Dynamic Systems and Applications 22 (2013) 385-396

BOUNDARY VALUE PROBLEMS WITH NONSMOOTH POTENTIAL, CONSTRAINTS AND PARAMETERS

GABRIELE BONANNO[a], DUMITRU MOTREANU[b], AND PATRICK WINKERT[c]

[a]Department of Civil, Information Technology, Construction, Environmental Engineering and Applied Mathematics, University of Messina, 98166 - Messina, Italy
bonanno@unime.it

[b]Département de Mathématiques, Université de Perpignan, Avenue Paul Alduy 52, 66860 Perpignan, France
motreanu@univ-perp.fr

[c]Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany
winkert@math.tu-berlin.de

ABSTRACT. The paper focuses on the existence of multiple solutions of variational- hemivariational inequalities depending on parameters and involving the p-Laplacian operator on a bounded domain $\Omega \subset \mathbb{R}^N$. The parameters relevant for the solvability are precisely estimated. In the previous works such results have been obtained by assuming that $N < p$. Here we treat the case $N > p$, which is the main novelty of our work. The paper contains results regarding positive solutions as well.

AMS Subject Classification. 35J87, 49J40, 49J52, 49J53

1. INTRODUCTION

The theory of hemivariational and variational-hemivariational inequalities deals with various problems in the form of inequalities containing nonlinear discontinuities and constraints. A multitude of techniques have been developed to study this type of problems, among which we mention variational methods connected to nonsmooth critical point theory (see, e.g., [12], [13]). In this respect, nonsmooth versions of the variational principle of Ricceri [15] have been utilized to show the existence of multiple solutions for hemivariational and variational-hemivariational inequalities formulated as boundary value problems and depending on parameters. These results enable us to find estimates for the range of parameters where the corresponding problems have multiple solutions. We refer to [1], [2], [3], [7], [8], [11], [14] for recent results in this direction. All these results establish the existence of multiple solutions of variational-hemivariational inequalities depending on parameters and involving the p-Laplacian

Received January 30, 2013

operator on a bounded domain $\Omega \subset \mathbb{R}^N$ under the key assumption $N < p$. The goal of the present paper is to obtain such a result when $N > p$.

We describe the problem that we study here. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $N \geq 2$ and a C^1-boundary $\partial\Omega$ and let $1 < p < N$. We denote by $W_0^{1,p}(\Omega)$ the usual Sobolev space consisting of the elements of $W^{1,p}(\Omega)$ with zero traces on $\partial\Omega$. Given a closed, convex set $K \subset W_0^{1,p}(\Omega)$, consider the following inequality problem with constraints in K and depending on a real parameter $\lambda > 0$: Find $u \in K$ such that

$$\begin{aligned}&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot(\nabla v(x)-\nabla u(x))dx\\&\quad+\lambda\int_\Omega \alpha(x)F^\circ(u(x);v(x)-u(x))dx \geq 0 \qquad \text{for all } v \in K.\end{aligned} \tag{1.1}$$

Problems like (1.1) are called variational-hemivariational inequalities. In (1.1) it is supposed that $\alpha \in L^1(\Omega)$ satisfies $\operatorname{ess\,inf}_{x\in\Omega}\alpha(x) > 0$ and F° stands for Clarke's generalized directional derivative of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ for which we assume the subcritical growth condition

$$|\xi| \leq b_1 + b_2|t|^{s-1} \quad \text{ for all } t \in \mathbb{R},\ \xi \in \partial F(t), \tag{1.2}$$

with constants $b_1, b_2 \geq 0$ and $1 < s < p^* := \frac{Np}{N-p}$. The notation ∂F in (1.2) means the generalized gradient of F. We note that under assumption (1.2), the integrals in (1.1) are well defined.

Our main result, which is stated as Theorem 3.1, provides a precise interval for the parameter λ such that the corresponding problem (1.1) for such a λ admits at least three distinct weak solutions. The main novelty of this result is the fact that it holds in the case $N > p$. We point out that a natural choice for the set of constraints K in (1.1) is the cone of nonnegative elements of $W_0^{1,p}(\Omega)$, that is

$$K = \{u \in W_0^{1,p}(\Omega) : u \geq 0 \text{ for a.a. } x \in \Omega\}. \tag{1.3}$$

With the same data α and F as above, we also consider the (unconstrained) problem depending on a real parameter $\lambda > 0$: Find $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx\\&\quad+\lambda\int_\Omega \alpha(x)F^\circ(u(x);v(x))dx \geq 0 \quad \text{for all } v \in W_0^{1,p}(\Omega).\end{aligned} \tag{1.4}$$

Problems as the one stated in (1.4) are called hemivariational inequalities. There is a fundamental difference between the problems (1.1) and (1.4), which consists in the fact that (1.1) is verified through the set of constraints K whose elements act as test functions, whereas in (1.4) we act with any element of $W_0^{1,p}(\Omega)$. We want to explore the connection between the two problems related to the nonnegative solutions. Namely, take in (1.1) as set of constraints K the set introduced in (1.3), so a solution of (1.1) is a nonnegative function. The question that we address is under what conditions this nonnegative function is also a (nonnegative) solution of (1.4). The converse assertion is clearly true, but generally not the raised one because the test functions in (1.4) are in the whole space $W_0^{1,p}(\Omega)$. Our results on this topic are given in Theorems 4.1 and 4.2, where we show that, under a verifiable condition on the generalized gradient ∂F, the question above has a positive answer meaning that the nonnegative solutions of (1.1) (with K in (1.3)) become solutions of (1.4).

The rest of the paper is organized as follows. Section 2 introduces the necessary mathematical background to be used later in the paper. Section 3 presents our main result on the multiple solutions of problem (1.1). Section 4 examines the connection between the nonnegative solutions of problems (1.1) and (1.4).

2. **PRELIMINARIES**

Let us start by recalling some basic notions in non-smooth analysis that are required in the sequel. For a real Banach space $(X, \|\cdot\|)$, we denote by X^* its dual space and by $\langle\cdot,\cdot\rangle$ the duality pairing between X and X^*. A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exist a neighborhood U_x of x and a constant $L_x \geq 0$ such that

$$|f(y) - f(z)| \leq L_x\|y - z\| \quad \text{for all } y, z \in U_x.$$

For a locally Lipschitz function $f : X \to \mathbb{R}$ on a Banach space X, the generalized directional derivative of f at the point $x \in X$ along the direction $y \in X$ is defined by

$$f^\circ(x; y) := \limsup_{z\to x, t\to 0^+} \frac{f(z + ty) - f(z)}{t}$$

(see [6, Chapter 2]). If f_1, $f_2 : X \to \mathbb{R}$ are locally Lipschitz functions, then we have

$$(f_1 + f_2)^\circ(x, y) \leq f_1^\circ(x, y) + f_2^\circ(x, y) \quad \text{for all } x, y \in X. \tag{2.1}$$

The generalized gradient of a locally Lipschitz function $f : X \to \mathbb{R}$ at $x \in X$ is the set

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y\rangle \leq f^\circ(x; y) \ \text{ for all } y \in X\}.$$

An element $x \in X$ is said to be a critical point of a locally Lipschitz function $f : X \to \mathbb{R}$ if there holds

$$f^\circ(x; y) \geq 0 \quad \text{for all } y \in X$$

or, equivalently, $0 \in \partial f(x)$ (see [5]). More generally, for a function $I : X \to]-\infty, +\infty]$ expressed as $I = f + j$ with $f : X \to \mathbb{R}$ locally Lipschitz and $j : X \to]-\infty, +\infty]$ convex, lower semicontinuous function, $\not\equiv +\infty$, an element $u \in X$ is called a critical point of I if

$$f^\circ(u; v - u) + j(v) - j(u) \geq 0 \quad \text{for all } v \in X$$

is satisfied (see [12, Chapter 3] and [13]).

Our approach in studying problem (1.1) relies on an abstract three critical points theorem that we now describe. On a reflexive Banach space X, there are given a sequentially weakly lower semicontinuous and coercive function $\Phi : X \to \mathbb{R}$, a sequentially weakly upper semicontinuous function $\Upsilon : X \to \mathbb{R}$ and a convex, lower semicontinuous function $j : X \to]-\infty, +\infty]$ whose effective domain $D(j) = \{x \in X : j(x) < +\infty\}$ fulfills

$$D(j) \cap \Phi^{-1}(]-\infty, r[) \neq \emptyset \qquad \text{for all } r > \inf_X \Phi.$$

Set

$$\Psi := \Upsilon - j$$

and, for a real parameter $\lambda > 0$,

$$J_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j,$$

$$\varphi_1(r) = \inf_{y\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{x\in\Phi^{-1}(]-\infty,r[)} \Psi(x)\right) - \Psi(y)}{r - \Phi(y)} \quad \text{for all } r > \inf_X \Phi,$$

$$\varphi_2(r) = \sup_{y\in\Phi^{-1}(]r,+\infty[)} \frac{\Psi(y) - \left(\sup_{x\in\Phi^{-1}(]-\infty,r])} \Psi(x)\right)}{\Phi(y) - r} \quad \text{for all } r < \sup_X \Phi.$$

The following result was recently proved in [3, Theorem 2.1].

Theorem 2.1. *Assume that there is $r \in]\inf_X \Phi, \sup_X \Phi[$ such that $\varphi_1(r) < \varphi_2(r)$ and the functional J_λ is bounded from below and satisfies the (PS)-condition for each $\lambda \in \Lambda := \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[$. Then, for each $\lambda \in \Lambda$, J_λ has at least three distinct critical points.*

Such results originate in Ricceri's work (see [15] and the references therein). Nonsmooth versions can be found in [1] and [11].

3. MAIN RESULT

The Sobolev embedding theorem ensures the existence of a constant $c_{p^*} > 0$ such that

$$\|u\|_{L^{p^*}(\Omega)} \le c_{p^*}\|u\|_{W_0^{1,p}(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{3.1}$$

The expression of the best such constant is known

$$c_{p^*} = \frac{1}{\sqrt{\pi}}\frac{1}{N^{\frac{1}{p}}}\left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}}\left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(1+N-\frac{N}{p}\right)}\right)^{\frac{1}{N}}$$

(see Talenti [16]). As $s < p^*$ in (1.2), we have that

$$\|u\|_{L^s(\Omega)} \le c_s\|u\|_{W_0^{1,p}(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega), \tag{3.2}$$

with a positive constant c_s which can be evaluated through Hölder's inequality and (3.1) as follows

$$c_s \leq |\Omega|^{\frac{p^*-s}{p^*s}} \frac{1}{\sqrt{\pi}} \frac{1}{N^{\frac{1}{p}}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p}\right)\Gamma\left(1+N-\frac{N}{p}\right)}\right)^{\frac{1}{N}}.$$

Denote

$$D := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega), \tag{3.3}$$

so it is clear that there is $x_0 \in \Omega$ such that the open ball $B(x_0, D)$ is contained in Ω. Using the positive constants b_1, b_2 in (1.2), c_1, c_s in (3.2) and D in (3.3), we set

$$\begin{aligned} K_1 &:= \frac{\|\alpha\|_{L^\infty(\Omega)} b_1 c_1 p^{\frac{1}{p}} p^{p-1}(p^N-(p-1)^N)}{(p-1)^N D^p}, \\ K_2 &:= \frac{\|\alpha\|_{L^\infty(\Omega)} b_2 c_s^s p^{\frac{s}{p}} p^{p-1}(p^N-(p-1)^N)}{s(p-1)^N D^p}, \end{aligned} \tag{3.4}$$

$$\kappa := \frac{Dp^{\frac{1}{p}}}{p\pi^{\frac{N}{2p}}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)}{D^N - \left(\frac{(p-1)D}{p}\right)^N}\right)^{\frac{1}{p}}. \tag{3.5}$$

In view of the above remarks, the numbers K_1, K_2, κ in (3.4), (3.5) can be effectively estimated.

Now we state our main result.

Theorem 3.1. *Assume that $\alpha \in L^\infty(\Omega)$ fulfills* $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$, *$F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, with $F(0) = 0$, satisfying the subcritical growth condition (1.2) and K is a nonempty, closed, convex subset of $W_0^{1,p}(\Omega)$. In addition, we suppose:*

(H1) $K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} < \operatorname*{ess\,inf}_{x\in\Omega} \alpha(x) \frac{[-F(a_2)]}{a_2^p}$ *for positive constants a_1, a_2 with $a_2 > \kappa a_1$ (see (3.5)), and $K_i, i = 1, 2$, given in (3.4);*

(H2) $\limsup_{|\xi|\to+\infty} \frac{-F(\xi)}{|\xi|^p} \leq 0$*;*

(H3) $-F(t) \geq 0$ *for all $t \in [0, a_2]$;*

(H4) $u_{a_2} \in K$*, where*

$$u_{a_2}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, D), \\ \frac{pa_2}{D}(D - |x-x_0|) & \text{if } x \in B(x_0, D) \setminus B\left(x_0, \frac{(p-1)D}{p}\right), \\ a_2 & \text{if } x \in B\left(x_0, \frac{(p-1)D}{p}\right) \end{cases}$$

for all $x \in \Omega$*. Then, for every* $\lambda \in \Lambda$ *with*

$$\Lambda := \left] \frac{p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p} \frac{1}{\text{ess inf}_{x\in\Omega}\, \alpha(x)} \frac{a_2^p}{[-F(a_2)]}, \right. \\ \left. \frac{p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p} \frac{1}{K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}} \right[, \tag{3.6}$$

problem (1.1) *possesses at least three distinct weak solutions.*

Proof. Fix $\lambda \in \Lambda$. We are going to apply Theorem 2.1 on the space $X = W_0^{1,p}(\Omega)$. To this end, for any $u \in W_0^{1,p}(\Omega)$ we define

$$\Phi(u) := \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}, \qquad j(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\Upsilon(u) = \int_\Omega \alpha(x)[-F(u(x))]dx.$$

As in Section 1 we set

$$\Psi(u) = \Upsilon(u) - j(u), \qquad J_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

which results in

$$J_\lambda(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \lambda \int_\Omega \alpha(x)[-F(u(x))]dx.$$

Recall the function φ_1 in Section 2. Since $F(0) = 0$, it follows that

$$\varphi_1(r) \le \frac{\sup_{\Phi(u)<r} \Upsilon(u)}{r} \text{ for all } r > 0. \tag{3.7}$$

From (1.2), (3.2) and (3.4), we derive the estimate

$$\begin{aligned} \Upsilon(u) &\le \|\alpha\|_{L^\infty(\Omega)} b_1 \|u\|_{L^1(\Omega)} + \|\alpha\|_{L^\infty(\Omega)} \frac{b_2}{s} \|u\|^s_{L^s(\Omega)} \\ &\le \|\alpha\|_{L^\infty(\Omega)} b_1 c_1 \|u\|_{W_0^{1,p}(\Omega)} + \|\alpha\|_{L^\infty(\Omega)} \frac{b_2}{s} c_s^s \|u\|^s_{W_0^{1,p}(\Omega)} \\ &= \frac{(p-1)^N D^p K_1}{p^{\frac{1}{p}} p^{p-1}(p^N - (p-1)^N)} \|u\|_{W_0^{1,p}(\Omega)} \\ &\quad + \frac{(p-1)^N D^p K_2}{p^{\frac{s}{p}} p^{p-1}(p^N - (p-1)^N)} \|u\|^s_{W_0^{1,p}(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.8}$$

Then (3.7) and (3.8) yield

$$\varphi_1(a_1^p) \le \frac{(p-1)^N D^p}{p^{p-1}(p^N - (p-1)^N)} \left(K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} \right). \tag{3.9}$$

By the definition of the function u_{a_2} we obtain

$$\begin{aligned}\Phi(u_{a_2}) &= \frac{1}{p}\int_{B(x_0,D)\setminus B\left(x_0,\frac{(p-1)D}{p}\right)} \frac{(pa_2)^p}{D^p}dx \\ &= \frac{1}{p}\frac{(pa_2)^p}{D^p}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right))}\left(D^N - \left(\frac{(p-1)D}{p}\right)^N\right).\end{aligned} \tag{3.10}$$

From hypothesis (H1) we know that $a_2 > \kappa a_1$, so (3.5) implies

$$\Phi(u_{a_2}) > a_1^p. \tag{3.11}$$

On the other hand, thanks to (H3) and (H4) we may write

$$\Psi(u_{a_2}) \geq -F(a_2)\operatorname*{ess\,inf}_{x\in\Omega}\alpha(x)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\frac{(p-1)^N D^N}{p^N}. \tag{3.12}$$

Combining (3.10) and (3.12) leads to

$$\frac{\Psi(u_{a_2})}{\Phi(u_{a_2})} \geq \frac{D^p}{p^{p-1}}\frac{(p-1)^N}{p^N-(p-1)^N}\operatorname*{ess\,inf}_{x\in\Omega}\alpha(x)\frac{[-F(a_2)]}{a_2^p}. \tag{3.13}$$

Now, on the basis of (H1), we obtain from (3.9) and (3.13) that

$$\varphi_1\left(a_1^p\right) < \frac{(p-1)^N D^p}{p^{p-1}(p^N-(p-1)^N)}\operatorname*{ess\,inf}_{x\in\Omega}\alpha(x)\frac{[-F(a_2)]}{a_2^p} \leq \frac{\Psi(u_{a_2})}{\Phi(u_{a_2})}. \tag{3.14}$$

Using (3.8), hypothesis (H1) and (3.13) ensures that

$$\begin{aligned}\frac{\sup_{\Phi(u)\leq a_1^p}\Psi(u)}{a_1^p} &\leq \frac{(p-1)^N D^p}{p^{p-1}(p^N-(p-1)^N)}\left(K_1\frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}\right) \\ &< \frac{(p-1)^N D^p}{p^{p-1}(p^N-(p-1)^N)}\operatorname*{ess\,inf}_{x\in\Omega}\alpha(x)\frac{[-F(a_2)]}{a_2^p} \\ &\leq \frac{\Psi(u_{a_2})}{\Phi(u_{a_2})}.\end{aligned} \tag{3.15}$$

Let us recall the function φ_2 introduced in Section 2. Then (3.11) and (3.15) enable us to get

$$\varphi_2(a_1^p) \geq \frac{\Psi(u_{a_2}) - \sup_{\Phi(u)\leq a_1^p}\Psi(u)}{\Phi(u_{a_2}) - a_1^p} > \frac{\Psi(u_{a_2}) - a_1^p\frac{\Psi(u_{a_2})}{\Phi(u_{a_2})}}{\Phi(u_{a_2}) - a_1^p} = \frac{\Psi(u_{a_2})}{\Phi(u_{a_2})}. \tag{3.16}$$

It turns out from (3.14), (3.16) and (3.11) that

$$\varphi_1(a_1^p) < \varphi_2(a_1^p) \quad\text{and}\quad a_1^p \in]0, \Phi(u_{a_2})[.$$

Hence the assumption of Theorem 2.1 holds true for $r = a_1^p$.

Let us prove the coercivity of J_λ for every $\lambda\in\Lambda$ with Λ given in (3.6). Fix $\lambda\in\Lambda$ and a constant $\tau > 0$ satisfying

$$\tau < \frac{1}{\lambda p c_p^p \|\alpha\|_{L^\infty(\Omega)}}.$$

By (1.2) and (H2), we have the estimate

$$-F(t) \le \tau|t|^p + \tilde{c} \quad \text{for all } t \in \mathbb{R} \tag{3.17}$$

with a constant $\tilde{c} \ge 0$. Then (3.17) and (3.2) with $s = p$ imply

$$J_\lambda(u) = \Phi(u) - \lambda\Psi(u) \ge \left[\frac{1}{p} - \lambda\tau c_p^p\|\alpha\|_{L^\infty(\Omega)}\right]\|u\|_{W^{1,p}(\Omega)}^p - \lambda\tilde{c}\|\alpha\|_{L^\infty(\Omega)},$$

for all $u \in W_0^{1,p}(\Omega)$. This shows that J_λ is coercive, so the Palais-Smale condition for J_λ follows (see [11, Proposition 2.3]). Therefore all the assumptions of Theorem 2.1 are fulfilled, so Theorem 2.1 with $r = a_1^p$ applies.

Gathering some estimates obtained in (3.16), (3.15) and (3.9) we infer that

$$\begin{aligned}\frac{1}{\varphi_2(a_1^p)} &< \frac{\Phi(u_{a_2})}{\Psi(u_{a_2})} \le \frac{p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p}\frac{a_2^p}{(\operatorname{ess\,inf}_{x\in\Omega}\alpha(x))[-F(a_2)]}\\ &< \frac{p^{p-1}(p^N - (p-1)^N)}{(p-1)^N D^p}\frac{1}{K_1\frac{1}{a_1^{p-1}} + K_2 a_1^{s-p}} \le \frac{1}{\varphi_1(a_1^p)}.\end{aligned}$$

It follows that the interval Λ obtained in Theorem 2.1 with $r = a_1^p$ contains the interval Λ in (3.6). We have thus shown that for any λ belonging to the interval Λ in (3.6) there exist at least three distinct critical points of J_λ.

Let u be a critical point of J_λ, which reads as

$$(\Phi - \lambda\Upsilon)^\circ(u; v-u) + \lambda j(v) - \lambda j(u) \ge 0 \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

This amounts to saying that $u \in K$ and

$$(\Phi - \lambda\Upsilon)^\circ(u; v-u) \ge 0 \quad \text{for all } v \in K. \tag{3.18}$$

Thanks to (2.1), it is seen from (3.18) that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot(\nabla v(x) - \nabla u(x))dx + \lambda(-\Upsilon)^\circ(u; v-u) \ge 0 \quad \text{for all } v \in K. \tag{3.19}$$

We note that assumption (1.2) guarantees the applicability of formula (2) in [6, p. 77] to infer that

$$(-\Upsilon)^\circ(u; v-u) \le \int_\Omega \alpha(x)F^\circ(u(x); v(x) - u(x))dx. \tag{3.20}$$

Then (3.19) and (3.20) lead to

$$\begin{aligned}&\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot(\nabla v(x) - \nabla u(x))dx\\ &\qquad + \lambda\int_\Omega \alpha(x)F^\circ(u(x); v(x) - u(x))dx \ge 0 \quad \text{for all } v \in K.\end{aligned}$$

This means exactly that u is a solution of problem (1.1). The proof is thus complete. $\square$

It is interesting that the interval Λ determined in Theorem 3.1 does not depend on the set of constrains K provided assumptions (H1)-(H4) are fulfilled. A meaningful case of Theorem 3.1 is when the closed convex set K is the cone of nonnegative functions.

Corollary 3.2. *Let the functions α and F be as in Theorem* 3.1 *and satisfy hypotheses* (H1)–(H3). *If $\lambda \in \Lambda$ for Λ in* (3.6)*, then problem* (1.1) *with K in* (1.3) *possesses at least three distinct nonnegative weak solutions.*

Proof. Note that condition (H4) holds true for the set K in (1.3). Since the assumptions of Theorem 3.1 are satisfied, we may apply Theorem 3.1 which yields the stated result. □

4. NONNEGATIVE SOLUTIONS

In the setting of Corollary 3.2 we can compare the obtained solutions with the nonnegative solutions of the hemivariational inequality (1.4). Obviously, every nonnegative solution of (1.4) is a solution of (1.1) with K in (1.3). The next theorem addresses the converse assertion which occurs by strengthening hypothesis (H3).

Theorem 4.1. *Assume that $\alpha \in L^\infty(\Omega)$ satisfies* $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$ *and $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, with $F(0) = 0$, for which the subcritical growth condition* (1.2) *holds true and in addition*

(H3') *there is a constant $a_2 > 0$ such that*

$$\xi \leq 0 \quad \text{for all } \xi \in \partial F(t) \text{ with } t \in [0, a_2]. \tag{4.1}$$

Then every solution $u \geq 0$ of problem (1.1)*, with K in* (1.3) *and any $\lambda > 0$, that is*

$$\begin{aligned} &\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla(v(x) - u(x))dx \\ &\quad + \lambda \int_\Omega \alpha(x) F^\circ(u(x); v(x) - u(x))dx \geq 0 \end{aligned} \tag{4.2}$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geq 0$, and which satisfies $u(x) \in [0, a_2]$ for all $x \in \Omega$, is a (nonnegative) solution of problem (1.4).

Proof. Let u be a solution of problem (1.1) with K in (1.3) satisfying $u(x) \in [0, a_2]$ for all $x \in \Omega$. We have to show that u is a solution of (1.4) if assumption (H3') is satisfied. To this end we follow an idea in the proof of Ma [10, Theorem 1.7]. Let $\varepsilon > 0$ and $w \in W_0^{1,p}(\Omega)$. Setting $w_\varepsilon = -\min\{0, u + \varepsilon w\}$, we have that $v = u + \varepsilon w + w_\varepsilon \in K$ (see (1.3)). Then (4.2) and (4.1), in conjunction with the subadditivity of $F^0(u; \cdot)$ and a basic property of the generalized gradient ∂F (see [6, p. 10]), yield

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx + \lambda \int_\Omega \alpha(x) F^\circ(u(x); w(x)) dx$$

$$
\begin{aligned}
&\geq -\varepsilon^{-1} \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w_\varepsilon(x) dx - \varepsilon^{-1} \lambda \int_\Omega \alpha(x) F^\circ(u(x); w_\varepsilon(x)) dx \\
&\geq \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx \\
&\quad - \varepsilon^{-1} \lambda \int_{\{u+\varepsilon w<0\}} \alpha(x) \max_{\xi \in \partial F(u(x))} [\xi(-u(x) - \varepsilon w(x))] \, dx \\
&= \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx \\
&\quad + \varepsilon^{-1} \lambda \int_{\{u+\varepsilon w<0\}} \alpha(x) \left(\max \partial F(u(x))\right) (u(x) + \varepsilon w(x)) dx \\
&\geq \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx.
\end{aligned}
$$

Since $\nabla u = 0$ on $\{u = 0\}$ (see, e.g., [9, Lemma 7.7]), letting $\varepsilon \to 0$ in the inequality above leads to the conclusion that u solves problem (1.4). □

Next we set forth a further connection between the problems (1.1) and (4.2) which is related to a priori estimates of the solutions.

Theorem 4.2. *Assume that $\alpha \in L^\infty(\Omega)$ satisfies $\operatorname{ess\,inf}_{x\in\Omega} \alpha(x) > 0$, $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, with $F(0) = 0$, for which the subcritical growth condition (1.2) holds true, and in addition*

(H5) *there is a constant $a_3 > 0$ such that*

$$\max \partial F(t) \geq 0 \quad \text{for all } t \geq a_3.$$

Then for every solution $u \geq 0$ of problem (1.1) with K in (1.3), the following alternative hold: either $\inf u < a_3$ or u solves problem (1.4).

Proof. Let u be a solution of problem (1.1) with K in (1.3) and some $\lambda > 0$ (equivalently, u solves problem (4.2)) such that $u \geq a_3$. For any $\varepsilon > 0$ and $w \in W_0^{1,p}(\Omega)$, taking the same test function $v = u + \varepsilon w + w_\varepsilon$ as in the proof of Theorem 4.2, the calculation therein gives

$$
\begin{aligned}
&\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx + \lambda \int_\Omega \alpha(x) F^\circ(u(x); w(x)) dx \\
&\geq \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx \\
&\quad + \varepsilon^{-1} \lambda \int_{\{u+\varepsilon w<0\}} \alpha(x) \left(\max \partial F(u(x))\right) (u(x) + \varepsilon w(x)) dx \\
&\geq \int_{\{u+\varepsilon w<0\}} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx \\
&\quad + \lambda \int_{\{u+\varepsilon w<0\}} \alpha(x) \left(\max \partial F(u(x))\right) w(x) dx.
\end{aligned}
$$

The last inequality is valid due to hypothesis (H5) because $u \geq a_3 > 0$. Letting $\varepsilon \to 0$ in the inequalities above we derive that u is a solution of problem (1.4), which completes the proof. $\square$

Remark 4.3. It is shown in [4] that, if the growth condition (1.2) holds with $s = p$, then (1.4) is equivalent to a differential inclusion problem involving the generalized gradient ∂F.

Remark 4.4. Hypothesis (H3') is a condition stronger than (H3) provided $F(0) \leq 0$. Indeed, by Lebourg's mean value theorem (see [6, Theorem 2.3.7]), for every $t \in (0, a_2]$ there is $\zeta \in (0, t)$ such that $F(t) = F(0) + \zeta t \leq \zeta t$. Thus (H3') implies (H3).

We provide an example for which all our hypotheses are satisfied, so all our results apply.

Example 4.5. Using the notation in Theorem 3.1 let us fix constants $a_1 > 0$, $a_2 > \kappa a_1$, $1 < r_1 < r_2 < p^*$ and

$$\begin{aligned} c > \max \Bigg\{ & a_2^{p-1} \left(\operatorname*{ess\,inf}_{x \in \Omega} \alpha(x) \right)^{-1} \left(K_1 \frac{1}{a_1^{p-1}} + K_2 a_1^{s-p} \right) \\ & + \max\{a_2^{r_1-1}, a_2^{r_2-1}\}, \max\{r_1 a_1^{r_1-1}, r_2 a_2^{r_2-1}\} \Bigg\}. \end{aligned} \tag{4.3}$$

Define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0 \\ -c\xi + \max\{\xi^{r_1}, \xi^{r_2}\} & \text{if } \xi > 0. \end{cases}$$

Then F is locally Lipschitz with the generalized gradient

$$\partial F(\xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ [-c, 0] & \text{if } \xi = 0 \\ -c + r_1 \xi^{r_1-1} & \text{if } 0 < \xi < 1 \\ [-c + r_1, -c + r_2] & \text{if } \xi = 1 \\ -c + r_2 \xi^{r_2-1} & \text{if } \xi > 1. \end{cases}$$

The fact that (H1) is satisfied follows directly from (4.3) and the expression of F. Furthermore, we derive that

$$\limsup_{\xi \to +\infty} \frac{-F(\xi)}{\xi^p} = \limsup_{\xi \to +\infty} \left[\frac{c}{\xi^{p-1}} - \max\{\xi^{r_1-p}, \xi^{r_2-p}\} \right] \leq 0,$$

so (H2) holds true. Hypothesis (H3') is a straightforward consequence of the expression of ∂F and (4.3). In view of Remark 4.4, we infer that (H3) is valid. We have already noted that (H4) is true for K in (1.3). In order to check (H5), it is sufficient to take any a_3 with $r_2 a_3^{r_2-1} > c$, which in particular ensures through (4.3) that $a_3 > a_2$.

REFERENCES

[1] G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations **244** (2008), no. 12, 3031–3059.

[2] G. Bonanno, D. Motreanu, P. Winkert, *Variational-hemivariational inequalities with small perturbations of nonhomogeneous Neumann boundary conditions*, J. Math. Anal. Appl. **381** (2011), no. 2, 627–637.

[3] G. Bonanno, P. Winkert, *Multiplicity results to a class of variational-hemivariational inequalities*, Topol. Methods Nonlinear Anal., to appear.

[4] S. Carl, *Equivalence of some multi-valued elliptic variational inequalities and variational-hemivariational inequalities*, Adv. Nonlinear Stud. **11** (2011), no. 2, 247–263.

[5] K. C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.

[6] F. H. Clarke, Optimization and Nonsmooth Analysis, Classics in Applied Mathematics, vol. 5, SIAM, Philadelphia, 1990.

[7] G. D'Aguì, G. Molica Bisci, *Three non-zero solutions for elliptic Neumann problems*, Anal. Appl. (Singap.) **9** (2011), no. 4, 383–394.

[8] H. Lisei, C. Varga, *Multiple solutions for a differential inclusion problem with nonhomogeneous boundary conditions*, Numer. Funct. Anal. Optim. **30** (2009), no. 5-6, 566–581.

[9] D. Gilbard, N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[10] L. Ma, *Mountain pass on a closed convex set*, J. Math. Anal. Appl. **205** (1997), no. 2, 531–536.

[11] S. A. Marano, D. Motreanu, *On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems*, Nonlinear Anal. **48** (2002), no. 1, 37–52.

[12] D. Motreanu, P. D. Panagiotopoulos, *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Nonconvex Optim. Appl. **29** Kluwer, Dordrecht, 1998.

[13] D. Motreanu, V. Rădulescu, Variational and Non-variational Methods in Nonlinear Analysis and Boundary Value Problems, Nonconvex Optimization and Applications, Kluwer Academic Publishers 2003.

[14] D. Motreanu, P. Winkert, *Variational-hemivariational inequalities with nonhomogeneous Neumann boundary condition*, Matematiche (Catania) **65** (2010), no. 2, 109–119.

[15] B. Ricceri, *A further refinement of a three critical points theorem*, Nonlinear Anal. **74** (2011), no. 18, 7446–7454.

[16] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.