DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **5**, Number **4**, August **2012**

doi:10.3934/dcdss.2012.5.865

pp. **865–878**

# A PRIORI BOUNDS FOR WEAK SOLUTIONS TO ELLIPTIC EQUATIONS WITH NONSTANDARD GROWTH

Patrick Winkert

Technische Universität Berlin, Institut für Mathematik
Straße des 17. Juni 136
10623 Berlin, Germany

Rico Zacher

Martin-Luther-Universität Halle-Wittenberg, Institut für Mathematik
Theodor-Lieser-Straße 5
06120 Halle, Germany

Abstract. In this paper we study elliptic equations with a nonlinear conormal derivative boundary condition involving nonstandard growth terms. By means of the localization method and De Giorgi's iteration technique we derive global a priori bounds for weak solutions of such problems.

1. **Introduction.** The present paper is concerned with global a priori bounds for elliptic equations with nonlinear conormal derivative boundary conditions which may contain nonlinearities with variable growth exponents. More precisely, let $\Omega$ be a bounded domain in $\mathbb{R}^N, N > 1$, with Lipschitz boundary $\Gamma := \partial\Omega$ and let $p \in C(\overline{\Omega})$ be a function that satisfies $1 < p^- := \inf_{\overline{\Omega}} p(x)$. We deal with elliptic equations of the form

$$\begin{aligned} -\operatorname{div} \mathcal{A}(x,u,\nabla u) &= \mathcal{B}(x,u,\nabla u) && \text{in } \Omega,\\ \mathcal{A}(x,u,\nabla u)\cdot \nu &= \mathcal{C}(x,u) && \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\nu(x)$ denotes the outer unit normal of $\Omega$ at $x \in \Gamma$, and $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy suitable $p(x)$-structure conditions, see (H) below.

An important special case of (1.1) which fits in our setting is given by

$$-\Delta_{p(x)} u = \mathcal{B}(x,u,\nabla u) \quad \text{in } \Omega, \qquad |\nabla u|^{p(x)-2}\partial_\nu u = \mathcal{C}(x,u) \quad \text{on } \Gamma.$$

Here the operator $\operatorname{div}\mathcal{A}$ becomes the so-called $p(x)$-Laplacian

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

which reduces to the standard $p$-Laplacian if $p(x) \equiv p$.

In recent years there has been a growing interest in the study of elliptic problems with a $p(x)$-structure, which are also termed problems with nonstandard growth conditions. Equations of this type appear in the study of non-Newtonian fluids with thermo-convective effects (see Antontsev and Rodrigues [4], Zhikov [35]), electrorheological fluids (see Diening [8], Rajagopal and Růžička [29], Růžička [31]), the

2000 *Mathematics Subject Classification.* Primary: 35J60, 35B45; Secondary: 35J25.

*Key words and phrases.* A priori estimates, De Giorgi iteration, Elliptic equations, Nonstandard growth, Partition of unity, Variable exponent spaces.

thermistor problem (see Zhikov [36]), or the problem of image recovery (see Chen et al. [5]).

Throughout the paper we impose the following conditions.

(H) The functions $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, and $\mathcal{C} : \Gamma \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the subsequent structure conditions:

$$\begin{aligned}
&\text{(H1)} \quad |\mathcal{A}(x,s,\xi)| \le a_0|\xi|^{p(x)-1} + a_1|s|^{q_0(x)\frac{p(x)-1}{p(x)}} + a_2, && \text{for a.a. } x \in \Omega,\\
&\text{(H2)} \quad \mathcal{A}(x,s,\xi)\cdot\xi \ge a_3|\xi|^{p(x)} - a_4|s|^{q_0(x)} - a_5, && \text{for a.a. } x \in \Omega,\\
&\text{(H3)} \quad |\mathcal{B}(x,s,\xi)| \le b_0|\xi|^{p(x)\frac{q_0(x)-1}{q_0(x)}} + b_1|s|^{q_0(x)-1} + b_2, && \text{for a.a. } x \in \Omega,\\
&\text{(H4)} \quad |\mathcal{C}(x,s)| \le c_0|s|^{q_1(x)-1} + c_1, && \text{for a.a. } x \in \Gamma,
\end{aligned}$$

and for all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$. Here $a_i, b_j$ and $c_l$ are positive constants, $p \in C(\overline{\Omega})$ with $\inf_{\overline{\Omega}} p(x) > 1$, and $q_0 \in C(\overline{\Omega})$ as well as $q_1 \in C(\Gamma)$ are chosen such that

$$p(x) \le q_0(x) < p^*(x), \quad x \in \overline{\Omega}, \quad \text{and} \quad p(x) \le q_1(x) < p_*(x), \quad x \in \Gamma,$$

with the critical exponents

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N,\\ +\infty & \text{if } p(x) \ge N, \end{cases} \qquad p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N,\\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

A function $u \in W^{1,p(\cdot)}(\Omega)$ is said to be a **weak solution** (**subsolution, supersolution**) of equation (1.1) if

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi dx = (\le, \ge) \int_\Omega \mathcal{B}(x,u,\nabla u)\varphi dx + \int_\Gamma \mathcal{C}(x,u)\varphi d\sigma, \tag{1.2}$$

holds for all nonnegative test functions $\varphi \in W^{1,p(\cdot)}(\Omega)$, where $d\sigma$ denotes the usual $(N-1)$-dimensional surface measure.

This definition makes sense, since thanks to assumption (H) the integrals in (1.2) are finite, by Hölder's inequality and embedding results for $W^{1,p(\cdot)}(\Omega)$-functions, see below.

The main goal of this paper is to prove a priori bounds for weak sub- and supersolutions, in particular for weak solutions of problem (1.1). Using the notation $y_+ = \max(y,0)$, our main result reads as follows.

**Theorem 1.1.** *Let the assumptions in (H) be satisfied. Then there exist positive constants $\alpha = \alpha(p,q_0,q_1)$ and $C = C(p,q_0,q_1,a_3,a_4,a_5,b_0,b_1,b_2,c_0,c_1,N,\Omega)$ such that the following assertions hold.*

*(i) If $u \in W^{1,p(\cdot)}(\Omega)$ is a weak subsolution of (1.1) then*

$$\operatorname*{ess\,sup}_\Omega u \le 2\max\left(1, C\left[\int_\Omega u_+^{q_0(x)}dx + \int_\Gamma u_+^{q_1(x)}d\sigma\right]^\alpha\right).$$

*(ii) If $u \in W^{1,p(\cdot)}(\Omega)$ is a weak supersolution of (1.1) then*

$$\operatorname*{ess\,inf}_\Omega u \ge -2\max\left(1, C\left[\int_\Omega (-u)_+^{q_0(x)}dx + \int_\Gamma (-u)_+^{q_1(x)}d\sigma\right]^\alpha\right).$$

Note that the constants $a_0, a_1, a_2$, which appear in (H1), do not play any role in determining the constants $\alpha$ and $C$. The finiteness of the right-hand sides in (i) and (ii) is a consequence of the compact embedding $W^{1,p(\cdot)}(\Omega) \to L^{q_0(\cdot)}(\Omega)$ and the fact that the trace operator is a bounded operator from $W^{1,p(\cdot)}(\Omega)$ into $L^{q_1(\cdot)}(\Gamma)$ (see

Fan et al. [15, Theorem 1.3] and Fan [12, Corollary 2.4]). We further point out that we merely assume continuity for the variable exponents $p, q_0$, and $q_1$; log-Hölder continuity conditions are not required.

Our proof of Theorem 1.1 uses De Giorgi's iteration technique and the localization method. By means of the latter we are able to reduce the estimates involving variable exponents to ones with constant exponents, which then also allows us to apply classical embedding results. This crucial step in the proof is achieved by means of an appropriate partition of unity.

By the definition of sub- and supersolution of (1.1) one easily verifies that a weak solution is both a weak subsolution and a weak supersolution. Hence we have the following.

**Corollary 1.2.** *Let the assumptions (H) be satisfied and let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak solution of (1.1). Then $u \in L^\infty(\Omega)$ and the estimates in (i) and (ii) from Theorem 1.1 are valid.*

The main novelty of the paper consists in the generality of the assumptions needed to establish the boundedness of weak solutions to (1.1). In particular the assumptions on the nonlinearity $\mathcal{C}$ are rather general, allowing for a growth term with variable exponent, which seems to be optimal. Another novelty is the use of the localization technique in the context of global a priori estimates for problems with variable exponents and nonlinear conormal derivative boundary conditions.

Let us comment on some relevant known results on elliptic problems with $p(x)$-structure. Local boundedness of solutions to the equation

$$-\operatorname{div} \mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u) \quad \text{ in } \Omega, \tag{1.3}$$

has been studied by Fan and Zhao [16]. There it is shown that under suitable structure conditions every weak solution $u$ of (1.3) (corresponding to test functions $\varphi \in W_0^{1,p(x)}(\Omega)$) belongs to $L^\infty_{\text{loc}}(\Omega)$, and if in addition $u$ is bounded on the boundary $\Gamma$, then $u \in L^\infty(\Omega)$. The proof uses De Giorgi iterations as well. Recently, Gasiński and Papageorgiou (see [19, Proposition 3.1]) studied global a priori bounds for weak solutions to the equation

$$\begin{aligned} -\Delta_{p(x)} u &= g(x,u) && \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \Gamma, \end{aligned} \tag{1.4}$$

where the Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a subcritical growth condition. They proved that every weak solution $u \in W^{1,p(\cdot)}(\Omega)$ of problem (1.4) belongs to $L^\infty(\Omega)$ provided $p \in C^1(\overline{\Omega})$ satisfying $1 < \min_{x\in\overline{\Omega}} p(x)$.

$L^\infty$-estimates for solutions of (1.1) in case $p(x) \equiv p$ with $q_0(x) = q_1(x) \equiv p$ have been established by the first author in [33, 34] following Moser's iteration technique (for constant $p$ see also Pucci and Servadei [28]).

Concerning boundedness and regularity results for problems of type (1.3), in particular for the special case

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0,$$

we further refer to Acerbi and Mingione [1, 2], Antontsev and Consiglieri [3], Chiadò Piat and Coscia [6], Diening et al. [9], Eleuteri and Habermann [11], Fan [13, 14], Fan and Zhao [18], Habermann and Zatorska-Goldstein [20], Harjulehto et al. [21], Liskevich and Skrypnik [25], Lukkari [26, 27] and the references given therein.

The paper is organized as follows. In Section 2 we fix some notation and recall the definition of the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. We further state a lemma on sequences of numbers which will be needed for the De Giorgi iterations. The main result is proved in Section 3. From the structure conditions we first derive truncated energy estimates. These are then used, together with the localization method and embedding results, to prove suitable iterative inequalities, which in turn imply the desired a priori bounds.

2. **Notations and preliminaries.** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\Gamma$ and let $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$. We set $p^- := \min_{x\in\overline{\Omega}} p(x)$ and $p^+ := \max_{x\in\overline{\Omega}} p(x)$, then $p^- > 1$ and $p^+ < \infty$. By $L^{p(\cdot)}(\Omega)$ we identify the variable exponent Lebesgue space which is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \;\middle|\; u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)}.$$

For more information and basic properties of variable exponent spaces we refer the reader to the papers of Fan and Zhao [17], Kováčik and Rákosník [22] and the recent monograph of Diening et al. [10]. If $p(x) \equiv p$ is a constant, the usual Sobolev space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left( \int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

For $q_0 \in C(\overline{\Omega})$ and $q_1 \in C(\Gamma)$ (as in (H)) we define

$$q_0^+ = \max_{\overline{\Omega}} q_0(x), \quad q_0^- = \min_{\overline{\Omega}} q_0(x),$$
$$q_1^+ = \max_{\Gamma} q_1(x), \quad q_1^- = \min_{\Gamma} q_1(x).$$

For $s \in [1,\infty)$ we further use the notation

$$s^* = \begin{cases} \frac{Ns}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \ge N, \end{cases} \qquad s_* = \begin{cases} \frac{(N-1)s}{N-s} & \text{if } s < N, \\ +\infty & \text{if } s \ge N. \end{cases}$$

The following lemma concerning the geometric convergence of sequences of numbers will be needed for the De Giorgi iteration arguments below. It can be found, for example in [32]. The case $\delta_1 = \delta_2$ is contained in [23, Chapter II, Lemma 5.6], see also [7, Chapter I, Lemma 4.1].

**Lemma 2.1.** *Let $\{Y_n\}, n = 0, 1, 2, \ldots,$ be a sequence of positive numbers, satisfying the recursion inequality*

$$Y_{n+1} \le K b^n \left( Y_n^{1+\delta_1} + Y_n^{1+\delta_2} \right), \quad n = 0, 1, 2, \ldots,$$

*for some* $b > 1$, $K > 0$, *and* $\delta_2 \geq \delta_1 > 0$. *If*

$$Y_0 \leq (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}},$$

*then*

$$Y_n \leq (2K)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}} b^{-\frac{n}{\delta_1}}, \quad n \in \mathbb{N},$$

*in particular* $\{Y_n\} \to 0$ *as* $n \to \infty$.

3. **Truncated energy estimates and proof of Theorem 1.1.** Our proof of the sup-bounds for weak subsolutions of (1.1) is based on the following lemma on truncated energy estimates.

**Lemma 3.1.** *Let the conditions in (H) be satisfied. If* $u$ *is a weak subsolution of (1.1), there holds*

$$\int_{A_k} |\nabla u|^{p(x)} dx \leq d_1 \int_{A_k} u^{q_0(x)} dx + d_2 \int_{\Gamma_k} u^{q_1(x)} d\sigma,$$

*where*

$$A_k = \{x \in \Omega : u(x) > k\}, \qquad \Gamma_k = \{x \in \Gamma : u(x) > k\}, \quad k \geq 1,$$

*and* $d_1 = 2a_3^{-1}(a_4 + a_5 + b_1 + b_2 + b_0 \varepsilon^{-(q_0^+ + 1)})$, $d_2 = 2a_3^{-1}(c_0 + c_1)$, *and* $\varepsilon = \min(1, \frac{a_3}{2b_0})$.

*Proof.* Let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak subsolution of (1.1) and let $k \geq 1$. Taking $\varphi = (u-k)_+ = \max(u-k, 0) \in W^{1,p(\cdot)}(\Omega)$ (see [24, Lemma 3.2]) as test function in (1.2) with the '$\leq$'-sign we obtain

$$\begin{aligned} &\int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla(u-k) dx \\ &\leq \int_{A_k} \mathcal{B}(x, u, \nabla u)(u-k) dx + \int_{\Gamma_k} \mathcal{C}(x, u)(u-k) d\sigma. \end{aligned} \tag{3.1}$$

Using the structure condition (H2) we estimate the left-hand side of (3.1) as follows.

$$\begin{aligned} &\int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla(u-k) dx \\ &= \int_{A_k} \mathcal{A}(x, u, \nabla u) \cdot \nabla u dx \\ &\geq \int_{A_k} \left( a_3 |\nabla u|^{p(x)} - a_4 |u|^{q_0(x)} - a_5 \right) dx \\ &\geq a_3 \int_{A_k} |\nabla u|^{p(x)} dx - (a_4 + a_5) \int_{A_k} |u|^{q_0(x)} dx, \end{aligned} \tag{3.2}$$

as $u^{q_0(x)} > u > 1$ in $A_k$. Now, we are going to estimate the right-hand side of (3.1). By Young's inequality with $\varepsilon \in (0,1]$ and condition (H3) we have

$$
\begin{aligned}
&\int_{A_k} \mathcal{B}(x,u,\nabla u)(u-k)dx \\
&\leq \int_{A_k} \left[ b_0 |\nabla u|^{p(x)\frac{q_0(x)-1}{q_0(x)}} + b_1 |u|^{q_0(x)-1} + b_2 \right] (u-k)dx \\
&\leq b_0 \int_{A_k} \left[ \varepsilon^{\frac{q_0(x)-1}{q_0(x)}} |\nabla u|^{p(x)\frac{q_0(x)-1}{q_0(x)}} \varepsilon^{-\frac{q_0(x)-1}{q_0(x)}} u \right] dx + (b_1+b_2) \int_{A_k} |u|^{q_0(x)} dx \qquad (3.3)\\
&\leq b_0 \int_{A_k} \varepsilon |\nabla u|^{p(x)} dx + b_0 \int_{A_k} \varepsilon^{-(q_0(x)-1)} u^{q_0(x)} dx + (b_1+b_2) \int_{A_k} |u|^{q_0(x)} dx \\
&\leq \varepsilon b_0 \int_{A_k} |\nabla u|^{p(x)} dx + \left( b_0 \varepsilon^{-(q_0^+-1)} + b_1 + b_2 \right) \int_{A_k} u^{q_0(x)} dx.
\end{aligned}
$$

Thanks to condition (H4), the boundary integral can be estimated through

$$
\begin{aligned}
\int_{\Gamma_k} \mathcal{C}(x,u)(u-k)d\sigma &\leq \int_{\Gamma_k} (c_0 |u|^{q_1(x)-1} + c_1)(u-k)d\sigma \\
&\leq (c_0+c_1) \int_{\Gamma_k} u^{q_1(x)} d\sigma,
\end{aligned} \qquad (3.4)
$$

as $u > 1$ on $\Gamma_k$. Combining (3.1)–(3.4) and choosing $\varepsilon = \min(1, \frac{a_3}{2b_0})$ gives

$$
\begin{aligned}
&\frac{a_3}{2} \int_{A_k} |\nabla u|^{p(x)} dx \\
&\leq \left( a_4 + a_5 + b_1 + b_2 + b_0 \varepsilon^{-(q_0^+ + 1)} \right) \int_{A_k} u^{q_0(x)} dx + (c_0 + c_1) \int_{\Gamma_k} u^{q_1(x)} d\sigma.
\end{aligned}
$$

Dividing the last inequality by $\frac{a_3}{2} > 0$ yields the assertion of the lemma. □

The corresponding result for supersolutions reads as follows.

**Lemma 3.2.** *Let the conditions in (H) be satisfied. If $u$ is a weak supersolution of (1.1), there holds*

$$
\int_{\tilde{A}_k} |\nabla u|^{p(x)} dx \leq d_1 \int_{\tilde{A}_k} (-u)^{q_0(x)} dx + d_2 \int_{\tilde{\Gamma}_k} (-u)^{q_1(x)} d\sigma,
$$

*where*

$$
\tilde{A}_k = \{x \in \Omega : -u(x) > k\}, \qquad \tilde{\Gamma}_k = \{x \in \Gamma : -u(x) > k\}, \quad k \geq 1,
$$

*and $d_1$ and $d_2$ are the same constants as in Lemma 3.1.*

*Proof.* The proof is analogous to the previous one. We take $\varphi = -(u+k)_- = -\min(u+k, 0) \geq 0$ as test function in (1.2), which now holds with the '$\geq$'-sign, and use the same arguments as in the proof of Lemma 3.1. This yields the asserted inequality. □

Now we are in position to prove the main result of this paper.

*Proof of Theorem 1.1.*

**(i) Definition of the iteration variables $Z_n$, $\tilde{Z}_n$, and basic estimates.** Let now

$$
k_n = k\left(2 - \frac{1}{2^n}\right), \quad n = 0, 1, 2, \dots,
$$

with $k \geq 1$ specified later and put

$$Z_n := \int_{A_{k_n}} (u - k_n)^{q_0(x)} dx, \qquad \tilde{Z}_n := \int_{\Gamma_{k_n}} (u - k_n)^{q_1(x)} d\sigma.$$

We have

$$\begin{aligned} Z_n &\geq \int_{A_{k_{n+1}}} (u - k_n)^{q_0(x)} dx \geq \int_{A_{k_{n+1}}} u^{q_0(x)} \left(1 - \frac{k_n}{k_{n+1}}\right)^{q_0(x)} dx \\ &\geq \int_{A_{k_{n+1}}} \frac{1}{2^{q_0(x)(n+2)}} u^{q_0(x)} dx, \end{aligned}$$

and thus

$$\int_{A_{k_{n+1}}} u^{q_0(x)} dx \leq 2^{q_0^+(n+2)} Z_n. \tag{3.5}$$

Analogously, we see that

$$\int_{\Gamma_{k_{n+1}}} u^{q_1(x)} d\sigma \leq 2^{q_1^+(n+2)} \tilde{Z}_n. \tag{3.6}$$

From (3.5)–(3.6) and Lemma 3.1 with $k$ being replaced by $k_{n+1} \geq 1$ it follows that

$$\int_{A_{k_{n+1}}} |\nabla(u - k_{n+1})|^{p(x)} dx \leq d_3 a^n (Z_n + \tilde{Z}_n), \tag{3.7}$$

where $d_3 = \max\left(d_1 2^{2q_0^+}, d_2 2^{2q_1^+}\right)$ and $a = \max\left(2^{q_0^+}, 2^{q_1^+}\right)$.

Furthermore, we have

$$\begin{aligned} |A_{k_{n+1}}| &\leq \int_{A_{k_{n+1}}} \left(\frac{u - k_n}{k_{n+1} - k_n}\right)^{q_0(x)} dx \\ &\leq \int_{A_{k_n}} \frac{2^{q_0(x)(n+1)}}{k^{q_0(x)}} (u - k_n)^{q_0(x)} dx \\ &\leq \frac{2^{q_0^+(n+1)}}{k^{q_0^-}} \int_{A_{k_n}} (u - k_n)^{q_0(x)} dx \\ &= \frac{2^{q_0^+(n+1)}}{k^{q_0^-}} Z_n. \end{aligned} \tag{3.8}$$

**(ii) Partition of unity.** By compactness of $\overline{\Omega}$, for any $R > 0$ there exists a finite open cover $\{B_i(R)\}_{i=1,\ldots,m}$ of balls $B_i := B_i(R)$ with radius $R$ such that $\overline{\Omega} \subset \bigcup_{i=1}^m B_i(R)$. Moreover, since $p \in C(\overline{\Omega})$, $q_0 \in C(\overline{\Omega})$, and $q_1 \in C(\Gamma)$, these functions are uniformly continuous on $\overline{\Omega}$ and $\Gamma$, respectively. Recalling that

$$p(x) \leq q_0(x) < p^*(x), \quad x \in \overline{\Omega}, \quad \text{ and } \quad p(x) \leq q_1(x) < p_*(x), \quad x \in \Gamma,$$

we may take $R > 0$ small enough such that

$$p_i^+ \leq q_{0,i}^+ < (p_i^-)^*, \qquad p_i^+ \leq q_{1,i}^+ < (p_i^-)_*, \qquad i = 1, \ldots, m,$$

where

$$\begin{aligned} p_i^+ &= \max_{B_i \cap \overline{\Omega}} p(x), \quad q_{0,i}^+ = \max_{B_i \cap \overline{\Omega}} q_0(x), \\ p_i^- &= \min_{B_i \cap \overline{\Omega}} p(x), \quad q_{1,i}^+ = \max_{B_i \cap \Gamma} q_1(x). \end{aligned}$$

We next choose a partition of unity $\{\xi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^N)$ associated to the open cover $\{B_i(R)\}_{i=1,\dots,m}$ (see e.g. [30, Thm. 6.20]), that is, we have

$$\operatorname{supp} \xi_i \subset B_i, \quad 0 \le \xi_i \le 1, \ \ i = 1, \dots, m, \quad \text{and} \ \sum_{i=1}^m \xi_i = 1 \ \text{ on } \overline{\Omega}.$$

Let $L > 0$ be a positive constant such that

$$|\nabla \xi_i| \le L, \quad i = 1, \dots, m.$$

**(iii) Estimating the gradient term in (3.7) from below.** Using the partition of unity from step (ii) we have

$$\begin{aligned}
&\int_{A_{k_{n+1}}} |\nabla(u-k_{n+1})|^{p(x)} dx \\
&= \int_{A_{k_{n+1}}} |\nabla(u-k_{n+1})|^{p(x)} \sum_{i=1}^m \xi_i dx \\
&\ge \sum_{i=1}^m \int_{A_{k_{n+1}}} (|\nabla(u-k_{n+1})|^{p_i^-} - 1)\xi_i dx \\
&\ge \sum_{i=1}^m \int_{A_{k_{n+1}}} |\nabla(u-k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx - m|A_{k_{n+1}}|,
\end{aligned} \tag{3.9}$$

as $\xi_i \ge \xi_i^{p_i^-}$. From (3.9) we trivially deduce that for all $i = 1, \dots, m$,

$$\int_{A_{k_{n+1}}} |\nabla(u-k_{n+1})|^{p(x)} dx \ge \int_{A_{k_{n+1}}} |\nabla(u-k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx - m|A_{k_{n+1}}|. \tag{3.10}$$

Combining (3.7) and (3.10) and using (3.8) yields

$$\int_{A_{k_{n+1}}} |\nabla(u-k_{n+1})|^{p_i^-} \xi_i^{p_i^-} dx \le d_4 a^n (Z_n + \tilde{Z}_n) \tag{3.11}$$

for any $i = 1, \dots, m$, with the positive constant $d_4 = d_3 + m2^{q_0^+}$.

**(iv) Estimating $Z_{n+1}$.** Next we want to derive a suitable estimate for the term $Z_{n+1}$ from above. To this end, we make again use of the partition of unity introduced in step (ii). We have

$$
\begin{aligned}
Z_{n+1} &= \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_0(x)} dx \\
&= \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_0(x)} \left(\sum_{i=1}^m \xi_i\right)^{q_0^+} dx \\
&\le \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_0(x)} m^{q_0^+} \sum_{i=1}^m \xi_i^{q_0^+} dx \\
&\le m^{q_0^+} \sum_{i=1}^m \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_0(x)} \xi_i^{q_0(x)} dx \\
&\le m^{q_0^+} \sum_{i=1}^m \left[\int_{A_{k_{n+1}}} (u-k_{n+1})^{q_{0,i}^+} \xi_i^{q_{0,i}^+} dx + \int_{A_{k_{n+1}}} (u-k_{n+1})^{q_{0,i}^-} \xi_i^{q_{0,i}^-} dx\right],
\end{aligned} \tag{3.12}
$$

where we have set $q_{0,i}^- = \min_{B_i \cap \overline{\Omega}} q_0(x)$. Observe that $p_i^- \le q_{0,i}^- \le q_{0,i}^+ < (p_i^-)^*$ for all $i = 1, \ldots, m$.

Let now $i \in \{1, \ldots, m\}$ be fixed, and suppose that $r \in \{q_{0,i}^-, q_{0,i}^+\}$. Then $p_i^- \le r < (p_i^-)^*$ and $r \le q^+$, where $q^+ = \max(q_0^+, q_1^+)$. By Hölder's inequality and the continuous embedding $W^{1,p_i^-}(\Omega) \hookrightarrow L^{(p_i^-)^*}(\Omega)$, we may estimate as follows.

$$
\begin{aligned}
&\int_\Omega (u-k_{n+1})_+^r \xi_i^r dx \\
&\le \left(\int_\Omega (u-k_{n+1})_+^{(p_i^-)^*} \xi_i^{(p_i^-)^*} dx\right)^{\frac{r}{(p_i^-)^*}} |A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}} \\
&\le C^{q^+} \left(\int_\Omega \Big[|\nabla[(u-k_{n+1})_+\xi_i]|^{p_i^-} \right. \\
&\qquad \left. + |(u-k_{n+1})_+\xi_i|^{p_i^-}\Big] dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}},
\end{aligned} \tag{3.13}
$$

where $C = \max(1, C(p_1^-, N), \ldots, C(p_m^-, N))$ with $C(p_j^-, N)$ being the embedding constant corresponding to the embedding $W^{1,p_j^-}(\Omega) \hookrightarrow L^{(p_j^-)^*}(\Omega)$, $j = 1, \ldots, m$. Thus $C$ is independent of $i$. A simple calculation shows that the right-hand side of (3.13) can be estimated further to obtain

$$
\begin{aligned}
&\int_\Omega (u-k_{n+1})_+^r \xi_i^r dx \\
&\le d_5 \left(\int_\Omega |\nabla(u-k_{n+1})_+|^{p_i^-} \xi_i^{p_i^-} dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}} \\
&\quad + d_6 \left(\int_{A_{k_{n+1}}} u^{q_0(x)} dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}}.
\end{aligned} \tag{3.14}
$$

Here $d_5 = d_5(q^+, C)$ and $d_6 = d_6(p^+, q^+, C, L)$ are positive constants, where $L$ is the constant introduced in step (ii).

Using (3.5), (3.11), (3.12), and (3.14) with $r = q_{0,i}^+$ and $r = q_{0,i}^-$, respectively, for $i = 1, \dots, m$, we get

$$
\begin{aligned}
Z_{n+1} \le m^{q_0^+} \sum_{i=1}^m \Bigg[ & d_5 \left( d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{q_{0,i}^+}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{q_{0,i}^+}{(p_i^-)^*}} \\
& + d_6 \left( 2^{q_0^+(n+2)} Z_n \right)^{\frac{q_{0,i}^+}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{q_{0,i}^+}{(p_i^-)^*}} \\
& + d_5 \left( d_4 a^n (Z_n + \tilde{Z}_n) dx \right)^{\frac{q_{0,i}^-}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{q_{0,i}^-}{(p_i^-)^*}} \\
& + d_6 \left( 2^{q_0^+(n+2)} Z_n \right)^{\frac{q_{0,i}^-}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{q_{0,i}^-}{(p_i^-)^*}} \Bigg].
\end{aligned} \tag{3.15}
$$

Setting

$$
Y_n := Z_n + \tilde{Z}_n
$$

and $\eta = \max \left( \frac{q_{0,1}^+}{(p_1^-)^*}, \dots, \frac{q_{0,m}^+}{(p_m^-)^*} \right)$ we have for $r \in \{q_{0,i}^+, q_{0,i}^-\}$

$$
\begin{aligned}
\left( d_4 a^n (Z_n + \tilde{Z}_n) \right)^{\frac{r}{p_i^-}} &\le (d_4)^{\frac{q^+}{p^-}} \left( a^{\frac{q^+}{p^-}} \right)^n (Y_n + Y_n^{\frac{q^+}{p^-}}), \\
\left( 2^{q_0^+(n+2)} Z_n \right)^{\frac{r}{p_i^-}} &\le \left( 2^{\frac{(q^+)^2(n+2)}{p^-}} \right) (Y_n + Y_n^{\frac{q^+}{p^-}}), \\
|A_{k_{n+1}}|^{1-\frac{r}{(p_i^-)^*}} &\le 2^{q^+(n+1)} \left( \frac{1}{k^{q_0^-}} \right)^{1-\eta} (Y_n + Y_n^{1-\eta}).
\end{aligned} \tag{3.16}
$$

Using these estimates, we conclude from (3.15) that

$$
Z_{n+1} \le d_7 d_8^n \frac{1}{k^{q_0^-(1-\eta)}} \left( Y_n^2 + Y_n^{2-\eta} + Y_n^{1+\frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\eta} \right) \tag{3.17}
$$

where $d_7$ and $d_8$ are positive constants that only depend on the data.

**(v) Estimating $\tilde{Z}_{n+1}$.** We proceed similarly as in step (iv). Analogously to (3.12) we get an estimate for $\tilde{Z}_{n+1}$ of the form

$$
\tilde{Z}_{n+1} \le m^{q_1^+} \sum_{i=1}^m \left[ \int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^{q_{1,i}^+} \xi_i^{q_{1,i}^+} d\sigma + \int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^{q_{1,i}^-} \xi_i^{q_{1,i}^-} d\sigma \right], \tag{3.18}
$$

where $q_{1,i}^- = \min_{B_i \cap \Gamma} q_1(x)$. Note that $p_i^- \le q_{1,i}^- \le q_{1,i}^+ < (p_i^-)_*$ for $i = 1, \dots, m$.

Let now $i \in \{1, \dots, m\}$ be fixed, and suppose that $r \in \{q_{1,i}^-, q_{1,i}^+\}$, that is we have $p_i^- \le r < (p_i^-)_*$ and $r \le q^+$. Define $s = s_i(r) \in (1, N)$ by means of

$$
s_* = \begin{cases} \frac{r + (p_i^-)_*}{2} & \text{if } (p_i^-)_* < \infty, \\ r + 1 & \text{if } (p_i^-)_* = \infty. \end{cases}
$$

Then $s < p_i^- \le r < s_* < (p_i^-)_*$. Since the trace operator maps $W^{1,s}(\Omega)$ boundedly into $L^r(\Gamma)$, we have, using Hölder's inequality,

$$\begin{aligned}
&\int_\Gamma ((u-k_{n+1})_+\xi_i)^r d\sigma \\
&\le \hat{C}^r \left[\int_\Omega (|\nabla[(u-k_{n+1})_+\xi_i]|^s + |(u-k_{n+1})_+\xi_i|^s)\, dx\right]^{\frac{r}{s}} \\
&\le \hat{C}^{q^+} \left[\int_\Omega \left(|\nabla[(u-k_{n+1})_+\xi_i]|^{p_i^-} + |(u-k_{n+1})_+\xi_i|^{p_i^-}\right) dx\right]^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s}{p_i^-}\right)\frac{r}{s}},
\end{aligned}$$

where $\hat{C}$ denotes the maximum of 1 and the norms of the trace maps $\gamma : W^{1,s}(\Omega) \to L^r(\Gamma)$ when $r$ runs through set $\bigcup_{j=1}^m \{q_{1,j}^-, q_{1,j}^+\}$. The right-hand side of the last inequality can be estimated to get

$$\begin{aligned}
&\int_\Gamma ((u-k_{n+1})_+\xi_i)^r d\sigma \\
&\le d_9 \left(\int_\Omega |\nabla(u-k_{n+1})_+|^{p_i^-} \xi_i^{p_i^-}\, dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s}{p_i^-}\right)\frac{r}{s}} \\
&\quad + d_{10} \left(\int_{A_{k_{n+1}}} u^{q_0(x)} dx\right)^{\frac{r}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s}{p_i^-}\right)\frac{r}{s}}
\end{aligned} \tag{3.19}$$

with positive constants $d_9, d_{10}$ that only depend on the data.

From (3.5), (3.11), (3.18), and (3.19) with $r = q_{1,i}^+$ and $r = q_{1,i}^-$, respectively, for $i = 1, \dots, m$, we infer that

$$\begin{aligned}
\tilde{Z}_{n+1} \le m^{q_1^+} \sum_{i=1}^m \Bigg[ & d_9 \left(d_4 a^n (Z_n + \tilde{Z}_n)\right)^{\frac{q_{1,i}^+}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s_i(q_{1,i}^+)}{p_i^-}\right)\frac{q_{1,i}^+}{s_i(q_{1,i}^+)}} \\
& + d_{10} \left(2^{q_0^+(n+2)} Z_n\right)^{\frac{q_{1,i}^+}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s_i(q_{1,i}^+)}{p_i^-}\right)\frac{q_{1,i}^+}{s_i(q_{1,i}^+)}} \\
& + d_9 \left(d_4 a^n (Z_n + \tilde{Z}_n)\right)^{\frac{q_{1,i}^-}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s_i(q_{1,i}^-)}{p_i^-}\right)\frac{q_{1,i}^-}{s_i(q_{1,i}^-)}} \\
& + d_{10} \left((2^{q_0^+(n+2)} Z_n\right)^{\frac{q_{1,i}^-}{p_i^-}} |A_{k_{n+1}}|^{\left(1-\frac{s_i(q_{1,i}^-)}{p_i^-}\right)\frac{q_{1,i}^-}{s_i(q_{1,i}^-)}} \Bigg].
\end{aligned} \tag{3.20}$$

Put $\tilde{\eta} = \max\left(\frac{s_1(q_{1,1}^+)}{p_1^-}, \dots, \frac{s_m(q_{1,m}^+)}{p_m^-}\right)$. Similarly to (3.16) we have for $r \in \{q_{1,i}^+, q_{1,i}^-\}$

$$\begin{aligned}
&\left(d_4 a^n (Z_n + \tilde{Z}_n)\right)^{\frac{r}{p_i^-}} \le (d_4)^{\frac{q^+}{p^-}} \left(a^{\frac{q^+}{p^-}}\right)^n (Y_n + Y_n^{\frac{q^+}{p^-}}), \\
&\left(2^{q_0^+(n+2)} Z_n\right)^{\frac{r}{p_i^-}} \le \left(2^{\frac{(q^+)^2(n+2)}{p^-}}\right) (Y_n + Y_n^{\frac{q^+}{p^-}}), \\
&|A_{k_{n+1}}|^{\left(1-\frac{s_i(r)}{p_i^-}\right)\frac{r}{s_i(r)}} \le 2^{q^+(n+1)} \left(\frac{1}{k^{q_0^-}}\right)^{1-\tilde{\eta}} (Y_n^{q^+} + Y_n^{1-\tilde{\eta}}).
\end{aligned} \tag{3.21}$$

Finally, (3.20) and (3.21) imply that

$$\tilde{Z}_{n+1} \le d_{11} d_{12}^n \frac{1}{k^{q_0^-(1-\tilde{\eta})}} \left( Y_n^{q^+1} + Y_n^{2-\tilde{\eta}} + Y_n^{q^+ + \frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-} - \tilde{\eta}} \right) \tag{3.22}$$

where $d_{11}$ and $d_{12}$ are positive constants that only depend on the data.

**(vi) The iterative inequality for $Y_n$.** Recall that $Y_n = Z_n + \tilde{Z}_n$. Hence (3.17) and (3.22) yield

$$\begin{aligned} Y_{n+1} &\le K b^n \frac{1}{k^{q_0^-(1-\hat{\eta})}} \left( Y_n^2 + Y_n^{2-\eta} + Y_n^{1+\frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\eta} \right. \\ &\qquad \left. + Y_n^{q^+} + Y_n^{2-\tilde{\eta}} + Y_n^{q^+ + \frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\tilde{\eta}} \right) \\ &\le 8 K b^n \frac{1}{k^{q_0^-(1-\hat{\eta})}} \left( Y_n^{1+\delta_1} + Y_n^{1+\delta_2} \right) \end{aligned}$$

with $K = \max(d_7, d_{11}), b = \max(d_8, d_{12})$, $\hat{\eta} = \max(\eta, \tilde{\eta})$, and where $0 < \delta_1 \le \delta_2$ are given by

$$\begin{aligned} \delta_1 &= \min \left( 1, 1-\eta, \frac{q^+}{p^-}, \frac{q^+}{p^-} - \eta, q^+, 1-\tilde{\eta}, q^+ + \frac{q^+}{p^-} - 1, \frac{q^+}{p^-} - \tilde{\eta} \right), \\ \delta_2 &= \max \left( 1, 1-\eta, \frac{q^+}{p^-}, \frac{q^+}{p^-} - \eta, q^+, 1-\tilde{\eta}, q^+ + \frac{q^+}{p^-} - 1, \frac{q^+}{p^-} - \tilde{\eta} \right). \end{aligned}$$

Without loss of generality we may assume that $b > 1$. Now we may apply Lemma 2.1, which says that $Y_n \to 0$ as $n \to \infty$ provided

$$Y_0 = \int_\Omega (u-k)_+^{q_0(x)} dx + \int_\Gamma (u-k)_+^{q_1(x)} d\sigma \le \left( \frac{16K}{k^{q_0^-(1-\hat{\eta})}} \right)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}}. \tag{3.23}$$

Relation (3.23) is clearly satisfied if

$$\int_\Omega u_+^{q_0(x)} dx + \int_\Gamma u_+^{q_1(x)} d\sigma \int_\Gamma \le \left( \frac{16K}{k^{q_0^-(1-\hat{\eta})}} \right)^{-\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1^2}}. \tag{3.24}$$

Hence, if we choose $k$ such that

$$k = \max \left( 1, \left[ (16K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1^2}} \left( \int_\Omega u_+^{q_0(x)} dx + \int_\Gamma u_+^{q_1(x)} d\sigma \right) \right]^{\frac{\delta_1}{q_0^-(1-\hat{\eta})}} \right), \tag{3.25}$$

then (3.24) and in particular (3.23) are satisfied. Since $k_n \to 2k$ as $n \to \infty$ we obtain

$$\operatorname*{ess\,sup}_\Omega u \le 2k = 2 \max \left( 1, \left[ (16K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1^2}} \left( \int_\Omega u_+^{q_0(x)} dx + \int_\Gamma u_+^{q_1(x)} d\sigma \right) \right]^{\frac{\delta_1}{q_0^-(1-\hat{\eta})}} \right).$$

Tracing back the constants, we see that the first part of the theorem is proved. The supersolution case can be done analogously, replacing $u$ with $-u$ and $A_k$ with $\tilde{A}_k$, and using Lemma 3.2 instead of Lemma 3.1. This completes the proof. $\square$

## REFERENCES


[1] E. Acerbi and G. Mingione, *Regularity results for a class of functionals with non-standard growth*, Arch. Ration. Mech. Anal., **156** (2001), 121–140.

[2] E. Acerbi and G. Mingione, *Regularity results for electrorheological fluids: The stationary case*, C. R. Math. Acad. Sci. Paris, **334** (2002), 817–822.

[3] S. N. Antontsev and L. Consiglieri, *Elliptic boundary value problems with nonstandard growth conditions*, Nonlinear Anal., **71** (2009), 891–902.

[4] S. N. Antontsev and J. F. Rodrigues, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **52** (2006), 19–36.

[5] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., **66** (2006), 1383–1406.

[6] V. Chiadò Piat and A. Coscia, *Hölder continuity of minimizers of functionals with variable growth exponent*, Manuscripta Math., **93** (1997), 283–299.

[7] E. DiBenedetto, "Degenerate Parabolic Equations," Universitext, Springer-Verlag, New York, 1993.

[8] L. Diening, "Theoretical and Numerical Results for Electrorheological Fluids," Ph.D thesis, Univ. Freiburg in Breisgau, Mathematische Fakultät, (2002), 156 pp.

[9] L. Diening, F. Ettwein and M. Růžička, *$C^{1,\alpha}$-regularity for electrorheological fluids in two dimensions*, NoDEA Nonlinear Differential Equations Appl., **14** (2007), 207–217.

[10] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, "Lebesgue and Sobolev spaces with variable exponents," Lecture Notes in Mathematics, **2017**, Springer-Verlag, Heidelberg, 2011.

[11] M. Eleuteri and J. Habermann, *Regularity results for a class of obstacle problems under nonstandard growth conditions*, J. Math. Anal. Appl., **344** (2008), 1120–1142.

[12] X. Fan, *Boundary trace embedding theorems for variable exponent Sobolev spaces*, J. Math. Anal. Appl., **339** (2008), 1395–1412.

[13] X. Fan, *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations, **235** (2007), 397–417.

[14] X. Fan, *Local boundedness of quasi-minimizers of integral functions with variable exponent anisotropic growth and applications*, NoDEA Nonlinear Differential Equations Appl., **17** (2010), 619–637.

[15] X. Fan and J. Shen and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl., **262** (2001), 749–760.

[16] X. Fan and D. Zhao, *A class of De Giorgi type and Hölder continuity*, Nonlinear Anal., **36** (1999), 295–318.

[17] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., **263** (2001), 424–446.

[18] X. Fan and D. Zhao, *The quasi-minimizer of integral functionals with $m(x)$ growth conditions*, Nonlinear Anal., **39** (2000), 807–816.

[19] L. Gasiński and N. S. Papageorgiou, *Anisotropic nonlinear Neumann problems*, Calc. Var. Partial Differential Equations, **42** (2011), 323–354.

[20] J. Habermann and A. Zatorska-Goldstein, *Regularity for minimizers of functionals with nonstandard growth by $\mathcal{A}$-harmonic approximation*, NoDEA Nonlinear Differential Equations Appl., **15** (2008), 169–194.

[21] P. Harjulehto, J. Kinnunen and T. Lukkari, *Unbounded supersolutions of nonlinear equations with nonstandard growth*, Bound. Value Probl., (2007), Art. ID 48348, 20 pp.

[22] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., **41(116)** (1991), 592–618.

[23] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural′ceva, "Linear and Quasilinear Equations of Parabolic Type," Translations of Mathematical Monographs, **23**, American Mathematical Society, Providence, R.I., 1967.

[24] V. K. Le, *On a sub-supersolution method for variational inequalities with Leray-Lions operators in variable exponent spaces*, Nonlinear Anal., **71** (2009), 3305–3321.

[25] V. Liskevich and I. I. Skrypnik, *Harnack inequality and continuity of solutions to elliptic equations with nonstandard growth conditions and lower order terms*, Ann. Mat. Pura Appl. (4), **189** (2010), 333–356.

[26] T. Lukkari, *Boundary continuity of solutions to elliptic equations with nonstandard growth*, Manuscripta Math., **132** (2010), 463–482.

[27] T. Lukkari, *Singular solutions of elliptic equations with nonstandard growth*, Math. Nachr., **282** (2009), 1770–1787.

[28] P. Pucci and R. Servadei, *Regularity of weak solutions of homogeneous or inhomogeneous quasilinear elliptic equations*, Indiana Univ. Math. J., **57** (2008), 3329–3363.

[29] K. R. Rajagopal and M. Růžička, *Mathematical modeling of electrorheological materials*, Cont. Mech. and Thermodyn., **13** (2001), 59–78.

[30] W. Rudin, "Functional Analysis," McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[31] M. Růžička, "Electrorheological Fluids: Modeling and Mathematical Theory," Lecture Notes in Mathematics, **1748**, Springer-Verlag, Berlin, 2000.

[32] V. Vergara and R. Zacher, *A priori bounds for degenerate and singular evolutionary partial integro-differential equations*, Nonlinear Anal., **73** (2010), 3572–3585.

[33] P. Winkert, *Constant-sign and sign-changing solutions for nonlinear elliptic equations with Neumann boundary values*, Adv. Differential Equations, **15** (2010), 561–599.

[34] P. Winkert, *$L^\infty$-estimates for nonlinear elliptic Neumann boundary value problems*, NoDEA Nonlinear Differential Equations Appl., **17** (2010), 289–302.

[35] V. V. Zhikov, *Meyer-type estimates for solving the nonlinear Stokes system*, Differ. Equ., **33** (1997), 108–115.

[36] V. V. Zhikov, *On some variational problems*, Russian J. Math. Phys., **5** (1997), 105–116.

Received March 2011; revised July 2011.

*E-mail address*: winkert@math.tu-berlin.de
*E-mail address*: rico.zacher@mathematik.uni-halle.de