1}{R^{\frac{(N-sp)(\vartheta-p)}{p-1}}} \inf_{|x|\geq R} |x|^{\frac{(N-sp)(\vartheta-p)}{p-1}} V(x) \geq \Lambda,$$

where $\vartheta > p$ given by (f_1).

Now, we state the second result of this paper:

Theorem 1.2. *Suppose hypotheses* (V_1), (V_3), (f_1)–(f_2), *and* (f_3) *with* $S_0 = 0$ *hold. Then there exists* $\widetilde{\Lambda}^* > 0$ *such that* (1.1) *has a positive solution for every* $\Lambda \geq \widetilde{\Lambda}^*$.

To strengthen the interaction between the theoretical behavior of the nonlinear term and the decay of the potential, by conditions (f_1) and (V_2), we give a result in which the function f approaches zero at the origin: Assume that f and V satisfy:

($\widehat{f_1}$) There are constants $\vartheta, \varsigma > 0$ such that

$$\limsup_{z\to 0} |f(x,z)| e^{\frac{\varsigma}{|z|^\vartheta}} < +\infty \quad \text{uniformly in } \mathbb{R}^N.$$

(V_4) There are constants $\Lambda > 0$, $\mu > 0$ and $R > 0$ ($R > |x_0| + r_0$, for $r_0 > 0$ and $x_0 \in \mathbb{R}^N$ given by (1.5), if $V \geq 0$) such that

$$\inf_{|x|\geq R} e^{\mu|x|^{\frac{(N-sp)\vartheta}{p-1}}} V(x) \geq \Lambda,$$

where ϑ given by ($\widehat{f_1}$).

We can state the following result.

Theorem 1.3. *Suppose hypotheses* (V_1), (V_4), ($\widehat{f_1}$) *and* (f_2)–(f_3) *hold. Then there exist constants* $\Lambda^*, \mu^* > 0$ *such that* (1.1) *has a positive solution for every* $\Lambda \geq \Lambda^*$ *and* $0 < \mu \leq \mu^*$.

Similar to Theorem 1.3, if (f_3) holds with $S_0 = 0$ and V satisfies

(V_5) There are constants $\Lambda > 0$, $\mu > 0$ and $R > 0$ ($R > |x_0| + r_0$, for $r_0 > 0$ and $x_0 \in \mathbb{R}^N$ given by (1.5), if $V \geq 0$) such that

$$\inf_{|x|\geq R} e^{\mu(\frac{|x|}{R})^{\frac{(N-sp)\vartheta}{p-1}}} V(x) \geq \Lambda,$$

with ϑ given by ($\widehat{f_1}$).

Then we may take $\widehat{\Lambda}^* > 0$, which does not depend on R, such that problem (1.1) has a positive solution for each $\Lambda > \widehat{\Lambda}^*$. More precisely, using the arguments employed in the proof of Theorem 1.3, we also have the following theorem.

Theorem 1.4. *Suppose hypotheses* (V_1), (V_5), ($\widehat{f_1}$), (f_2) *and* (f_3) *with* $S_0 = 0$ *hold. Then there are constants* $\widehat{\Lambda}^*, \widehat{\mu}^* > 0$ *such that* (1.1) *has a positive solution for every* $\Lambda \geq \widehat{\Lambda}^*$ *and* $0 < \mu \leq \widehat{\mu}^*$.

Remark that, under the above hypotheses, we may actually obtain solutions u^+ and u^- of problem (1.1) with $u^+ > 0$ and $u^- < 0$ in $\mathbb{R}^N$. Suppose now f is odd with respect to the second variable, that is:
(f_4) $f(x, -z) = -f(x, z)$, for every $(x, z) \in \mathbb{R}^N \times \mathbb{R}$.
Then we may use a version of the penalization technique and a minimax critical point theorem for functional with symmetry to get the subsequent results.

Theorem 1.5. *Suppose hypotheses* (V_1)–(V_2), (f_1)–(f_3) *and* (f_4) *hold. Then there exists constant* $\Lambda^* > 0$ *such that* (1.1) *has infinitely many nontrivial solutions for every* $\Lambda \geq \Lambda^*$.

Theorem 1.6. *Suppose hypotheses* (V_1), (V_3), (f_1)–(f_2), (f_4) *and* (f_3) *with* $S_0 = 0$ *hold. Then there exists* $\widetilde{\Lambda}^* > 0$ *such that* (1.1) *has infinitely many nontrivial solutions for every* $\Lambda \geq \widetilde{\Lambda}^*$.

Theorem 1.7. *Suppose hypotheses* (V_1), (V_4), ($\widehat{f_1}$) *and* (f_2)–(f_4) *hold. Then there exist constants* $\Lambda^*, \mu^* > 0$ *such that* (1.1) *has infinitely many nontrivial solutions for every* $\Lambda \geq \Lambda^*$ *and* $0 < \mu \leq \mu^*$.

Theorem 1.8. *Suppose hypotheses* (V_1), (V_5), ($\widehat{f_1}$), (f_2), (f_4) *and* (f_3) *with* $S_0 = 0$ *hold. Then there are constants* $\widehat{\Lambda}^*$, $\widehat{\mu}^* > 0$ *such that* (1.1) *has a positive solution for every* $\Lambda \geq \widehat{\Lambda}^*$ *and* $0 < \mu \leq \widehat{\mu}^*$.

We know that (f_3) is the classical (AR) condition, and it only considers the case $\theta > 2p$. When $p < \theta \leq 2p$, we can obtain a similar existence result considering the following hypothesis:
($\widetilde{f_3}$) There exist $p < \theta \leq 2p$ and $S_0 \geq 0$ such that

$$zf(x,z) \geq \theta F(x,z) > 0 \quad \text{for every } |z| \geq S_0,\ x \in \mathbb{R}^N,$$

where $F(x,z) := \int_0^z f(x,t)\,\mathrm{d}t$.
Then we have the following theorem.

Theorem 1.9. *Suppose hypotheses* (V_1)–(V_2), (f_1)–(f_2) *and* ($\widetilde{f_3}$) *hold. Then there exist* $b^* > 0$ *and* $\Lambda^* > 0$ *such that* (1.1) *has a positive solution for every* $b \in (0, b^*)$ *and* $\Lambda \geq \Lambda^*$.

It is not difficult to verify that, as a direct consequence of Theorem 1.9, versions of Theorems 1.2–1.4 hold under condition ($\widetilde{f_3}$):

Theorem 1.10. *Suppose hypotheses* (V_1), (V_3), (f_1)–(f_2), *and* ($\widetilde{f_3}$) *with* $S_0 = 0$ *hold. Then there exist constants* $b^* > 0$ *and* $\widetilde{\Lambda}^* > 0$ *such that* (1.1) *has a positive solution for every* $b \in (0, b^*)$ *and* $\Lambda \geq \widetilde{\Lambda}^*$.

Theorem 1.11. *Suppose hypotheses* (V_1), (V_4), ($\widehat{f_1}$), (f_2) *and* ($\widetilde{f_3}$) *hold. Then there exist constants* $b^* > 0$, Λ^* *and* $\mu^* > 0$ *such that* (1.1) *has a positive solution for every* $b \in (0, b^*)$, $\Lambda \geq \Lambda^*$ *and* $0 < \mu \leq \mu^*$.

Theorem 1.12. *Suppose hypotheses* (V_1), (V_5), ($\widehat{f_1}$), (f_2) *and* ($\widetilde{f_3}$) *with* $S_0 = 0$ *hold. Then there are constants* $b^* > 0$, $\widehat{\Lambda}^* > 0$ *and* $\widehat{\mu}^* > 0$ *such that* (1.1) *has a positive solution for every* $b \in (0, b^*)$, $\Lambda \geq \widehat{\Lambda}^*$ *and* $0 < \mu \leq \widehat{\mu}^*$.

Remark 1.13. The paper by de B. Silva and Soares [24] established the same conclusions for a semilinear elliptic problem involving the Laplacian operator. More precisely, they considered only the case $a = 1$, $b = 0$, $p = 2$ and $s \to 1^-$. Obviously, our results are more general than those of [24].

Remark 1.14. By the subcritical and Kirchhoff problem, we will use the following techniques:
(i) In order to prove Theorems 1.1–1.4, we use the penalization argument explored by Alves and Souto [4], which consists of a modification of the original problem such that f to be controlled by a function at infinity.
(ii) Next, we use the Fountain Theorem to obtain the multiplicity of solutions.
(iii) When $p < \theta \leq 2p$, the mountain pass geometry and the boundedness of the $(PS)_c$- or $(C)_c$-sequence $\{u_n\}_{n\in\mathbb{N}}$ is very difficult to prove by a standard argument. In order to show Theorem 1.9, we also use the truncation

technique in Zhang and Du [55] to prove the boundedness of $(\mathrm{C})_c$-sequences and later we prove that every $(\mathrm{C})_c$-sequence contains a convergent subsequence.

This paper is organized as follows. In Section 2, we give a detailed description of the properties of the function space defined by the energy functional. In Section 3, considering the case $\theta > 2p$, we introduce the version of the penalization argument used for proving our results and establish the existence of a positive solution for the penalized problem. Then we present an estimate for the L^∞ norm for the solution to the modified problem. Finally, we obtain the positive solution of the original problem and the multiplicity of the solutions. In Section 4, when $p < \theta \leq 2p$, we provide the proof of Theorem 1.9.

2 Preliminaries

In this section, let us first recall some basic results related to the fractional Sobolev spaces. Let $u\colon \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We say that u belongs to the space $W^{s,p}(\mathbb{R}^N)$ if and only if $u \in L^p(\mathbb{R}^N)$ and

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} := \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

The space $W^{s,p}(\mathbb{R}^N)$ is a Banach space endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = [|u|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}]^{\frac{1}{p}}.$$

Moreover, $L^t(\mathbb{R}^N)$ denotes the Lebesgue space with norm

$$|u|_{L^t(\mathbb{R}^N)} = \bigg(\int\limits_{\mathbb{R}^N} |u|^t\,\mathrm{d}x\bigg)^{\frac{1}{t}}$$

for $1 \leq t < \infty$. Then $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ is continuous for any $t \in [p, p_s^*]$, that is, there exists a positive constant C_* such that

$$|u|_{L^t(\mathbb{R}^N)} \leq C_* \|u\|_{W^{s,p}(\mathbb{R}^N)} \quad \text{for all } u \in W^{s,p}(\mathbb{R}^N). \tag{2.1}$$

For detailed properties of $W^{s,p}(\mathbb{R}^N)$, we refer the reader to the work of Di Nezza, Palatucci and Valdinoci [26].

Now let E be the subspace of $W^{s,p}(\mathbb{R}^N)$ defined by

$$E = \bigg\{u \in W^{s,p}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x < \infty\bigg\}.$$

Under hypothesis (V_1), we can introduce a new norm $\|\cdot\|$ on E given by

$$\|u\| = \|u\|_{E(\mathbb{R}^N)} := \bigg[a[u]^p_{W^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x\bigg]^{\frac{1}{p}}.$$

Lemma 2.1. *Let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $N > sp$. Under hypothesis (V_1), the embedding $E \hookrightarrow W^{s,p}(\mathbb{R}^N)$ is continuous in such a way that E is a Banach space that is continuously embedded into $L^t(\mathbb{R}^N)$ for all $t \in [p, p_s^*]$. In particular, there exists a constant $C_t > 0$ such that*

$$|u|_{L^t(\mathbb{R}^N)} \leq C_t\|u\| \quad \textit{for all } u \in E.$$

If $t \in [1, p_s^*)$, then the embedding $E \hookrightarrow\hookrightarrow L^t(B_R)$ is compact for any $R > 0$.

Proof. Since the result is trivially verified if $V \geq 0$ in $\mathbb{R}^N$, it suffices to suppose that $\Omega \neq \emptyset$. Given $u \in W^{s,p}(\mathbb{R}^N)$, we may use Hölder's inequality and the estimate $|u|^p_{L^{p_s^*}(\Omega)} \leq \mathcal{S}^{-1}[u]^p_{W^{s,p}(\mathbb{R}^N)}$ to get

$$\int\limits_\Omega |u|^p\,\mathrm{d}x \leq \bigg(\int\limits_\Omega |u|^{p_s^*}\,\mathrm{d}x\bigg)^{\frac{p}{p_s^*}}\bigg(\int\limits_\Omega 1^{\frac{N}{sp}}\,\mathrm{d}x\bigg)^{\frac{sp}{N}} = |\Omega|^{\frac{sp}{N}}|u|^p_{L^{p_s^*}(\Omega)} \leq \frac{|\Omega|^{\frac{sp}{N}}}{\mathcal{S}}[u]^p_{W^{s,p}(\mathbb{R}^N)}. \tag{2.2}$$

From (V_1), there is $\alpha > 0$ such that

$$\inf_{x\in\Omega} V(x) \ge -\alpha > -\frac{\mathcal{S}}{|\Omega|^{\frac{sp}{N}}}. \tag{2.3}$$

Then we may invoke (2.2) to obtain

$$a[u]^p_{W^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V(x)|u|^p \,\mathrm{d}x \ge \Big(a - \frac{\alpha|\Omega|^{\frac{sp}{N}}}{\mathcal{S}}\Big)[u]^p_{W^{s,p}(\mathbb{R}^N)} > 0.$$

Consequently, the first part follows.

Now, fix $R > 0$ and note that

$$\Big(\|u\|^p_{L^p(B_R)} + \iint_{B_R\times B_R} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}$$

is an equivalent norm on $W^{s,p}(B_R)$ and the embedding $E \hookrightarrow W^{s,p}(B_R)$ is continuous. By Di Nezza, Palatucci and Valdinoci [26, Corollary 7.2], the embedding $W^{s,p}(B_R) \hookrightarrow\hookrightarrow L^t(B_R)$ is compact. Thus, the embedding $E \hookrightarrow\hookrightarrow L^t(B_R)$ is also compact by the first part of the Lemma. This proves the assertion. □

Remark 2.2. In this paper we take $\alpha = 0$ and $\Omega = \emptyset$ whenever $V \ge 0$ in $\mathbb{R}^N$. Note that in this setting the above estimates are satisfied for those values of α and Ω.

The Euler–Lagrange functional associated with (1.1) is given by

$$\Phi(u) = \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{b}{2p}([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \quad \text{for all } u \in E.$$

From the conditions on f, it is easy to see that the functional Φ belongs to $C^1(E,\mathbb{R})$. Now we give the definition of solutions for problem (1.1).

Definition 2.3. We say that $u \in E$ is a weak solution of equation (1.1) if

$$\Big(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\Big)\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(\varphi(x)-\varphi(y))\,\mathrm{d}x\,\mathrm{d}y\Big) + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)\varphi\,\mathrm{d}x$$

holds for any $\varphi \in E$.

Moreover, since in Theorems 1.1–1.4 we intend to prove the existence of a positive solution, we let $f(x,z) = 0$ for every $(x,z) \in \mathbb{R}^N \times (-\infty, 0]$.

3 The case $\theta > 2p$

3.1 The penalized problem

In this section, we adopt a version of the penalization argument employed in Alves and Souto [4]. To this end, for $\theta > p$ and $R > 0$ given by conditions (f_3) and (V_2), respectively, we take $k = \frac{p\theta}{\theta - p}$ and consider, for every $(x,z) \in \mathbb{R}^N \times (0,\infty)$,

$$\tilde{f}(x,z) = \begin{cases} -\frac{1}{k}V(x)|z|^{p-1} & \text{if } kf(x,z) < -V(x)|z|^{p-1}, \\ f(x,z) & \text{if } -V(x)|z|^{p-1} \le kf(x,z) \le V(x)|z|^{p-1}, \\ \frac{1}{k}V(x)|z|^{p-1} & \text{if } kf(x,z) > V(x)|z|^{p-1}. \end{cases}$$

Furthermore, set $\tilde{f}(x,z) = 0$ for every $(x,z) \in \mathbb{R}^N \times (-\infty, 0]$, and define

$$g(x,z) = \begin{cases} f(x,z) & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| \le R, \\ \tilde{f}(x,z) & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| > R. \end{cases}$$

A direct computation shows that g is a Carathéodory function and the following hold:

$$\begin{cases} g(x,z) = 0 & \text{for } (x,z) \in \mathbb{R}^N \times (-\infty, 0], \\ g(x,z) = f(x,z) & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| \le R, \\ |g(x,z)| \le |f(x,z)| & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R}, \\ |g(x,z)| \le \dfrac{1}{k} V(x)|z|^{p-1} & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| > R, \end{cases} \tag{3.1}$$

and

$$\begin{cases} G(x,z) = F(x,z) & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| \le R, \\ G(x,z) \le \dfrac{1}{pk} V(x)|z|^p & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| > R, \end{cases} \tag{3.2}$$

where $G(x,z) := \int_0^z g(x,t)\,\mathrm{d}t$.

The auxiliary problem that we will consider is the following one:

$$\begin{cases} \left(a + b \iint\limits_{\mathbb{R}^{2N}} \dfrac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)(-\Delta)_p^s u + V(x)|u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in E. \end{cases} \tag{3.3}$$

Remark 3.1. Observe that any positive solution u of (3.3) that satisfies $k|f(x,u)| \le V(x)|u|^{p-1}$ for $|x| \ge R$ is a solution of (1.1).

Due to (3.3), the associated Euler–Lagrange functional $\mathcal{J}\colon E \to \mathbb{R}$ given by

$$\mathcal{J}(u) = \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{b}{2p}([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x - \int\limits_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x$$

is well defined and of class $C^1(E,\mathbb{R})$ and its Gateaux derivative is

$$\begin{aligned} \mathcal{J}'(u)v = {} & \left(a + b \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(v(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y\right) \\ & + \int\limits_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,\mathrm{d}x - \int\limits_{\mathbb{R}^N} g(x,u)v\,\mathrm{d}x \end{aligned} \tag{3.4}$$

for all $u, v \in E$. Therefore, it is easy to see that the solutions of (3.3) correspond to the critical points of the energy functional $\mathcal{J}$.

Under our assumptions, we can show that functional has the mountain pass geometry.

Lemma 3.2. *Suppose V satisfies* (V_1)–(V_2) *and f satisfies* (f_1)–(f_3). *Then the following hold:*
(1) *There exist $\beta, \rho > 0$ such that $\mathcal{J}(u) \ge \beta$ for every $u \in E$ such that $\|u\| = \rho$;*
(2) *There exists a function $e \in E$ with $\|u\| \ge \rho$, such that $\mathcal{J}(e) < 0$.*

Proof. The proof for (1) is standard and follows well-known arguments. We give a proof for the case $\Omega \ne \emptyset$. By (V_1) and (V_2), $\Omega \subset B_R(0)$ and $V(x) > 0$ for every $|x| \ge R$. Note that, by (f_1)–(f_2), it follows that for each $\eta > 0$, there exists $C_\eta > 0$ such that

$$|F(x,z)| \le \eta|z|^p + C_\eta|z|^{p_s^*} \quad \text{for every } (x,z) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, there are positive constants $d_1 = d_1(R)$ and $d_2 = d_2(\eta)$ such that

$$\int\limits_{B_R(0)} F(x,u)\,\mathrm{d}x \le \eta d_1\|u\|^p + d_2\|u\|^{p_s^*} \quad \text{for all } u \in E. \tag{3.5}$$

Then, combining (2.3) with (3.2), we obtain

$$\begin{aligned}\mathcal{J}(u) &= \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{b}{2p}([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x - \int\limits_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x \\ &\geq \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x - \int\limits_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x \\ &\geq \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x - \int\limits_{B_R(0)} F(x,u)\,\mathrm{d}x - \frac{1}{pk}\int\limits_{\mathbb{R}^N\setminus\Omega} V(x)|u|^p\,\mathrm{d}x \\ &\geq \frac{a}{p}\left(1-\frac{\alpha|\Omega|^{\frac{sp}{N}}}{a\mathcal{S}}\right)[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{k-1}{pk}\int\limits_{\mathbb{R}^N\setminus\Omega} V(x)|u|^p\,\mathrm{d}x - \int\limits_{B_R(0)} F(x,u)\,\mathrm{d}x \\ &\geq d_3\left(a[u]^p_{W^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x\right) - \int\limits_{B_R(0)} F(x,u)\,\mathrm{d}x \\ &= d_3\|u\|^p - \int\limits_{B_R(0)} F(x,u)\,\mathrm{d}x, \end{aligned} \tag{3.6}$$

where

$$d_3 := \min\left\{\frac{1}{p}\left(1-\frac{\alpha|\Omega|^{\frac{sp}{N}}}{a\mathcal{S}}\right), \frac{k-1}{pk}\right\}.$$

From the above estimates (3.5) and (3.6), one has

$$\mathcal{J}(u) \geq d_3\|u\|^p - \eta d_1\|u\|^p - d_2\|u\|^{p_s^*} \quad \text{for every } u \in E.$$

By using the above estimate and taking $\eta > 0$ sufficiently small, statement (1) follows by finding appropriated values of $\beta, \rho > 0$.

On the other hand, by hypotheses (V_1)–(V_2) and taking $V_\infty = 0$, if $\Omega \neq \emptyset$, we suppose that $B_{r_0}(x_0) \subset B_R(0)$ and $V(x) \leq V_\infty$, for each $x \in B_{r_0}(x_0)$. Note that, by (f_2) and (f_3), there exist constants $C_1, C_2 > 0$, depending on r_0, such that

$$F(x,z) \geq C_1|z|^\theta - C_2 \quad \text{for every } (x,z) \in B_{r_0}(x_0)\times[0,\infty). \tag{3.7}$$

Then, considering a nonnegative function $\phi \in E\setminus\{0\}$ such that $\operatorname{supp}(\phi) \subset B_{r_0}(x_0)$, we obtain

$$\mathcal{J}(t\phi) \leq \frac{at^p}{p}[\phi]^p_{W^{s,p}(B_{r_0}(x_0))} + \frac{bt^{2p}}{2p}([\phi]^p_{W^{s,p}(B_{r_0}(x_0))})^2 + t^p\int\limits_{B_{r_0}(x_0)} V_\infty|\phi|^p\,\mathrm{d}x - \int\limits_{B_{r_0}(x_0)} F(x,t\phi)\,\mathrm{d}x \tag{3.8}$$

for every $t \geq 0$. Combining (3.7) with (3.8), we have

$$\mathcal{J}(t\phi) \leq \frac{at^p}{p}[\phi]^p_{W^{s,p}(B_{r_0}(x_0))} + \frac{bt^{2p}}{2p}([\phi]^p_{W^{s,p}(B_{r_0}(x_0))})^2 + t^p\int\limits_{B_{r_0}(x_0)} V_\infty|\phi|^p\,\mathrm{d}x - C_1t^\theta\int\limits_{B_{r_0}(x_0)} |\phi|^\theta\,\mathrm{d}x + C_2|B_{r_0}(x_0)|, \tag{3.9}$$

which implies that $\mathcal{J}(t\phi) \to -\infty$ as $t \to +\infty$, since $\theta > 2p$. Hence, taking $e = t\phi$, with $t > 0$ sufficiently large, we have that $\|e\| > \rho$ and $\mathcal{J}(e) < 0$. The proof is complete. □

Consequently, using a version of the Mountain Pass Theorem (see Willem [48]), there exists a $(\mathrm{PS})_c$ sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ such that

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad \mathcal{J}'(u_n) \to 0,$$

where the minimax value c is given by

$$c := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} \mathcal{J}(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0,\ \mathcal{J}(\gamma(1)) < 0\}.$$

Lemma 3.3. *There exist constants $\beta_1, \beta_2 > 0$ such that $\beta_1 \le c \le \beta_2$.*

Proof. Note that by Lemma 3.2, $c \ge \beta > 0$, and we take $\beta_1 \in (0, \beta)$. On the other hand, fix $\varphi \in E \setminus \{0\}$. Then, for all $t > 0$, as in (3.9) we can get

$$\mathcal{J}(t\varphi) \le \frac{at^p}{p}[\varphi]^p_{W^{s,p}(B_{r_0}(x_0))} + \frac{bt^{2p}}{2p}([\varphi]^p_{W^{s,p}(B_{r_0}(x_0))})^2 + t^p \int\limits_{B_{r_0}(x_0)} V_\infty |\varphi|^p \,\mathrm{d}x - C_1 t^\theta \int\limits_{B_{r_0}(x_0)} |\varphi|^\theta \,\mathrm{d}x + C_2 |B_{r_0}(x_0)| \to -\infty$$

as $t \to +\infty$. Thus, if $\beta_2 =: \max_{t>0} \mathcal{J}(t\phi) > 0$, it follows from the definition of c that $c \le \beta_2$. The proof of the lemma is complete. □

Lemma 3.4. *The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E.*

Proof. Note that by (f$_1$) and (f$_3$), there exists a positive constant $C = C(R)$ such that

$$\frac{1}{\theta} f(x,z)z - F(x,z) \ge -C \quad \text{for every } (x,z) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.10}$$

Hence, it follows from (3.1), (3.2) and (3.10) that

$$\begin{aligned}
c + o_n(1) &= \mathcal{J}(u_n) - \frac{1}{\theta}\mathcal{J}'(u_n)u_n \\
&= \Big(\frac{a}{p} - \frac{a}{\theta}\Big)[u]^p_{W^{s,p}(\mathbb{R}^N)} + \Big(\frac{b}{2p} - \frac{b}{\theta}\Big)([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 \\
&\quad + \Big(\frac{1}{p} - \frac{1}{\theta}\Big) \int\limits_{\mathbb{R}^N} V(x)|u_n|^p \,\mathrm{d}x + \int\limits_{\mathbb{R}^N} \Big(\frac{1}{\theta} g(x,u_n)u_n - G(x,u_n)\Big) \mathrm{d}x \\
&\ge \Big(\frac{a}{p} - \frac{a}{\theta}\Big)[u]^p_{W^{s,p}(\mathbb{R}^N)} + \Big(\frac{b}{2p} - \frac{b}{\theta}\Big)([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \Big(\frac{1}{p} - \frac{1}{\theta}\Big) \int\limits_{\mathbb{R}^N} V(x)|u_n|^p \,\mathrm{d}x \\
&\quad + \int\limits_{B_R(0)} \Big(\frac{1}{\theta} f(x,u_n)u_n - F(x,u_n)\Big) \mathrm{d}x + \frac{p-\theta}{\theta p k} \int\limits_{\mathbb{R}^N \setminus B_R(0)} V(x)|u_n|^p \,\mathrm{d}x \\
&\ge \Big(\frac{1}{p} - \frac{1}{\theta}\Big) a[u]^p_{W^{s,p}(\mathbb{R}^N)} + \Big(\frac{b}{2p} - \frac{b}{\theta}\Big)([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \Big(\frac{1}{p} - \frac{1}{\theta}\Big) \int\limits_{B_R(0)} V(x)|u_n|^p \,\mathrm{d}x \\
&\quad + \frac{(\theta-p)(\theta p - \theta + p)}{(\theta p)^2} \int\limits_{\mathbb{R}^N \setminus B_R(0)} V(x)|u_n|^p \,\mathrm{d}x - C|B_R(0)| \\
&\ge K\bigg(a[u]^p_{W^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} V(x)|u_n|^p \,\mathrm{d}x\bigg) - C|B_R(0)|,
\end{aligned}$$

where

$$K = \min\Big\{\frac{1}{p} - \frac{1}{\theta}, \frac{(\theta-p)(\theta p - \theta + p)}{(\theta p)^2}\Big\} > 0.$$

Consequently, by $a > 0$ and $\theta > 2p$, we obtain

$$c + o_n(1) \ge K\|u_n\|^p - C|B_R(0)|,$$

which means that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E. □

Remark 3.5. Note that by Lemmas 3.3 and 3.4, there is $L > 0$ such that $\|u_n\| \le L$ for every n.

By Lemma 3.4, the embeddings of E in $W^{s,p}(\mathbb{R}^N)$ and the Sobolev embedding theorem, up to a subsequence, we may suppose that there exists $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } E, \\ u_n \to u & \text{strongly in } L^t_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } t \in [1, p_s^*), \\ u_n(x) \to u(x) & \text{a.e. } x \in \mathbb{R}^N. \end{cases} \tag{3.11}$$

Now we give several useful conclusions.

Lemma 3.6. *Assume that conditions* (f_1)–(f_3) *and* (V_1)–(V_2) *hold. Then for any* $\varepsilon > 0$, *there exists* $r = r(\varepsilon) > R > 0$ *such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_r} \left(a \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y + V(x)|u_n|^p \right) \mathrm{d}x < \varepsilon, \tag{3.12}$$

$$\limsup_{n\to\infty} \int_{B_r} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\,\mathrm{d}x = 0, \tag{3.13}$$

$$u_n \to u \quad \textit{strongly in } L^t(\mathbb{R}^N) \textit{ for all } t \in [p, p_s^*). \tag{3.14}$$

Proof. First, we consider $r > R$ and a function $\psi = \psi_r \in C_0^\infty(B_r^c)$ such that $\psi \equiv 0$ if $x \in B_r(0)$, $\psi \equiv 1$ if $x \notin B_{2r}(0)$ with $0 \le \psi(x) \le 1$, and $|\nabla\psi(x)| \le \frac{C}{r}$, where C is a constant independent of r, for all $x \in \mathbb{R}^N$. As $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E, the sequence $\{\psi u_n\}_{n\in\mathbb{N}}$ is also bounded. This shows that $I'(u_n)(\psi u_n) = o_n(1)$, namely,

$$\begin{aligned}
&(a + b[u_n]^p_{W^{s,p}(\mathbb{R}^N)}) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\psi(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p\psi\,\mathrm{d}x \\
&\quad = o_n(1) + \int_{\mathbb{R}^N} g(x,u_n)\psi u_n\,\mathrm{d}x \\
&\qquad - (a + b[u_n]^p_{W^{s,p}(\mathbb{R}^N)}) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+ps}} u_n(y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Then, by the definition of ψ and (3.1), we obtain

$$\begin{aligned}
&a \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\psi(x)\,\mathrm{d}x\,\mathrm{d}y + \left(1-\frac{1}{k}\right)\int_{\mathbb{R}^N} V(x)|u_n|^p\psi\,\mathrm{d}x \\
&\quad \le o_n(1) - (a + b[u_n]^p_{W^{s,p}(\mathbb{R}^N)}) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}} u_n(y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.15}$$

Due to the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in E, we can suppose that $a + b[u_n]^p_{W^{s,p}(\mathbb{R}^N)} \to \ell \in (0,\infty)$. From Lemma 3.4 and Hölder's inequality, we get

$$\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}} u_n(y)\,\mathrm{d}x\,\mathrm{d}y \right| \le C\left(\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}|u_n(y)|^p\,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}}. \tag{3.16}$$

In addition, by the definition of ψ, Lemma 3.4 and the polar coordinates, it follows that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}|u_n(x)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^N}\int_{|y-x|>r} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}|u_n(x)|^p\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N}\int_{|y-x|\leqslant r} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}|u_n(x)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \le C\int_{\mathbb{R}^N} |u_n(x)|^p\left(\int_{|y-x|>r} \frac{\mathrm{d}y}{|x-y|^{N+sp}} \right)\mathrm{d}x + \frac{C}{r^p}\int_{\mathbb{R}^N} |u_n(x)|^p\left(\int_{|y-x|\leqslant r} \frac{\mathrm{d}y}{|x-y|^{N+sp-p}} \right)\mathrm{d}x \\
&\quad \le C\int_{\mathbb{R}^N} |u_n(x)|^p\left(\int_{|z|>r} \frac{\mathrm{d}z}{|z|^{N+sp}} \right)\mathrm{d}x + \frac{C}{r^p}\int_{\mathbb{R}^N} |u_n(x)|^p\left(\int_{|z|\leqslant r} \frac{dz}{|z|^{N+sp-p}} \right)\mathrm{d}x \\
&\quad \le C\int_{\mathbb{R}^N} |u_n(x)|^p\,\mathrm{d}x\left(\int_r^\infty \frac{\mathrm{d}\rho}{\rho^{sp+1}} \right) + \frac{C}{r^p}\int_{\mathbb{R}^N} |u_n(x)|^p\,\mathrm{d}x\left(\int_0^r \frac{d\rho}{\rho^{sp-p+1}} \right) \\
&\quad \le \frac{C}{r^{sp}}\int_{\mathbb{R}^N} |u_n(x)|^p\,\mathrm{d}x + \frac{C}{r^p}r^{-sp+p}\int_{\mathbb{R}^N} |u_n(x)|^p\,\mathrm{d}x \le \frac{C}{r^{sp}}\int_{\mathbb{R}^N} |u_n(x)|^p\,\mathrm{d}x \le \frac{C}{r^{sp}} \to 0
\end{aligned} \tag{3.17}$$

as $r \to \infty$. Using (3.15), (3.16) and (3.17), we conclude that (3.12) is verified.

On the other hand, since $u_n \to u$ in $L^t(B_r)$, for all $t \in [1, p_s^*)$, by Lebesgue's Dominated Convergence Theorem, we obtain that

$$\lim_{n\to\infty} \int_{B_r} V(x)|u_n|^t \,dx = \int_{B_r} V(x)|u|^t \,dx,$$

which shows that (3.13) holds. The proof of part (b) is finished.

In particular, it follows from (3.12) and Fatou's lemma that

$$\int_{\mathbb{R}^N\setminus B_r} \left(a \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} \,dy + V(x)|u_n|^p \right) dx < \varepsilon. \tag{3.18}$$

For any n large enough, by (3.18), we obtain

$$\begin{aligned}|u_n-u|^p_{L^p(\mathbb{R}^N)} &= |u_n-u|^p_{L^p(B_r)} + |u_n-u|^p_{L^p(\mathbb{R}^N\setminus B_r)} \le \varepsilon + |u_n-u|^p_{L^p(\mathbb{R}^N\setminus B_r)} \le \varepsilon + \frac{1}{V_0}\int_{\mathbb{R}^N\setminus B_r} V(x)|u_n-u|^p \,dx \\ &\le \varepsilon + C\int_{\mathbb{R}^N\setminus B_r}\left(a\int_{\mathbb{R}^N} \frac{|(u_n(x)-u(x))-(u_n(y)-u(y))|^p}{|x-y|^{N+sp}}\,dy + V(x)|u_n-u|^p\right) dx \\ &\le (1+C)\varepsilon,\end{aligned}$$

where $V_0 = \inf_{x\in\mathbb{R}^N\setminus B_R(0)} V(x) > 0$. This implies that $u_n \to u$ in $L^p(\mathbb{R}^N)$. Then, by interpolation, we have that (3.14) holds, which shows part (c). □

Lemma 3.7. *Assume that conditions* (f$_1$)–(f$_3$) *and* (V$_1$)–(V$_2$) *hold. Then the functional* $\mathcal{I}$ *satisfies the* $(\mathrm{PS})_c$ *condition.*

Proof. The proof is based on the proofs of Lemmas 3.4–3.6. Indeed, from Lemma 3.4 and the growth assumptions on g, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} (g(x,u_n)u_n - g(x,u)u)(u_n-u)\,dx\right| &\le (|u_n|^{p-1}_{L^p(\mathbb{R}^N)} + |u|^{p-1}_{L^p(\mathbb{R}^N)})|u_n-u|_{L^p(\mathbb{R}^N)} \\ &\quad + C(|u_n|^{q-1}_{L^q(\mathbb{R}^N)} + |u|^{q-1}_{L^q(\mathbb{R}^N)})|u_n-u|_{L^q(\mathbb{R}^N)} \\ &\le C|u_n-u|_{L^p(\mathbb{R}^N)} + C|u_n-u|_{L^q(\mathbb{R}^N)},\end{aligned}$$

where q is given by (f$_2$). The above estimate and (3.14) provide

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (g(x,u_n)u_n - g(x,u)u)(u_n-u)\,dx = 0. \tag{3.19}$$

Now, we prove that $\|u_n-u\| \to 0$ as $n\to\infty$. Consider $\varphi \in E$ to be fixed and $\mathcal{B}_\varphi : E \to \mathbb{R}$ the linear functional on E defined as

$$\mathcal{B}_\varphi(v) := \iint_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy \quad \text{for all } v \in E.$$

Note that, by Hölder's inequality, $\mathcal{B}_\varphi$ is continuous on E, which shows using $u_n \to u$ in E that

$$\lim_{n\to\infty}((a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)}) - (a+b[u]^p_{W^{s,p}(\mathbb{R}^N)}))\mathcal{B}_u(u_n-u) = 0, \tag{3.20}$$

where we have that $(a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)}) - (a+b[u]^p_{W^{W^{s,p}(\mathbb{R}^N)}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$. Moreover, since $u_n \to u$ in E, $\mathcal{I}'(u_n) \to 0$, and (3.14), we have that $\langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n-u\rangle \to 0$, as $n\to\infty$. Then, by (3.19) and (3.20), one has

$$\begin{aligned} o_n(1) &= \langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n-u\rangle \\ &= (a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)})\mathcal{B}_{u_n}(u_n-u) - (a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)})\mathcal{B}_u(u_n-u) \\ &\qquad + ((a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)}) - (a+b[u]^p_{W^{s,p}(\mathbb{R}^N)}))\mathcal{B}_u(u_n-u) \\ &\qquad + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\,dx - \int_{\mathbb{R}^N}(g(x,u_n)u_n - g(x,u)u)(u_n-u)\,dx \\ &= (a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)})(\mathcal{B}_{u_n}(u_n-u) - \mathcal{B}_u(u_n-u)) + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\,dx + o_n(1),\end{aligned}$$

and so,

$$\lim_{n\to\infty}\left((a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)})(\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u))+\int_{\mathbb{R}^N}V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\,\mathrm{d}x\right)=0.$$

Then, by the inequality

$$(|x|^{p-2}x-|y|^{p-2}y)(z-w)\geq 0\quad\text{for all } x,y\in\mathbb{R},$$

it follows that

$$(a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)})(\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u))\geq 0,$$

and we also obtain

$$V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\geq 0\quad\text{if } x\in\mathbb{R}^N\setminus\Omega.$$

Thus we may invoke (3.13) to get

$$\begin{aligned}&\lim_{n\to\infty}(a+b[u_n]^p_{W^{s,p}(\mathbb{R}^N)})(\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u))=0,\\ &\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus\Omega}V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\,\mathrm{d}x=0.\end{aligned}\tag{3.21}$$

Let us recall the Simon's inequalities [44] given as

$$|\xi-\eta|^p\leq\begin{cases}c_p(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta) & \text{if } p\geq 2,\\ C_p[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta)]^{\frac{p}{2}}(|\xi|^p+|\eta|^p)^{\frac{2-p}{2}} & \text{if } 1<p<2\end{cases}\tag{3.22}$$

for all $\xi,\eta\in\mathbb{R}^N$ with positive constants c_p and C_p depending only on p.

Case (i). Suppose that $p\geq 2$. Then, by (3.21) and (3.22), it follows that

$$\begin{aligned}[u_n-u]^p_{W^{s,p}(\mathbb{R}^N)}&=\iint_{\mathbb{R}^{2N}}|u_n(x)-u_n(y)-u(x)+u(y)|^p|x-y|^{-(N+sp)}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq c_p\iint_{\mathbb{R}^{2N}}[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\\ &\qquad-|u(x)-u(y)|^{p-2}(u(x)-u(y))](u_n(x)-u_n(y)-u(x)+u(y))|x-y|^{-(N+sp)}\,\mathrm{d}x\,\mathrm{d}y\\ &=c_p[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)]=o_n(1).\end{aligned}$$

Similarly, by (3.21), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}V(x)|u_n-u|^p\,\mathrm{d}x&\leq\int_{\mathbb{R}^N\setminus\Omega}V(x)|u_n-u|^p\,\mathrm{d}x\\ &\leq c_p\int_{\mathbb{R}^N\setminus\Omega}V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\mathrm{d}x=o_n(1).\end{aligned}$$

In conclusion, $\|u_n-u\|\to 0$ as $n\to\infty$.

Case (ii). Suppose that $1<p<2$. Since $u_n\rightharpoonup u$ in E, there exists $\varrho>0$ such that $\|u_n\|\leq\varrho$ for all $n\in\mathbb{N}$. Then, applying the inequality

$$(a+b)^{\frac{2-p}{2}}\leq a^{\frac{2-p}{2}}+b^{\frac{2-p}{2}}\quad\text{for all } a,b\geq 0,\ 1<p<2,$$

it follows from (3.21), (3.22) and Hölder's inequality that

$$\begin{aligned}[u_n-u]^p_{W^{s,p}(\mathbb{R}^N)}&\leq C_p(\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u))^{\frac{p}{2}}\Big([u_n]^p_{W^{s,p}(\mathbb{R}^N)}+[u]^p_{W^{s,p}(\mathbb{R}^N)}\Big)^{\frac{2-p}{2}}\\ &\leq C_p(\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u))^{\frac{p}{2}}\Big([u_n]^{\frac{p(2-p)}{2}}_{W^{s,p}(\mathbb{R}^N)}+[u]^{\frac{p(2-p)}{2}}_{W^{s,p}(\mathbb{R}^N)}\Big)\\ &\leq C_p'(\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u))^{\frac{p}{2}}=o_n(1).\end{aligned}$$

Similarly, we also get that

$$\int_{\mathbb{R}^N} V(x)|u_n - u|^p \,\mathrm{d}x \le \int_{\mathbb{R}^N\setminus\Omega} V(x)|u_n - u|^p \,\mathrm{d}x$$
$$\le C_p''\bigg(\int_{\mathbb{R}^N\setminus\Omega} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,\mathrm{d}x\bigg)^{\frac{p}{2}} = o_n(1).$$

Then $\|u_n - u\| \to 0$ as $n \to \infty$. This fact implies that $u_n \to u$ strongly in E. □

Remark 3.8. Actually, in the proofs of Lemmas 3.2–3.7, we have only used (V_1) and the fact that V is positive on $\mathbb{R}^N \setminus B_R(0)$. The decay of V at infinity is not needed.

As a byproduct of Lemmas 3.2–3.7 and the Mountain Pass Theorem (see Ambrosetti and Rabinowitz [7]), there exists $u \in E$ such that

$$\mathcal{J}(u) = c > 0 \quad \text{and} \quad \mathcal{J}'(u) = 0,$$

which shows that u is a weak solution of problem (3.3).

Furthermore, $u^- = \min\{u, 0\} = 0$. Indeed, by the definition of u^-, (3.4) and the fact u is a weak solution to (3.3), we obtain that

$$\mathcal{J}'(u^-)u^- = 0,$$

which together with (3.1) and (3.11) yields

$$\begin{aligned}\|u^-\|^p &= a\iint_{\mathbb{R}^{2N}} \frac{|u^-(x) - u^-(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u^-|^p\,\mathrm{d}x\\ &= \mathcal{J}'(u^-)u^- - b([u^-]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \int_{\mathbb{R}^N} g(x,u^-)u^-\,\mathrm{d}x\\ &= -b([u^-]^p_{W^{s,p}(\mathbb{R}^N)})^2\\ &\le 0.\end{aligned}$$

This implies that $u^- = 0$. Since $c > 0$, the function u is a nontrivial and nonnegative weak solution of (3.3). Consequently, from a Moser iteration argument, we can prove that $u \in L^\infty(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ (see Lemma 3.11 below). Then, by the maximum principle (see Del Pezzo and Quaas [25]), we can get that u is positive in $\mathbb{R}^N$. It remains to prove that u is also a positive solution of problem (1.1).

Let us denote

$$d := \sup_{t\ge 0}\bigg[\frac{at^p}{p}[\phi]^p_{W^{s,p}(B_0)} + \frac{bt^{2p}}{2p}([\phi]^p_{W^{s,p}(B_0)})^2 + t^p\int_{B_0} V_\infty|\phi|^p\,\mathrm{d}x - C_1t^\theta\int_{B_0}|\phi|^\theta\,\mathrm{d}x + C_2|B_0|\bigg],$$

where the constants C_1, C_2 are given in the proof of Lemma 3.2 and $B_0 := B_{r_0}(x_0)$.

Lemma 3.9. *Any solution u of* (3.3) *satisfies the estimate*

$$\|u\|^p \le K^{-1}(d + C|B_R(0)|),$$

where C, K are given by the proof of Lemma 3.4, respectively.

Proof. Note that, by (3.14), we obtain that $c \le d$. In addition, arguing as in the proof of Lemma 3.4, we have

$$c \ge K\|u\|^p - C|B_R(0)|.$$

Thus, $\|u\|^p \le K^{-1}(c + C|B_R(0)|) \le K^{-1}(d + C|B_R(0)|)$. □

Remark 3.10. If we suppose (f_3) with $S_0 = 0$, the estimate provided by Lemma 3.9 is independent of R. Indeed, since in this case the constant C given by (3.10) is zero, we get $\|u\|^p \le K^{-1}d$.

3.2 A priori estimates of the solution of the penalized problem

In this part, we establish an estimate for the L^∞ norm of the solutions u in terms of its $L^{p_s^*}$ norm. Here, we shall consider the problem (3.3) with V satisfying (V_1) and $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function fulfilling the subsequent assumptions:

(g_1) There exist $R > 0$ and $k > 1$ such that

$$|g(x,z)| \le \frac{1}{k} V(x)|z|^{p-1}$$

for all $z \in \mathbb{R}$, for all $x \in \mathbb{R}^N \setminus B_R(0)$.

(g_2) There exist $a_1 > 0$, $a_2 \ge 0$, and $q \in (p, p_s^*)$ such that

$$|g(x,z)| \le a_1|z|^{q-1} + a_2$$

for all $z \in \mathbb{R}$ and for all $x \in \mathbb{R}^N$.

Note that for (V_1) and (g_1) we have that $\Omega \subset B_R(0)$ whenever $\Omega \ne \emptyset$. For our problem, we shall adopt some ideas found in Alves and Souto [4] and Ambrosio, Isernia and Rădulescu [14].

Lemma 3.11. *Suppose* (V_1) *and* (g_1)–(g_2) *hold. Let* $u \in E$ *be a solution of problem* (3.3). *Then* $u \in L^\infty(\mathbb{R}^N)$ *and*

$$|u|_{L^\infty(\mathbb{R}^N)} \le M.$$

Proof. It is sufficient to prove $u^+ \in L^\infty(\mathbb{R}^N)$. In addition, we shall prove the lemma under the hypothesis $\Omega \ne \emptyset$.

For each $L > 0$, let $u_L := \min\{u, L\}$ and denote the function

$$\ell(u) := \ell_{L,\sigma}(u) = u u_L^{p(\sigma-1)} \in E,$$

with $\sigma > 1$ to be determined later. Note that ℓ is increasing, thus we have

$$(a-b)(\ell(a) - \ell(b)) \ge 0 \quad \text{for any } a, b \in \mathbb{R}.$$

Consider the functions

$$\mathcal{Q}(t) := \frac{|t|^p}{p} \quad \text{and} \quad \mathcal{L}(t) := \int_0^t (\ell'(\tau))^{\frac{1}{p}}\,\mathrm{d}\tau,$$

and note that

$$\mathcal{L}(u) \ge \frac{1}{\sigma} u u_L^{\sigma-1}. \tag{3.23}$$

Hence, from (2.1) and (3.23), we obtain

$$[\mathcal{L}(u)]^p_{W^{s,p}(\mathbb{R}^N)} \ge C_*^{-1} |\mathcal{L}(u)|^p_{L^{p_s^*}(\mathbb{R}^N)} \ge C_*^{-1} \frac{1}{\sigma^p} |u u_L^{\sigma-1}|^p_{L^{p_s^*}(\mathbb{R}^N)}. \tag{3.24}$$

In addition, for any $a, b \in \mathbb{R}$, it holds

$$\mathcal{Q}'(a-b)(\ell(a) - \ell(b)) \ge |\mathcal{L}(a) - \mathcal{L}(b)|^p.$$

In fact, suppose that $a > b$, it follows from Jensen's inequality that

$$\begin{aligned}
\mathcal{Q}'(a-b)(\ell(a) - \ell(b)) &= (a-b)^{p-1}(\ell(a) - \ell(b)) \\
&= (a-b)^{p-1} \int_b^a \ell'(\tau)\,\mathrm{d}\tau \\
&= (a-b)^{p-1} \int_b^a (\mathcal{L}'(\tau))^p\,\mathrm{d}\tau \\
&\ge \Big(\int_b^a \mathcal{L}'(\tau)\,\mathrm{d}\tau \Big)^p \\
&= (\mathcal{L}(a) - \mathcal{L}(b))^p.
\end{aligned}$$

A similar argument holds if $a \le b$. Thus, we infer that

$$|\mathcal{L}(u)(x) - \mathcal{L}(u)(y)|^p \le |u(x) - u(y)|^{p-2}(u(x) - u(y))\big(u(x)u_L^{p(\sigma-1)}(x) - u(y)u_L^{p(\sigma-1)}(y)\big).$$

Using $\ell(u)$ as the test function in (3.3), in view of the above inequality and (g_1), we get that

$$\begin{aligned}
&a[\mathcal{L}(u)]^p_{W^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V(x)|u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x \\
&\le a \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(u(x)u_L^{p(\sigma-1)}(x) - u(y)u_L^{p(\sigma-1)}(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + b[u]^p_{W^{s,p}(\mathbb{R}^N)} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(u(x)u_L^{p(\sigma-1)}(x) - u_n(y)u_L^{p(\sigma-1)}(y))}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \int_{\mathbb{R}^N} V(x)|u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} g(x,u)uu_L^{p(\sigma-1)}\,\mathrm{d}x \\
&\le \int_{B_R(0)} |g(x,u)|uu_L^{p(\sigma-1)}\,\mathrm{d}x + \frac{1}{k}\int_{\mathbb{R}^N\setminus B_R(0)} V(x)|u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x.
\end{aligned}$$

By the fact that $\Omega \subset B_R(0)$, we have

$$a[\mathcal{L}(u)]^p_{W^{s,p}(\mathbb{R}^N)} + \int_{\Omega} V(x)|u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x + \frac{k-1}{k}\int_{\mathbb{R}^N\setminus\Omega} V(x)|u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x \le \int_{B_R(0)} |g(x,u)|uu_L^{p(\sigma-1)}\,\mathrm{d}x,$$

which leads to

$$a[\mathcal{L}(u)]^p_{W^{s,p}(\mathbb{R}^N)} \le \int_{B_R(0)} |g(x,u)|uu_L^{p(\sigma-1)}\,\mathrm{d}x + \alpha\int_{\Omega} |u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x,$$

where α is given by (2.3). The above estimate and (3.24) provide

$$\begin{aligned}
|uu_L^{\sigma-1}|^p_{L^{p_s^*}(\mathbb{R}^N)} &\le \sigma^p C_*[\mathcal{L}(u)]^p_{W^{s,p}(\mathbb{R}^N)} \\
&\le \frac{\sigma^p C_*}{a}\Big(\int_{B_R(0)} |g(x,u)|uu_L^{p(\sigma-1)}\,\mathrm{d}x + \alpha\int_{\Omega} |u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x\Big) \\
&\le C\sigma^p\Big(\int_{B_R(0)} |g(x,u)|uu_L^{p(\sigma-1)}\,\mathrm{d}x + \alpha\int_{\Omega} |u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x\Big).
\end{aligned} \tag{3.25}$$

On the other hand, by the growth assumptions on g and (3.25), it follows that

$$|uu_L^{\sigma-1}|^p_{L^{p_s^*}(\mathbb{R}^N)} \le C\sigma^p\Big(a_1\int_{B_R(0)} |u|^q u_L^{p(\sigma-1)}\,\mathrm{d}x + a_2\int_{B_R(0)} uu_L^{p(\sigma-1)}\,\mathrm{d}x + \alpha\int_{\Omega} |u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x\Big). \tag{3.26}$$

Applying Hölder's inequality, we have that

$$\begin{aligned}
&\int_{B_R(0)} |u|^q u_L^{p(\sigma-1)}\,\mathrm{d}x \le \Big(\int_{B_R(0)} u^{p_s^*}\,\mathrm{d}x\Big)^{\frac{q-p}{p_s^*}}\Big(\int_{B_R(0)} (uu_L^{\sigma-1})^{\frac{pp_s^*}{p_s^*-(q-p)}}\,\mathrm{d}x\Big)^{\frac{p_s^*-(q-p)}{p_s^*}}, \\
&\int_{B_R(0)} uu_L^{p(\sigma-1)}\,\mathrm{d}x \le |B_R(0)|^{\frac{q-p}{p_s^*}}\Big(\int_{B_R(0)} (uu_L^{p(\sigma-1)})^{\frac{p_s^*}{p_s^*-(q-p)}}\,\mathrm{d}x\Big)^{\frac{p_s^*-(q-p)}{p_s^*}}, \\
&\int_{\Omega} |u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x \le |\Omega|^{\frac{q-p}{p_s^*}}\Big(\int_{B_R(0)} (uu_L^{\sigma-1})^{\frac{pp_s^*}{p_s^*-(q-p)}}\,\mathrm{d}x\Big)^{\frac{p_s^*-(q-p)}{p_s^*}},
\end{aligned}$$

where $p < \frac{pp_s^*}{p_s^*-(q-p)} < p_s^*$. Since $u \geq u_L$ in $\mathbb{R}^N$, using Hölder's inequality one more time, we have

$$\int_{B_R(0)} |u|^q u_L^{p(\sigma-1)}\,\mathrm{d}x \leq \Big(\int_{B_R(0)} u^{p_s^*}\,\mathrm{d}x\Big)^{\frac{q-p}{p_s^*}} \Big(\int_{B_R(0)} |u|^{\frac{pp_s^*\sigma}{p_s^*-(q-p)}}\,\mathrm{d}x\Big)^{\frac{p_s^*-(q-p)}{p_s^*}},$$

$$\begin{aligned}\int_{B_R(0)} u u_L^{p(\sigma-1)}\,\mathrm{d}x &\leq |B_R(0)|^{\frac{q-p}{p_s^*}} \Big(\int_{B_R(0)} (uu^{p(\sigma-1)})^{\frac{p_s^*}{p_s^*-(q-p)}}\,\mathrm{d}x\Big)^{\frac{p_s^*-(q-p)}{p_s^*}} \\ &\leq |B_R(0)|^{\frac{q-p}{p_s^*}+\frac{p-1}{p\sigma}\cdot\frac{p_s^*-(q-p)}{p_s^*}} \Big(\int_{B_R(0)} |u|^{\frac{pp_s^*\sigma}{p_s^*-(q-p)}}\,\mathrm{d}x\Big)^{\frac{p(\sigma-1)+1}{p\sigma}\cdot\frac{p_s^*-(q-p)}{p_s^*}},\end{aligned}$$

$$\int_{\Omega} |u|^p u_L^{p(\sigma-1)}\,\mathrm{d}x \leq |\Omega|^{\frac{q-p}{p_s^*}} \Big(\int_{B_R(0)} |u|^{\frac{pp_s^*\sigma}{p_s^*-(q-p)}}\,\mathrm{d}x\Big)^{\frac{p_s^*-(q-p)}{p_s^*}},$$

which together with (3.26) implies

$$|u|^{p\sigma}_{L^{p_s^*\sigma}(\mathbb{R}^N)} \leq C\sigma^p\big(a_3|u|^{p\sigma}_{\sigma\alpha^*} + a_4|u|^{p(\sigma-1)+1}_{\sigma\alpha^*}\big) \leq C\sigma^p\big(|u|^{p\sigma}_{\sigma\alpha^*} + |u|^{p(\sigma-1)+1}_{\sigma\alpha^*}\big),$$

where

$$\alpha^* = \frac{pp_s^*}{p_s^*-(q-p)}, \quad a_3 = a_1\Big(\int_{B_R(0)} u^{p_s^*}\,\mathrm{d}x\Big)^{\frac{q-p}{p_s^*}} + \alpha|\Omega|^{\frac{q-p}{p_s^*}} \quad \text{and} \quad a_4 = a_2|B_R(0)|^{\frac{q-p}{p_s^*}+\frac{p-1}{\alpha^*\sigma}}.$$

Now, taking $\sigma = \frac{p_s^*}{\alpha^*}$, we have

$$|u|^{p\sigma}_{L^{p_s^*\sigma}(\mathbb{R}^N)} \leq C\sigma^p\big(|u|^{p\sigma}_{L^{p_s^*}(\mathbb{R}^N)} + |u|^{p(\sigma-1)+1}_{L^{p_s^*}(\mathbb{R}^N)}\big),$$

and replacing σ by σ^j, $j \in \mathbb{N}$, in the above inequality, we obtain that

$$|u|^{p\sigma^j}_{L^{p_s^*\sigma^j}(\mathbb{R}^N)} \leq C\sigma^{jp}\big(|u|^{p\sigma^j}_{L^{p_s^*}(\mathbb{R}^N)} + |u|^{p(\sigma^j-1)+1}_{L^{p_s^*}(\mathbb{R}^N)}\big).$$

Then, by an argument of induction, we may verify that

$$|u|_{L^{p_s^*\sigma^j}(\mathbb{R}^N)} \leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}+\cdots+\frac{j}{\sigma^j}}(pC+1)^{\frac{1}{p}(\frac{1}{\sigma}+\frac{1}{\sigma^2}+\cdots+\frac{1}{\sigma^j})}(1+|u|_{L^{p_s^*}(\mathbb{R}^N)}) \tag{3.27}$$

for every $j \in \mathbb{N}$. Note that

$$\sum_{j=1}^{\infty}\frac{1}{\sigma^j} = \frac{1}{\sigma-1} \quad \text{and} \quad \sum_{j=1}^{\infty}\frac{i}{\sigma^j} = \frac{\sigma}{(\sigma-1)^2}.$$

Since $\sigma > 1$, passing to the limit as $j \to \infty$ in (3.27), we may infer that $u \in L^\infty(\mathbb{R}^N)$ and

$$|u|_{L^\infty(\mathbb{R}^N)} \leq \sigma^{\frac{\sigma}{(\sigma-1)^2}}(pC+1)^{\frac{1}{\sigma-1}}(1+|u|_{L^{p_s^*}(\mathbb{R}^N)}). \tag{3.28}$$

From inequality (3.28) and the argument used at the end of the proof of [14, Lemma 2.8], we can conclude that $u \in L^\infty(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$. □

Lemma 3.12. *Suppose* (V_1) *and* (g_1)–(g_2) *hold. Let $u \in E$ be a solution of problem* (3.3). *Then*

$$|u(x)| \leq M\Big(\frac{R}{|x|}\Big)^{\frac{N-sp}{p-1}} \quad \textit{for all } x \in \mathbb{R}^N \textit{ and for all } |x| \geq R,$$

where $R > 0$ is given by (g_1) *and M is given by Lemma 3.11.*

Proof. Let $v \in C^\infty(\mathbb{R}^N \setminus \{0\})$ be the function

$$v(x) = M\Big(\frac{R}{|x|}\Big)^{\frac{N-sp}{p-1}},$$

for each $x \in \mathbb{R}^N$. Moreover, since $\frac{1}{|x|^{\frac{N-sp}{p-1}}}$ is s-harmonic (see for instance Bucur-Valdinoci [20]), it shows that

$(-\Delta)_p^s v = 0$ in $\mathbb{R}^N \setminus B_R(0)$. Obviously, by Lemma 3.11, we obtain the inequality

$$u(x) \leq |u|_{L^\infty(\mathbb{R}^N)} \leq M\Big(\frac{R}{|x|}\Big)^{\frac{N-sp}{p-1}} = v(x) \quad \text{for all } 0 < |x| \leq R.$$

Next, we define the function

$$w^+(x) = \begin{cases} (u(x) - v(x))^+ & \text{if } |x| \geq R, \\ 0 & \text{if } |x| < R. \end{cases}$$

Since $(-\Delta)_p^s v = 0$ in $\mathbb{R}^N \setminus B_R(0)$, $w^+ \in E$, $w^+(x) = 0$ for every $|x| \leq R$, and $w^+ \geq 0$, it follows from (g_1) that

$$\begin{aligned}(a + b[w^+]^p_{W^{s,p}(\mathbb{R}^N)})[w^+]^p_{W^{s,p}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} g(x, w^+)w^+ \,\mathrm{d}x - \int_{\mathbb{R}^N} V(x)|w^+|^p \,\mathrm{d}x \\ &\leq \Big(\frac{1}{k} - 1\Big) \int_{\mathbb{R}^N\setminus B_R(0)} V(x)|w^+|^p \,\mathrm{d}x \\ &\leq 0.\end{aligned}$$

Hence, we have $w^+ \equiv 0$, which implies that $u(x) \leq v(x)$ in $|x| \geq R$. Similarly, by defining

$$w^-(x) = \begin{cases} (-u(x) - v(x))^+ & \text{if } |x| \geq R, \\ 0 & \text{if } |x| < R, \end{cases}$$

we can also get $-u(x) \leq v(x)$ in $|x| \geq R$. Thus

$$|u(x)| \leq M\Big(\frac{R}{|x|}\Big)^{\frac{N-sp}{p-1}} \quad \text{for all } x \in \mathbb{R}^N \text{ and for all } |x| \geq R.$$

The proof is complete. □

3.3 Existence results for problem (1.1)

Now we present the proofs of Theorems 1.1–1.4.

Proof of Theorem 1.1. From Lemmas 3.2–3.9 and the estimate $|u|_{L^{p_s^*}(\Omega)} \leq \mathcal{S}^{-1}[u]^p_{W^{s,p}(\mathbb{R}^N)}$, problem (3.3) has a positive solution $u \in E$, which satisfies

$$|u|_{L^{p_s^*}(\mathbb{R}^N)} \leq \check{C} := [K^{-1}(a\mathcal{S} - \alpha|\Omega|^{\frac{sp}{N}})^{-1}(d + C|B_R(0)|)]^{\frac{1}{p}},$$

where C, K and d are given by Lemma 3.7. Next, using the hypotheses (f_1), (f_2), there exists a constant $C > 0$ such that

$$|f(x, z)| \leq C|z|^{\vartheta-1} \quad \text{for all } |x| \geq R.$$

Therefore, it is enough to show that an appropriate u satisfies the inequality

$$\frac{|f(x, u(x))|}{|u(x)|^{p-1}} \leq CM^{(\vartheta-p)}\Big(\frac{R}{|x|}\Big)^{\frac{(N-sp)(\vartheta-p)}{p-1}} \quad \text{for all } |x| \geq R.$$

Fixing $\Lambda^* = kCM^{(\vartheta-p)}R^{\frac{(N-sp)(\vartheta-p)}{p-1}}$ and $\Lambda \geq \Lambda^* > 0$, it follows from (V_2) that

$$|f(x, u(x))| \leq \frac{1}{k}V(x)|u(x)|^{p-1} \quad \text{for all } |x| \geq R.$$

This shows that u is a positive solution of (1.1). The proof of Theorem 1.1 is complete. □

When (f_3) holds with $S_0 = 0$, we may provide a relation between the parameter in hypothesis (V_2) and the value of R.

Proof of Theorem 1.2. Since f satisfies (f_3) with $S_0 = 0$, we may invoke Remark 3.10 and (3.28) to infer that the constants C and M, do not depend on the values of Λ and R. Consequently, supposing that (V_3) holds, the argument used in the proof of Theorem 1.1 shows that problem (1.1) has a positive solution for every $\Lambda \geq \widetilde{\Lambda}^* = kCM^{\vartheta-p}$. The proof is complete. □

Proof of Theorem 1.3. As $(\widehat{f_1})$ shows that (f_1) holds, we may exploit the arguments used in the proof of Theorem 1.1 to infer that problem (3.3) has a positive solution $u \in E$. Fixing $0 < \widehat{\varsigma} < \varsigma$, from $(\widehat{f_1})$ and (f_2) we may find $C > 0$ such that

$$|f(x,z)| \leq Ce^{-\frac{\widehat{\varsigma}}{|z|^{\vartheta}}}.$$

Consequently, we may obtain

$$|f(x,u(x))| \leq Ce^{-\mu^*|x|^{\frac{(N-sp)\vartheta}{p-1}}} \quad \text{for all } |x| \geq R,$$

where $\mu^* = \widehat{\varsigma}(MR^{\frac{N-sp}{p-1}})^{-\vartheta}$. Thereby, fixing $0 < \mu \leq \mu^*$, $\Lambda^* = kC$ and $\Lambda \geq \Lambda^* > 0$, it follows from (V_4) that

$$|f(x,u(x))| \leq \frac{1}{k}V(x)|u(x)|^{p-1} \quad \text{for all } |x| \geq R.$$

This shows that u is a positive solution of (1.1). □

Proof of Theorem 1.4. The proof is analogous to Lemma 1.3, we omit it here. □

3.4 Multiplicity of solutions

In this subsection, we give the proof for the multiplicity results. Before we prove Theorem 1.5, let us recall the following version of the Fountain Theorem which can be found in Willem [48].

Theorem 3.13. *Let X be a Banach space with the norm $\|\cdot\|$ and X_j a sequence of subspace of X with* $\dim X_j < \infty$ *for each $j \in \mathbb{N}$. Further, $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, the closure of the direct sum of all X_j. Set $Y_k = \bigoplus_{j=0}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. Consider an even functional $I \in C^1(X,\mathbb{R})$ (i.e. $I(-u) = I(u)$ for all $u \in X$). Suppose, for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that:*

(A_1) $a_k := \max_{u\in Y_k,\|u\|=\rho_k} I(u) \leq 0$,

(A_2) $b_k := \inf_{u\in Z_k,\|u\|=r_k} I(u) \to +\infty$ *as* $k \to \infty$,

(A_3) *the Palais–Smale condition holds above* 0, *i.e. any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X which satisfies $I(u_n) \to c > 0$ and $I'(u_n) \to 0$ contains a convergent subsequence.*

Then I possesses an unbounded sequence of critical values.

To apply the Fountain Theorem, we still consider the odd extension of the function $\hat{g}\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, which is a Carathéodory function satisfying

$$\begin{cases} \hat{g}(x,z) = f(x,z) & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| \leq R, \\ |\hat{g}(x,z)| \leq |f(x,z)| & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R}, \\ |\hat{g}(x,z)| \leq \dfrac{1}{k}V(x)|z|^{p-1} & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| > R, \end{cases}$$

and

$$\begin{cases} \hat{G}(x,z) = F(x,z) & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| \leq R, \\ \hat{G}(x,z) \leq \dfrac{1}{pk}V(x)z^p & \text{for } (x,z) \in \mathbb{R}^N \times \mathbb{R},\ |x| > R, \end{cases}$$

where $\hat{G}(x,z) := \int_0^z g(x,t)\,\mathrm{d}t$. The symmetric version of the auxiliary problem that we will consider is the following:

$$\begin{cases} \left(a + b \iint\limits_{\mathbb{R}^{2N}} \dfrac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)(-\Delta)_p^s u + V(x)|u|^{p-2}u = \hat{g}(x,u) & \text{in } \mathbb{R}^N, \\ u \in E. \end{cases} \tag{3.29}$$

Observe that any solution u of (3.29) satisfying $k|f(x,u)| \leq V(x)|u|^{p-1}$ for $|x| \geq R$ is a solution of problem (1.1).

Moreover, the associated Euler–Lagrange functional $\hat{\mathfrak{I}}\colon E \to \mathbb{R}$ given by

$$\hat{\mathfrak{I}}(u) = \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{b}{2p}([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,dx - \int_{\mathbb{R}^N} \hat{G}(x,u)\,dx$$

is of class $C^1(E,\mathbb{R})$ and the critical points of $\hat{\mathfrak{I}}$ are weak solutions of (3.29). By assumption (f_4), we know that $\hat{\mathfrak{I}}(0) = 0$ and $\hat{\mathfrak{I}}$ is an even functional.

We choose an orthogonal basis $\{e_j\}$ of $X := E$ and define

$$Y_k := \operatorname{span}\{e_1, \ldots, e_k\}, \quad Z_k := Y_{k-1}^{\perp}.$$

In addition, to complete the proof of our result, we need the following Lemma.

Lemma 3.14. *Suppose that* (V_1) *holds. Then, for $p \le t < p_s^*$, we have*

$$\beta_k := \sup_{u\in Z_k, \|u\|_{E(B_R)}=1} \|u\|_{L^t(B_R)} \to 0, \quad k \to \infty.$$

Proof. It is clear that $0 < \beta_{k+1} \le \beta_k$, so that $\beta_k \to \beta \ge 0$, as $k \to \infty$. For every $k \in \mathbb{N}$, there is $u_k \in Z_k$ such that $\|u_k\|_{L^t(B_R)} > \frac{\beta_k}{2}$ and $\|u_k\|_{E(B_R)} = 1$. By the definition of Z_k, we can obtain that $u_k \rightharpoonup 0$ in E. By Lemma 2.1, the Sobolev embedding theorem implies that $u_k \to 0$ in $L^t(B_R)$. Thus, taking $k \to \infty$, we have proved that $\beta = 0$, which completes the proof. □

Next, we will verify that the functional $\hat{\mathfrak{I}}$ satisfies the remaining conditions of Theorem 3.13.

Lemma 3.15. *Suppose* (V_1)–(V_2) *and* (f_1)–(f_3) *hold. Then the functional $\hat{\mathfrak{I}}$ satisfies* (A_1).

Proof. As in the proof of Lemma 3.2, we consider $B_0 := B_{r_0}(x_0) \subset B_R(0)$ such that $V(x) \le V_\infty$ for every $x \in B_0$, and take a function $\phi \in Y_k \setminus \{0\}$, such that $\operatorname{supp}(\phi) \subset B_{r_0}(x_0)$. Then

$$\hat{\mathfrak{I}}(t\phi) \le \frac{at^p}{p}[\phi]^p_{W^{s,p}(B_0)} + \frac{bt^{2p}}{2p}([\phi]^p_{W^{s,p}(B_0)})^2 + t^p\int_{B_0} V_\infty|\phi|^p\,dx - C_1t^\theta\int_{B_0}|\phi|^\theta\,dx + C_2|B_0|,$$

where C_1, C_2 are given by (3.7). Since on the finite-dimensional space Y_k all norms are equivalent, it follows from $\theta > 2p$ that

$$a_k := \max_{u\in Y_k, \|u\|=\rho_k} \hat{\mathfrak{I}}(u) \le 0$$

for some $\rho_k > 0$ large enough. □

Lemma 3.16. *Suppose* (V_1)–(V_2) *and* (f_1)–(f_3) *hold. Then the functional $\hat{\mathfrak{I}}$ satisfies* (A_2).

Proof. As in the proof of Lemma 3.2, we consider the case $\Omega \ne \emptyset$. By (V_1) and (V_2), $\Omega \subset B_R(0)$ and $V(x) > 0$ for every $|x| \ge R$. From (f_1)–(f_2), we get

$$\hat{\mathfrak{I}}(u) \ge d_3\|u\|^p - \eta d_1\|u\|^p - d_2(\eta)\|u\|^{p_s^*} \quad \text{for all } u \in Z_k,$$

where d_1, d_2 are given by (3.5), and

$$d_3 := \min\left\{\frac{1}{p}\left(1 - \frac{\alpha|\Omega|^{\frac{sp}{N}}}{a\mathcal{S}}\right), \frac{k-1}{pk}\right\} > 0.$$

Then we have

$$\hat{\mathfrak{I}}(u) \ge d_4\|u\|^p - d_2(\eta)\beta_k^{p_s^*}\|u\|^{p_s^*} \quad \text{for all } u \in Z_k,$$

for enough small $\eta > 0$ such that $d_4 = d_3 - \eta d_1 > 0$. Using the above estimate and choosing $r_k := \beta_k^{p_s^*/(p-p_s^*)}$, we have

$$b_k := \inf_{u\in Z_k, \|u\|=r_k} \hat{\mathfrak{I}}(u) = \inf_{u\in Z_k, \|u\|=r_k} \hat{\mathfrak{I}}(u)|_{B_R} + \inf_{u\in Z_k, \|u\|=r_k} \hat{\mathfrak{I}}(u)|_{B_R^c} \ge \inf_{u\in Z_k, \|u\|=r_k} \hat{\mathfrak{I}}(u)|_{B_R} \ge (d_4 - d_2(\eta))\beta_k^{\frac{pp_s^*}{p-p_s^*}}.$$

Since, by Lemma 3.14, $\beta_k \to 0$ as $k \to \infty$ and $p_s^* > p$, we obtain

$$b_k \to +\infty.$$

Thus, (A_2) is proved. □

Proof of Theorem 1.5. Let $E = Y_k \oplus Z_k$. By (f_4) and Lemma 3.7, the functional $\hat{\mathcal{I}}$ satisfies the $(PS)_c$ condition, and $\hat{\mathcal{I}}$ satisfies (I_3). Then Lemmas 3.15 and 3.16 imply that all conditions of Theorem 3.13 are satisfied. Thus, from the Fountain Theorem, problem (3.29) possesses infinitely many nontrivial solutions by using the estimate provided by Lemma 3.12. Hence, problem (1.1) also possesses infinitely many nontrivial solutions. □

4 The case $p < \theta \le 2p$

4.1 The penalized problem

In this section, we study the existence of a positive solution for (1.1) in the case $p < \theta \le 2p$. For this, we first define a cut-off function $\zeta \in \mathcal{C}^1([0,\infty), \mathbb{R})$ (see Zhang and Du [55]) which satisfies

$$\zeta(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t \ge 2, \\ \max_{t>0} |\zeta'(t)| \le 2, & t > 0, \\ \zeta'(t) \le 0, & t > 0. \end{cases}$$

Moreover, using ζ, for any $T > 0$, we then define the truncated functional $\mathcal{I}_b^T(u)\colon E \to \mathbb{R}$ by

$$\mathcal{I}_b^T(u) = \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{b}{2p}\zeta\Big(\frac{\|u\|^p}{T^p}\Big)([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx.$$

By a standard argument, we can infer that $\mathcal{I}_b^T \in \mathcal{C}^1(E, \mathbb{R})$ and its Gateaux differential is

$$\begin{aligned}(\mathcal{I}_b^T)'(u)v &= a \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(v(x)-v(y))\,dx\,dy \\ &\quad + b\zeta\Big(\frac{\|u\|^p}{T^p}\Big)[u]^p_{W^{s,p}(\mathbb{R}^N)} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(v(x)-v(y))\,dx\,dy \\ &\quad + \frac{b}{2T^p}\zeta'\Big(\frac{\|u\|^p}{T^p}\Big)\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(v(x)-v(y))\,dx\,dy \\ &\qquad + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,dx\,dy\Big)([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\,dx\,dy - \int_{\mathbb{R}^N} g(x,u)v\,dx\end{aligned}$$

for all $u, v \in E$. With this penalization, by choosing an appropriate $T > 0$ and restricting $b > 0$ small enough, we may obtain a Cerami sequence $\{u_n\}_{n\in\mathbb{N}}$ of $\mathcal{I}_b^T$ satisfying $\|u_n\| \le T$, and so $\{u_n\}$ is also a Cerami sequence of $\mathcal{I}$ satisfying $\|u_n\| \le T$. Also, we are able to find a critical point u of $\mathcal{I}_b^T$ such that $\|u\| \le T$ and so u is also a critical point of $\mathcal{I}$.

In order to obtain the critical point for $\mathcal{I}_b^T$, we show that $\mathcal{I}_b^T$ satisfies the mountain pass geometry.

Lemma 4.1. *Suppose* (V_1)–(V_2), (f_1), (f_2) *and* $(\widetilde{f_3})$ *hold. Then we have the following:*
(I) *For any $T > 0$ and $b > 0$, there exist $\beta, \rho > 0$ (independent of T and b) such that $\mathcal{I}_b^T(u) \ge \beta$ for every $u \in E$ such that $\|u\| = \rho$.*
(II) *There exist $\mathring{b} > 0$ and a function $e \in E$ with $\|u\| \ge \rho$, such that for each $T > 0$ and $b \in (0, \mathring{b})$, we have $\mathcal{I}_b^T(e) < 0$.*

Proof. Similar to the proof of (1) in Lemma 3.2, we also give a proof for the case $\Omega \ne \emptyset$. By (V_1) and (V_2), $\Omega \subset B_R(0)$ and $V(x) > 0$ for every $|x| \ge R$. Then, by (f_1)–(f_2), combining (2.3) with (3.5), we also obtain

$$\begin{aligned}\mathcal{I}_b^T(u) &= \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{b}{2p}\zeta\Big(\frac{\|u\|^p}{T^p}\Big)([u]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx \\ &\ge d_3\|u\|^p - \eta d_1\|u\|^p + d_2\|u\|^{p_s^*},\end{aligned}$$

where d_1, d_2 are given by (3.5), and

$$d_3 := \min\left\{\frac{1}{p}\left(1 - \frac{\alpha|\Omega|^{\frac{sp}{N}}}{a\mathcal{S}}\right), \frac{k-1}{pk}\right\} > 0.$$

Using the above estimate and taking $\eta > 0$ sufficiently small, the item (I) follows by finding appropriated values of $\beta, \rho > 0$.

On the other hand, by hypotheses (V_1)–(V_2) and taking $V_\infty = 0$, if $\Omega \neq \emptyset$, we suppose that $B_{r_0}(x_0) \subset B_R(0)$ and $V(x) \leq V_\infty$, for each $x \in B_{r_0}(x_0)$. We first define the functional $\mathfrak{I}(u) \colon E \to \mathbb{R}$ by

$$\mathfrak{I}(u) = \frac{a}{p}[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,\mathrm{d}x - \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x.$$

Then, by (3.7), we can choose a positive smooth function $\phi \in E \setminus \{0\}$ such that $\mathrm{supp}(\phi) \subset B_{r_0}(x_0)$, to get

$$\mathfrak{I}(t\phi) \leq \frac{at^p}{p}[\phi]^p_{W^{s,p}(B_{r_0}(x_0))} + t^p \int_{B_{r_0}(x_0)} V_\infty|\phi|^p\,\mathrm{d}x - C_1 t^\theta \int_{B_{r_0}(x_0)} |\phi|^\theta\,\mathrm{d}x + C_2|B_{r_0}(x_0)| \to -\infty$$

as $t \to \infty$, since $p < \theta \leq 2p$. Thus, there exist $t_0 > 0$ large enough and $e = t_0\phi$ such that $\mathfrak{I}(e) \leq -1$ with $\|e\| > \rho$. Since

$$\mathfrak{J}^T_b(e) = \mathfrak{I}(e) + \frac{b}{2p}\zeta\left(\frac{\|e\|^p}{T^p}\right)([e]^p_{W^{s,p}(\mathbb{R}^N)})^2 \leq -1 + \frac{b}{2p}([e]^p_{W^{s,p}(\mathbb{R}^N)})^2,$$

there exists $\mathring{b} = \frac{2p}{[e]^{2p}_{W^{s,p}(\mathbb{R}^N)}} > 0$ such that $\mathfrak{J}^T_b(e) < 0$ for each $T > 0$ and $b \in (0, \mathring{b})$. The proof is complete. □

Remark 4.2. We point out that the function $e \in E \setminus \{0\}$ is a positive smooth function and does not depend on T and b.

Next, we recall the following version of the Mountain Pass Theorem which can be found in Ekeland [27].

Theorem 4.3. *Let X be a Banach space with its dual space X^*, and suppose that $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ satisfies*

$$\max\{\Phi(0), \Phi(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} \Phi(u)$$

for some $\mu < \eta, \rho > 0$ and $e \in X$ with $\|e\| > \rho$. Let $c \geq \eta$ be characterized by

$$c = \inf_{\varpi\in\Gamma} \max_{t\in[0,1]} \Phi(\varpi(t)),$$

where $\Gamma = \{\varpi \in \mathcal{C}([0,1], X) : \varpi(0) = 0,\ \varpi(1) = e\}$ is the set of continuous paths joining 0 and e. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$$\Phi(u_n) \to c \geq \eta \quad \textit{and} \quad (1 + \|u_n\|)\|\Phi'(u_n)\|_{X^*} \to 0 \quad \textit{as } n \to \infty.$$

By Lemma 4.1, we consider the mountain pass value

$$c^T_b := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathfrak{J}^T_b(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

From Lemma 4.1 and Theorem 4.3, we deduce that for each $T > 0$ and $b \in (0, \mathring{b})$, there exists a Cerami sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ (here we do not write the dependence on T and b) such that

$$\mathfrak{J}^T_b(u_n) \to c^T_b \quad \text{and} \quad (1 + \|u_n\|)\|(\mathfrak{J}^T_b)'(u_n)\|_{E^*} \to 0. \tag{4.1}$$

The above sequence is called a $(\mathrm{C})_{c^T_b}$-sequence for $\mathfrak{J}^T_b$.

Lemma 4.4. *For each $T > 0$ and $b \in (0, \mathring{b})$, there exist constants $\beta^T_1, \beta^T_2 > 0$ such that $\beta^T_1 \leq c^T_b \leq \beta^T_2$.*

Proof. Note that by Lemma 4.1, $c_b^T \geq \beta > 0$, and we take $\beta_1^T \in (0, \beta)$. On the other hand, fix e as in Lemma 4.1. Then it is easy to see that

$$\mathcal{J}_b^T(te) \leq \frac{at^p}{p}[e]^p_{W^{s,p}(B_{r_0}(x_0))} + \frac{\mathring{b}t^{2p}}{2p}([e]^p_{W^{s,p}(B_{r_0}(x_0))})^2 + t^p \int\limits_{B_{r_0}(x_0)} V_\infty |e|^p \,\mathrm{d}x - C_1 t^\theta \int\limits_{B_{r_0}(x_0)} |e|^\theta \,\mathrm{d}x + C_2|B_{r_0}(x_0)|.$$

Consequently, there exists a constant $\beta_2^T > 0$ (independent of T and b) such that

$$c_b^T \leq \max_{t\in[0,1]} \mathcal{J}_b^T(te) \leq \beta_2^T.$$

The proof is complete. □

In the following lemma, we shall show that for a properly chosen $T > 0$, after passing to a subsequence, the sequence $\{u_n\}_{n\in\mathbb{N}}$ given by (4.1) satisfies $\|u_n\| \leq T$, and so $\{u_n\}_{n\in\mathbb{N}}$ is also a bounded Cerami sequence of $\mathcal{J}$ satisfying $\|u_n\| \leq T$.

Lemma 4.5. *If $\{u_n\}_{n\in\mathbb{N}} \subset E$ is a Cerami sequence satisfying* (4.1)*, then, up to a subsequence, there exists $b^* > 0$ such that for any $b \in (0, b^*)$, there holds*

$$\|u_n\| \leq T.$$

In particular, this sequence $\{u_n\}_{n\in\mathbb{N}}$ is also a Cerami sequence at level c_b^T for $\mathcal{J}$, i.e.,

$$\mathcal{J}(u_n) \to c_b^T \quad \textit{and} \quad (1 + \|u_n\|)\|\mathcal{J}'(u_n)\|_{E^*} \to 0.$$

Proof. Suppose by contradiction, for any $T > 0$, there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (still denoted by $\{u_n\}_{n\in\mathbb{N}}$) such that $\|u_n\| > T$. We divide the proof into the two cases $\|u_n\|^p \geq 2T^p$ and $T^p < \|u_n\|^p \leq 2T^p$.

Case (i). Assume that $\|u_n\|^p \geq 2T^p$ holds. By Lemma 4.4 and $k = p\theta/(\theta - p)$, we have

$$\begin{aligned}
\beta_2^T + 1 &\geq c_b^T + 1 \\
&\geq \mathcal{J}_b^T(u_n) - \frac{1}{\theta}(\mathcal{J}_b^T)'(u_n)u_n \\
&= \Big(\frac{a}{p} - \frac{a}{\theta}\Big)[u_n]^p_{W^{s,p}(\mathbb{R}^N)} + \Big(\frac{b}{2p} - \frac{b}{\theta}\Big)\zeta\Big(\frac{\|u_n\|^p}{T^p}\Big)([u_n]^p_{W^{s,p}(\mathbb{R}^N)})^2 \\
&\quad - \frac{b}{2\theta T^p}\zeta'\Big(\frac{\|u_n\|^p}{T^p}\Big)\|u_n\|^p([u_n]^p_{W^{s,p}(\mathbb{R}^N)})^2 + \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\int\limits_{\mathbb{R}^N} V(x)|u_n|^p \,\mathrm{d}x \\
&\quad + \int\limits_{\mathbb{R}^N} \Big(\frac{1}{\theta}g(x, u_n)u_n - G(x, u_n)\Big)\,\mathrm{d}x \\
&\geq \Big(\frac{a}{p} - \frac{a}{\theta}\Big)[u_n]^p_{W^{s,p}(\mathbb{R}^N)} + \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\int\limits_{\mathbb{R}^N} V(x)|u_n|^p \,\mathrm{d}x \\
&\quad + \int\limits_{B_R(0)} \Big(\frac{1}{\theta}f(x, u_n)u_n - F(x, u_n)\Big)\,\mathrm{d}x + \frac{p-\theta}{\theta p k}\int\limits_{\mathbb{R}^N\setminus B_R(0)} V(x)|u_n|^p \,\mathrm{d}x \\
&\geq \Big(\frac{1}{p} - \frac{1}{\theta}\Big)a[u_n]^p_{W^{s,p}(\mathbb{R}^N)} + \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\int\limits_{B_R(0)} V(x)|u_n|^p \,\mathrm{d}x + \frac{(\theta-p)(\theta p - \theta + p)}{(\theta p)^2}\int\limits_{\mathbb{R}^N\setminus B_R(0)} V(x)|u_n|^p \,\mathrm{d}x \\
&\geq K\Big(a[u_n]^p_{W^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} V(x)|u_n|^p \,\mathrm{d}x\Big) = K\|u_n\|^p,
\end{aligned}$$

where $K = \min\{\frac{1}{p} - \frac{1}{\theta}, \frac{(\theta-p)(\theta p-\theta+p)}{(\theta p)^2}\} > 0$, for n large enough, which is a contradiction when $T > 0$ is large enough.

Case (ii). Assume that $T^p < \|u_n\|^p \leq 2T^p$ holds. We have

$$0 \leq \zeta\Big(\frac{\|u_n\|^p}{T^p}\Big) \leq 1 \quad \text{and} \quad \zeta'\Big(\frac{\|u_n\|^p}{T^p}\Big) \leq 0 \leq \Big|\zeta'\Big(\frac{\|u_n\|^p}{T^p}\Big)\Big| \leq 2,$$

which shows that

$$\begin{aligned}
K\|u_n\|^p - \frac{1}{\theta}\|(\mathcal{I}_b^T)'(u_n)\|_{E^*}\|u_n\| &\le K\|u_n\|^p + \frac{1}{\theta}(\mathcal{I}_b^T)'(u_n)u_n \\
&\le \mathcal{I}_b^T(u_n) + \Big(\frac{b}{\theta} - \frac{b}{2p}\Big)\zeta\Big(\frac{\|u_n\|^p}{T^p}\Big)([u_n]^p_{W^{s,p}(\mathbb{R}^N)})^2 \\
&\quad + \frac{b}{2\theta T^p}\zeta'\Big(\frac{\|u_n\|^p}{T^p}\Big)\|u\|^p([u_n]^p_{W^{s,p}(\mathbb{R}^N)})^2 \\
&\le \mathcal{I}_b^T(u_n) + 4b\Big(\frac{3}{r} - \frac{1}{2p}\Big)T^{2p} \\
&=: \mathcal{I}_b^T(u_n) + CbT^{2p}.
\end{aligned} \tag{4.2}$$

Let $e \in E \setminus \{0\}$ be as in Lemma 4.1. By $\mathcal{I}_b^T(u_n) \to c_b^T$ as $n \to \infty$, we have

$$\mathcal{I}_b^T(u_n) \le 2c_b^T \le 2\max_{t\in[0,1]} \mathcal{I}_b^T(te) \le 2\beta_2^T \tag{4.3}$$

for n large enough. Moreover, we obtain

$$K\|u_n\|^p - \frac{1}{\theta}\|(\mathcal{I}_b^T)'(u_n)\|_{E^*}\|u_n\| \ge KT^p - T. \tag{4.4}$$

So from (4.2), (4.3) and (4.4) we get

$$KT^p - T \le 2\beta_2^T + CbT^{2p},$$

which is a contradiction if $b^T := \frac{1}{T^{2p}} > 0$, $b^* = \min\{\mathring{b}, b^T\}$ and $b \in (0, b^*)$ for T large enough. Thus, we obtain $\|u_n\| \le T$. □

By Lemma 4.5, the embeddings of E in $W^{s,p}(\mathbb{R}^N)$ and the Sobolev embedding theorem, up to a subsequence, we may suppose that there exists $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } E, \\ u_n \to u & \text{strongly in } L^t_{\mathrm{loc}}(\mathbb{R}^N), \text{ for all } t \in [1, p_s^*), \\ u_n(x) \to u(x) & \text{for a.a. } x \in \mathbb{R}^N. \end{cases}$$

Similar to the proof of Lemma 3.7, we can obtain that for all $b \in (0, b^*)$, $\{u_n\}_{n\in\mathbb{N}} \subset E$ contains a convergent subsequence. Furthermore, $u^- = \min\{u, 0\} = 0$. Since $c > 0$, from a Moser iteration argument and the maximum principle, we can get that u is a positive solution of problem (3.3). It remains to verify that u is also a positive solution of problem (1.1).

4.2 Existence results for problem (1.1)

Proof of Theorem 1.9. From Lemmas 4.1–4.5 and the estimate

$$|u|_{L^{p_s^*}(\Omega)} \le \mathcal{S}^{-1}[u]^p_{W^{s,p}(\mathbb{R}^N)} \quad \text{for every } b \in (0, b^*),$$

problem (3.3) has a positive solution $u \in E$. Next, using hypotheses (f_1)–(f_2), there exists a constant $C > 0$ such that

$$|f(x,z)| \le C|z|^{\vartheta-1} \quad \text{for all } |x| \ge R.$$

Thus, we still have the inequality

$$|f(x,u(x))| \le CM^{(\vartheta-p)}\Big(\frac{R}{|x|}\Big)^{\frac{(N-sp)(\vartheta-p)}{p-1}}|u(x)|^{p-1} \quad \text{for all } |x| \ge R.$$

Fixing $\Lambda^* = kCM^{(\vartheta-p)}R^{\frac{(N-sp)(\vartheta-p)}{p-1}}$ and $\Lambda \ge \Lambda^* > 0$, it follows from (V_2) that

$$|f(x,u(x))| \le \frac{1}{k}V(x)|u(x)|^{p-1} \quad \text{for all } |x| \ge R.$$

It follows that u is a positive solution of (1.1). The proof of Theorem 1.9 is complete. □

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