Calc. Var. (2025) 64:258
https://doi.org/10.1007/s00526-025-03079-2

Calculus of Variations

Least energy sign-changing solution for logarithmic double phase problems with nonlinear boundary condition

Franziska Borer[1] · Leszek Gasiński[2] · Matheus F. Stapenhorst[3] · Patrick Winkert[1]

Received: 4 April 2025 / Accepted: 30 June 2025

Abstract

In this paper we study logarithmic double phase problems with superlinear right-hand sides and nonlinear Neumann boundary condition. In particular, we show that the problem under consideration has a least energy sign-changing solution. The proof is based on the minimization of the energy functional over the related nodal Nehari manifold along with the Poincaré–Miranda existence theorem. As a result of independent interest, we prove the existence of a new and very general equivalent norm in the logarithmic Musielak–Orlicz Sobolev space. In addition, we present a priori bounds for a large class of logarithmic double phase problems involving convection terms for critical and subcritical situations.

Mathematics Subject Classification 35A01 · 35J20 · 35J25 · 35J62 · 35Q74

1 Introduction

In the last decade, the double phase operator has gained interest in many different research areas. This operator is defined by

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right), \quad 1<p<q, \tag{1.1}$$

Communicated by S. Terracini.

✉ Franziska Borer
borer@math.tu-berlin.de

Leszek Gasiński
leszek.gasinski@uken.krakow.pl

Matheus F. Stapenhorst
m.stapenhorst@unesp.br

Patrick Winkert
winkert@math.tu-berlin.de

1 Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

2 Department of Mathematics, University of the National Education Commission, Podchorazych 2, 30-084 Krakow, Poland

3 Departamento de Matemática e Computação, Universidade Estadual Paulista - Unesp, Presidente Prudente, SP 19060-900, Brazil

Published online: 16 September 2025

and arises from the study of general reaction-diffusion equations with nonhomogeneous diffusion and transport aspects. Applications can be found in biophysics, plasma physics and chemical reactions, with double phase features, where the function u corresponds to the concentration term, and the differential operator represents the diffusion coefficient. The related integral functional to (1.1) has the form

$$J(u) = \int_\Omega \left(\frac{|\nabla u|^p}{p} + \mu(x) \frac{|\nabla u|^q}{q} \right) \mathrm{d}x, \tag{1.2}$$

for a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary, and appeared for the first time in a work by Zhikov [54] in order to describe models for anisotropic materials. A first mathematical treatment of (1.2) concerning the regularity of local minimizers has been done in the groundbreaking papers by Baroni–Colombo–Mingione [4, 6] and Colombo–Mingione [14, 15], see also the works by Marcellini [35, 36] concerning general (p, q)-growth as well as the contributions by Beck–Mingione [7] and De Filippis–Mingione [17] for nonautonomous integrals. We also refer to the overview article by Mingione–Rădulescu [37] about recent developments in problems with nonstandard growth and nonuniform ellipticity. Furthermore, other applications related to the double phase operator and in general for problems with nonstandard growth can be found in the works by Bahrouni–Rădulescu–Repovš [3] on transonic flows, Benci–D'Avenia–Fortunato–Pisani [8] on quantum physics, Cherfils–Il'yasov [13] for reaction diffusion systems and Zhikov [55] on the Lavrentiev gap phenomenon, the thermistor problem and the duality theory. In this direction we also refer to the recent paper by Borowski–Chlebicka–De Filippis–Miasojedow [10] about the absence and presence of Lavrentiev's phenomenon for double phase functionals.

In a recent work by Arora–Crespo-Blanco–Winkert [2] a new double phase operator with logarithmic perturbation of the form

$$\begin{aligned} \operatorname{div} \mathcal{K}(u) := \operatorname{div} \Big(&|\nabla u|^{p(x)-2} \nabla u \\ &+ \mu(x) \left[\log(e + |\nabla u|) + \frac{|\nabla u|}{q(x)(e + |\nabla u|)} \right] |\nabla u|^{q(x)-2} \nabla u \Big), \end{aligned} \tag{1.3}$$

has been introduced, while the corresponding energy functional is given by

$$u \mapsto \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x) \frac{|\nabla u|^{q(x)}}{q(x)} \log(e + |\nabla u|) \right) \mathrm{d}x, \tag{1.4}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, e stands for Euler's number, $p, q \in C(\overline{\Omega})$ with $1 < p(x) \leq q(x)$ for all $x \in \overline{\Omega}$ and $\mu \in L^1(\Omega)$. Here, u belongs to the Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ which is generated by the generalized N-function

$$\mathcal{H}_{\log}(x, t) = t^{p(x)} + \mu(x) t^{q(x)} \log(e + t) \quad \text{for all } (x, t) \in \overline{\Omega} \times [0, \infty). \tag{1.5}$$

If $p = q$ are constant, then the functional (1.4) has the shape (we ignore the constants in front)

$$\omega \mapsto \int_\Omega \left[|\nabla \omega|^p + \mu(x) |\nabla \omega|^p \log(e + |\nabla \omega|) \right] \mathrm{d}x. \tag{1.6}$$

The functional (1.6) has been studied by Baroni–Colombo–Mingione [5] in order to prove the local Hölder continuity of the gradient of local minimizers of (1.6) provided $0 \leq \mu(\cdot) \in$

$C^{0,\alpha}(\overline{\Omega})$. Recently, De Filippis–Mingione [18] considered the functional

$$\omega \mapsto \int_\Omega \left[|\nabla \omega| \log(1+|\nabla \omega|) + \mu(x)|\nabla \omega|^q\right] \mathrm{d}x, \tag{1.7}$$

and proved the local Hölder continuity of the gradients of local minimizers of (1.7) whenever $0 \leq \mu(\cdot) \in C^{0,\alpha}(\overline{\Omega})$ and $1 < q < 1 + \frac{\alpha}{n}$. We point out that (1.7) has its origin in functionals with nearly linear growth given by

$$\omega \mapsto \int_\Omega |\nabla \omega| \log(1+|\nabla \omega|) \,\mathrm{d}x, \tag{1.8}$$

which has been discussed as a particular case by Fuchs–Mingione [27]. The authors proved that local minimizers of (1.8) have Hölder continuous first derivatives. It should be noted that functionals of the form (1.8) appear, for example, in the theory of plasticity with logarithmic hardening, see, Seregin–Frehse [45] and Fuchs–Seregin [28].

In this paper we are interested in elliptic equations driven by the logarithmic double phase operator (1.3) and with superlinear right-hand sides in the domain and on the boundary. In addition, we also prove some results of independent interest related to the underlying function space $W^{1,\mathcal{H}_{\log}}(\Omega)$ as well as a priori bounds for related weak solutions of problems involving (1.3). To be more precise, in the first part of the paper we are interested in an appropriate norm in the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$. Indeed, we are going to prove that

$$\begin{aligned}\|u\|_{1,\mathcal{H}_{\log}}^\circ = \inf \Bigg\{\lambda > 0\colon \int_\Omega \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\lambda}\right|^{q(x)} \log\left(e + \frac{|\nabla u|}{\lambda}\right)\right) \mathrm{d}x \\ + \int_\Omega \omega_1(x)\left|\frac{u}{\lambda}\right|^{\zeta_1(x)} \mathrm{d}x + \int_{\partial\Omega} \omega_2(x)\left|\frac{u}{\lambda}\right|^{\zeta_2(x)} \mathrm{d}\sigma \leq 1\Bigg\},\end{aligned}$$

is an equivalent norm on $W^{1,\mathcal{H}_{\log}}(\Omega)$ where we allow the exponents $1 \leq \zeta_1(\cdot), \zeta_2(\cdot) \in C(\overline{\Omega})$ to be critical with respect to the exponent $1 < p(\cdot) \in C(\overline{\Omega})$, that is $1 \leq \zeta_1(x) \leq p^*(x)$ and $1 \leq \zeta_2(x) \leq p_*(x)$ for all $x \in \overline{\Omega}$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N \end{cases}, \quad p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N \end{cases}. \tag{1.9}$$

Note that if $\zeta_1(x) = p^*(x)$ for some $x \in \overline{\Omega}$, then we have to suppose that $p \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, that is, p must be log-Hölder continuous, see Section 2 for the details. Similarly, if $\zeta_2(x) = p_*(x)$ for some $x \in \overline{\Omega}$, then $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$. These restriction in the critical cases are due to the Sobolev embedding theorem for variable exponents for which these additional regularity conditions are needed. The equivalent norm on $W^{1,\mathcal{H}_{\log}}(\Omega)$ given above seems to be the most general form for spaces generated by (1.5).

In the second part of this paper we discuss the boundedness of weak solutions of nonlinear Neumann problems in the general form

$$-\operatorname{div} \mathcal{K}(u) = \mathcal{B}(x,u,\nabla u) \quad \text{in } \Omega, \qquad \mathcal{K}(u)\cdot \nu = \mathcal{C}(x,u) \quad \text{on } \partial\Omega, \tag{1.10}$$

where $\operatorname{div} \mathcal{K}$ denotes the logarithmic double phase operator (1.3) while $\mathcal{B}\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{C}\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions that fulfill general growth conditions. We study both the critical and the subcritical case and prove that every weak solution of (1.10) is bounded in both $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$. In the subcritical case we can also give an explicit dependence of the norms on the data. The proofs of these results are mainly based on an appropriate version of De Giorgi's iteration along with localization arguments. Such results can be applied to several other problems of similar type involving the logarithmic double phase operator and general right-hand sides.

In the last part we are interested in the existence and multiplicity of solutions of nonhomogeneous Neumann problems involving the operator (1.3). Precisely, for a given bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we study the equation

$$\begin{aligned} -\operatorname{div} \mathcal{K}(u) + |u|^{p(x)-2}u &= f(x,u) && \text{in } \Omega, \\ \mathcal{K}(u) \cdot \nu &= g(x,u) - |u|^{p(x)-2}u && \text{on } \partial\Omega, \end{aligned} \tag{1.11}$$

where $\operatorname{div} \mathcal{K}$ denotes the logarithmic double phase operator with variable exponents given in (1.3), $\nu(x)$ is the outer unit normal of Ω at $x \in \partial\Omega$, and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ as well as $g\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions with certain conditions which will be specified below. For $r \in C(\overline{\Omega})$, we define

$$r_- = \min_{x \in \overline{\Omega}} r(x), \quad r_+ = \max_{x \in \overline{\Omega}} r(x), \quad C_+(\overline{\Omega}) = \{r \in C(\overline{\Omega}) \colon 1 < r_-\}.$$

We suppose the following conditions:

(H_1) $p, q \in C_+(\overline{\Omega})$ with $p(x) \leq q(x) < (p_-)_*$ for all $x \in \overline{\Omega}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$.

(H_2) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $g\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that the following hold, whereby $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ and $G(x,t) = \int_0^t g(x,s)\,\mathrm{d}s$:

(i) there exists $r, \ell \in C_+(\overline{\Omega})$ with $r_+ < (p_-)^*$ and $\ell_+ < (p_-)_*$ and constants $K_1, K_2 > 0$ such that

$$\begin{aligned} |f(x,t)| &\leq K_1 \left(1 + |t|^{r(x)-1}\right) && \text{for a.a. } x \in \Omega, \\ |g(x,t)| &\leq K_2 \left(1 + |t|^{\ell(x)-1}\right) && \text{for a.a. } x \in \partial\Omega, \end{aligned}$$

and for all $t \in \mathbb{R}$;

(ii)

$$\begin{aligned} \lim_{t \to \pm\infty} \frac{F(x,t)}{|t|^{q_+} \log(e + |t|)} &= +\infty && \text{uniformly for a.a. } x \in \Omega, \\ \lim_{t \to \pm\infty} \frac{G(x,t)}{|t|^{q_+} \log(e + |t|)} &= +\infty && \text{uniformly for a.a. } x \in \partial\Omega. \end{aligned}$$

(iii)

$$\begin{aligned} \lim_{t \to 0} \frac{F(x,t)}{|t|^{p(x)}} &= 0 && \text{uniformly for a.a. } x \in \Omega, \\ \lim_{t \to 0} \frac{G(x,t)}{|t|^{p(x)}} &= 0 && \text{uniformly for a.a. } x \in \partial\Omega; \end{aligned}$$

(iv) there exist $\alpha, \beta, \zeta, \theta \in C_+(\overline{\Omega})$ with

$$\begin{aligned} \min\{\alpha_-, \beta_-\} &\in \left((r_+ - p_-)\frac{N}{p_-}, r_+ \right), \\ \min\{\zeta_-, \theta_-\} &\in \left((\ell_+ - p_-)\frac{N-1}{p_- - 1}, \ell_+ \right), \end{aligned}$$

and $K_3, K_4 > 0$ such that

$$0 < K_3 \leq \liminf_{t \to +\infty} \frac{f(x,t)t - q_+ \left(1 + \frac{\kappa}{q_-}\right) F(x,t)}{|t|^{\alpha(x)}},$$

$$0 < K_3 \leq \liminf_{t \to -\infty} \frac{f(x,t)t - q_+ \left(1 + \frac{\kappa}{q_-}\right) F(x,t)}{|t|^{\beta(x)}},$$

uniformly for a.a. $x \in \Omega$ and

$$0 < K_4 \leq \liminf_{t \to +\infty} \frac{g(x,t)t - q_+ \left(1 + \frac{\kappa}{q_-}\right) G(x,t)}{|t|^{\zeta(x)}},$$

$$0 < K_4 \leq \liminf_{t \to -\infty} \frac{g(x,t)t - q_+ \left(1 + \frac{\kappa}{q_-}\right) G(x,t)}{|t|^{\theta(x)}},$$

uniformly for a.a. $x \in \partial\Omega$, where $\kappa = e/(e + t_0)$ with t_0 being the only positive solution of $t_0 = e \log(e + t_0)$, see Lemma 2.4;

Our first result is the following one.

Theorem 1.1 *Let hypotheses* (H_1) *and* (H_2) *be satisfied. Then there exist nontrivial weak solutions* $u_0, v_0 \in W^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ *of problem* (1.11) *such that* $u_0 \geq 0$ *and* $v_0 \leq 0$ *a.e. in* Ω.

In order to get a least energy sign-changing solution, we have to strengthen our hypotheses as follows.

(H_1') $p, q \in C_+(\overline{\Omega})$ with $p(x) \leq q(x) < q_+ + 1 < (p_-)_*$ for all $x \in \overline{\Omega}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$.

(H_2') $f: \Omega \times \mathbb{R} \to \mathbb{R}$ and $g: \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions that fulfill hypotheses (H_2)(i), (iii), (iv) (now denoted as (H_2')(i'), (iii'), (iv'), respectively), and

(ii') the functions

$$t \mapsto \frac{f(x,t)}{|t|^{q_+}} \quad \text{and} \quad t \mapsto \frac{g(x,t)}{|t|^{q_+}}$$

are increasing in $(-\infty, 0)$ and in $(0, \infty)$ for a.a. $x \in \Omega$ and for a.a. $x \in \partial\Omega$, respectively.

Remark 1.2 Note that hypothesis (H_2')(ii') implies (H_2)(ii).

Our main result concerning the existence of a sign-changing solution reads as follows.

Theorem 1.3 *Let hypotheses* (H_1') *and* (H_2') *be satisfied. Then there exists a nontrivial weak solution* $w_0 \in W^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ *of problem* (1.11) *which turns out to be a least energy sign-changing solution.*

The idea in the proof of Theorem 1.3 is to minimize the corresponding energy functional $\varphi(\cdot)$ of (1.11) over the nodal Nehari manifold

$$\mathcal{N}_0 = \left\{u \in W^{1,\mathcal{H}_{\log}}(\Omega) \colon \pm u^\pm \in \mathcal{N}\right\},$$

where $u^{\pm} = \max\{\pm u, 0\}$ and $\mathcal{N}$ is the classical Nehari manifold defined by

$$\mathcal{N} = \left\{u \in W^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\} : \langle \varphi'(u), u\rangle = 0\right\}.$$

It is easy to see that all sign-changing solutions of (1.11) belong to $\mathcal{N}_0$. Thus, the global minimizer of φ over $\mathcal{N}_0$ must be a least energy sign-changing solution of (1.11). In contrast to the work by Arora–Crespo-Blanco–Winkert [2], we do not need a monotonicity condition on the exponent p in the following sense: there exists a vector $y \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$ the function

$$h_x(t) = p(x + ty) \quad \text{with } y \in I_x = \{t \in \mathbb{R} : x + ty \in \Omega\}$$

is monotone. We overcome this fact by using the new equivalent norm obtained in Sect. 3 and the appearance of the terms $|u|^{p(x)-2}u$ in Ω and $\partial\Omega$, respectively, in problem (1.11). To the best of our knowledge, this is the first work for the logarithmic double phase operator given in (1.3) with a nonhomogeneous Neumann boundary condition. But also for homogeneous Dirichlet problem, only a few papers exist. Recently, Lu–Vetro–Zeng [34] introduced the operator

$$u \mapsto \Delta_{\mathcal{H}_L} u = \operatorname{div}\left(\frac{\mathcal{H}_L'(x, |\nabla u|)}{|\nabla u|}\nabla u\right), \quad u \in W^{1,\mathcal{H}_L}(\Omega), \tag{1.12}$$

where $\mathcal{H}_L : \Omega \times [0, \infty) \to [0, \infty)$ is given by

$$\mathcal{H}_L(x, t) = [t^{p(x)} + \mu(x)t^{q(x)}] \log(e + \alpha t),$$

with $\alpha \geq 0$ as well as $p, q \in C(\overline{\Omega})$ such that $1 < p(x) < N$ and $p(x) < q(x)$ for all $x \in \overline{\Omega}$, and $0 \leq \mu(\cdot) \in L^1(\Omega)$. Note that (1.12) is a different operator than the one in this paper. Moreover, the work by Lu–Vetro–Zeng [34] can be seen as the extension of Vetro–Zeng [47] from the constant exponent case to the variable one, see also the recent work by Cen–Lu–Vetro–Zeng [11] for multivalued problems with such operator. We also mention the work by Vetro–Winkert [46] who obtained the existence of a solution to the logarithmic problem with convection term of the form

$$-\operatorname{div}\mathcal{K}(u) = f(x, u, \nabla u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.13}$$

where $\operatorname{div}\mathcal{K}$ is as in (1.3) and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying a general growth condition. The authors prove the boundedness, closedness and compactness of the related solution set to (1.13). Furthermore, appropriate conditions are supposed in order to show the uniqueness of the solution of (1.13). Finally, we also mention some works dealing with double phase operators without logarithmic perturbation but with Neumann or Robin boundary condition. We refer to the papers by Amoroso–Crespo-Blanco–Pucci–Winkert [1], Borer–Pimenta–Winkert [9], El Manouni–Marino–Winkert [21], Farkas–Fiscella–Winkert [25], Fiscella–Marino–Pinamonti–Verzellesi [26], Gasiński–Winkert [29], Papageorgiou–Rădulescu– Repovš [38], Papageorgiou–Vetro–Vetro [42], Papageorgiou–Rădulescu–Zhang [40], Papageorgiou–Zhang [44], Zeng–Bai–Gasiński–Winkert [51], Zeng–Rădulescu–Winkert [52, 53], see also the very related works by Chen–Qin–Rădulescu–Tang [12], Fang–Rădulescu–Zhang [24], Liu–Pucci [33] and Papageorgiou–Rădulescu–Sun [41].

The paper is organized as follows. In Sect. 2 we recall the basic facts about the generalized N-function (1.5) and the related logarithmic double phase operator following the work by Arora–Crespo-Blanco–Winkert [2]. We also recall some tools which are needed in the sequel, for example, the Poincaré-Miranda existence theorem. In Sect. 3 we prove the existence of a new and very general equivalent norm in $W^{1,\mathcal{H}_{\log}}(\Omega)$ while Sect. 4 presents boundedness results in the critical and subcritical case for weak solutions of (1.11). Finally, in Sects. 5 and 6 we prove our existence results stated in Theorems 1.1 and 1.3.

2 Preliminaries

In this section we recall the basic facts about variable exponent Sobolev spaces and Musielak-Orlicz Sobolev spaces. We also mention some tools which are needed later. We refer to the monographs by Diening–Harjulehto–Hästö–Růžička [20] and Harjulehto–Hästö [30] as well as the recent paper by Arora–Crespo-Blanco–Winkert [2]. To this end, for $1 \leq r \leq \infty$, we denote by $L^r(\Omega)$ the usual Lebesgue spaces equipped with the norm $\|\cdot\|_r$ and by $W^{1,r}(\Omega)$ the Sobolev spaces endowed with the norm $\|\cdot\|_{1,r} = \|\nabla\cdot\|_r + \|\cdot\|_r$. Further, for $t \in \mathbb{R}$ we write $t^{\pm} = \max\{\pm t, 0\}$, i.e. $t = t^+ - t^-$ and $|t| = t^+ + t^-$ and so for any function $u\colon \Omega \to \mathbb{R}$, we denote $u^{\pm}(x) = [u(x)]^{\pm}$ for all $x \in \Omega$.

Let $r \in C_+(\overline{\Omega})$ and let $M(\Omega)$ be the set of all equivalence classes of measurable functions $u\colon \Omega \to \mathbb{R}$ which coincide almost everywhere. Then we denote by $L^{r(\cdot)}(\Omega)$ the Lebesgue space with variable exponent given by

$$L^{r(\cdot)}(\Omega) = \left\{u \in M(\Omega)\colon \varrho_{r(\cdot)}(u) < \infty\right\},$$

with the related modular

$$\varrho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)}\,\mathrm{d}x$$

and the norm

$$\|u\|_{r(\cdot)} = \inf\left\{\lambda > 0\colon \varrho_{r(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

We know that $L^{r(\cdot)}(\Omega)$ is a separable, uniformly convex and reflexive Banach space with dual space given by $\left[L^{r(\cdot)}(\Omega)\right]^* = L^{r'(\cdot)}(\Omega)$, where $r' \in C_+(\overline{\Omega})$ is the conjugate variable exponent of r defined by $r'(x) = r(x)/[r(x)-1]$ for all $x \in \overline{\Omega}$. We also have a Hölder type inequality given by

$$\int_\Omega |uv|\,\mathrm{d}x \leq \left[\frac{1}{r_-} + \frac{1}{r'_-}\right]\|u\|_{r(\cdot)}\|v\|_{r'(\cdot)} \leq 2\|u\|_{r(\cdot)}\|v\|_{r'(\cdot)}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r'(\cdot)}(\Omega)$. Also, if $r_1, r_2 \in C_+(\overline{\Omega})$ and $r_1(x) \leq r_2(x)$ for all $x \in \overline{\Omega}$, we have the continuous embedding $L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$.

The following proposition shows the relation between the norm and the modular, see Fan–Zhao [23, Theorems 1.2 and 1.3].

Proposition 2.1 *Let $r \in C_+(\overline{\Omega})$, $\lambda > 0$, and $u \in L^{r(\cdot)}(\Omega)$, then the following hold:*

(i) $\|u\|_{r(\cdot)} = \lambda$ *if and only if* $\varrho_{r(\cdot)}\left(\frac{u}{\lambda}\right) = 1$ *with* $u \neq 0$;
(ii) $\|u\|_{r(\cdot)} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\varrho_{r(\cdot)}(u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) if $\|u\|_{r(\cdot)} < 1$*, then* $\|u\|_{r(\cdot)}^{r_+} \leq \varrho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r_-}$;
(iv) if $\|u\|_{r(\cdot)} > 1$*, then* $\|u\|_{r(\cdot)}^{r_-} \leq \varrho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot)}^{r_+}$;
(v) $\|u\|_{r(\cdot)} \to 0$ *if and only if* $\varrho_{r(\cdot)}(u) \to 0$;
(vi) $\|u\|_{r(\cdot)} \to +\infty$ *if and only if* $\varrho_{r(\cdot)}(u) \to +\infty$.

The related Sobolev space $W^{1,r(\cdot)}(\Omega)$ for $r \in C_+(\overline{\Omega})$ is given by

$$W^{1,r(\cdot)}(\Omega) = \left\{ u \in L^{r(\cdot)}(\Omega) \colon |\nabla u| \in L^{r(\cdot)}(\Omega) \right\},$$

with modular

$$\varrho_{1,r(\cdot)}(u) = \varrho_{r(\cdot)}(u) + \varrho_{r(\cdot)}(\nabla u),$$

where $\varrho_{r(\cdot)}(\nabla u) = \varrho_{r(\cdot)}(|\nabla u|)$, and with the norm

$$\|u\|_{1,r(\cdot)} = \inf \left\{ \lambda > 0 \colon \varrho_{1,r(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

The space $W^{1,r(\cdot)}(\Omega)$ is a separable and reflexive Banach space.

For $r \in C_+(\overline{\Omega})$ we recall the critical Sobolev variable exponents r^* and r_* given by (1.9), hence

$$r^*(x) = \begin{cases} \frac{Nr(x)}{N-r(x)} & \text{if } r(x) < N \\ +\infty & \text{if } r(x) \geq N \end{cases}, \quad \text{for all } x \in \overline{\Omega},$$

$$r_*(x) = \begin{cases} \frac{(N-1)r(x)}{N-r(x)} & \text{if } r(x) < N \\ +\infty & \text{if } r(x) \geq N \end{cases}, \quad \text{for all } x \in \overline{\Omega}.$$

Furthermore, let σ be the $(N-1)$-dimensional Hausdorff measure on the boundary $\partial\Omega$ and indicate by $L^{r(\cdot)}(\partial\Omega)$ the boundary Lebesgue space endowed with the norm $\|\cdot\|_{r(\cdot),\partial\Omega}$ and related modular $\varrho_{r(\cdot),\partial\Omega}(\cdot)$, that is,

$$\varrho_{r(\cdot),\partial\Omega}(u) = \int_{\partial\Omega} |u|^{r(x)} \, \mathrm{d}\sigma \quad \text{and} \quad \|u\|_{r(\cdot),\partial\Omega} = \inf \left\{ \lambda > 0 \colon \varrho_{r(\cdot),\partial\Omega}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$$

whenever $u \in L^{r(\cdot)}(\partial\Omega)$ for $r \in C_+(\overline{\Omega})$. We can consider a trace operator, i.e., a continuous linear operator $T \colon W^{1,r(\cdot)}(\Omega) \to L^{m(\cdot)}(\partial\Omega)$ for all $m \in C(\overline{\Omega})$ with $1 \leq m(x) < r_*(x)$ for every $x \in \overline{\Omega}$, such that

$$T(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,r(\cdot)}(\Omega) \cap C(\overline{\Omega}).$$

If it also holds that $r \in W^{1,\gamma}(\Omega)$ with $\gamma > N$, then we can take any $m \in C(\overline{\Omega})$ with $1 \leq m(x) \leq r_*(x)$ for every $x \in \overline{\Omega}$. By the trace embedding theorem, it is known that γ is compact for any $r \in C(\overline{\Omega})$ with $1 \leq r(x) < r_*(x)$ for all $x \in \overline{\Omega}$, see Fan [22, Corollary 2.4]. In this paper we avoid the notation of the trace operator and we consider all the restrictions of Sobolev functions to the boundary $\partial\Omega$ in the sense of traces.

The following lemma can be proved similarly as Proposition 2.1.

Proposition 2.2 *Let $r \in C_+(\overline{\Omega})$, $\lambda > 0$, and $u \in L^{r(\cdot)}(\partial\Omega)$, then the following hold:*

(i) $\|u\|_{r(\cdot),\partial\Omega} = \lambda$ *if and only if* $\varrho_{r(\cdot),\partial\Omega}\left(\frac{u}{\lambda}\right) = 1$ *with* $u \neq 0$*;*
(ii) $\|u\|_{r(\cdot),\partial\Omega} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\varrho_{r(\cdot),\partial\Omega}(u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) if $\|u\|_{r(\cdot),\partial\Omega} < 1$*, then* $\|u\|_{r(\cdot),\partial\Omega}^{r_+} \leq \varrho_{r(\cdot),\partial\Omega}(u) \leq \|u\|_{r(\cdot),\partial\Omega}^{r_-}$*;*
(iv) if $\|u\|_{r(\cdot),\partial\Omega} > 1$*, then* $\|u\|_{r(\cdot),\partial\Omega}^{r_-} \leq \varrho_{r(\cdot)}(u) \leq \|u\|_{r(\cdot),\partial\Omega}^{r_+}$*;*
(v) $\|u\|_{r(\cdot),\partial\Omega} \to 0$ *if and only if* $\varrho_{r(\cdot),\partial\Omega}(u) \to 0$*;*
(vi) $\|u\|_{r(\cdot),\partial\Omega} \to +\infty$ *if and only if* $\varrho_{r(\cdot),\partial\Omega}(u) \to +\infty$*.*

Moreover, the space $C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$ is the set of all functions $h\colon \overline{\Omega} \to \mathbb{R}$ being log-Hölder continuous, i.e. there exists a constant $C > 0$ such that

$$|h(x) - h(y)| \leq \frac{C}{|\log|x-y||} \quad \text{for all } x, y \in \overline{\Omega} \text{ with } |x-y| < \frac{1}{2}.$$

Note that for a bounded domain $\Omega \subset \mathbb{R}^N$ and $\gamma > N$ we have the following inclusions

$$C^{0,1}(\overline{\Omega}) \subset W^{1,\gamma}(\Omega) \subset C^{0,1-\frac{N}{\gamma}}(\overline{\Omega}) \subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}). \tag{2.1}$$

Now, we consider the nonlinear function $\mathcal{H}_{\log}\colon \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{H}_{\log}(x,t) = t^{p(x)} + \mu(x)t^{q(x)}\log(e+t),$$

where e stands for Euler's number while we suppose hypotheses (H_1). Clearly, $\mathcal{H}_{\log}(\cdot,t)$ is measurable for all $t \geq 0$, $\mathcal{H}_{\log}(x,0) = 0$ and $\mathcal{H}_{\log}(x,t) > 0$ for all $t > 0$. Also, $\mathcal{H}_{\log}$ satisfies the Δ_2-condition, that is,

$$\mathcal{H}_{\log}(x,2t) \leq K\mathcal{H}_{\log}(x,t)$$

for a.a. $x \in \Omega$, for all $t > 0$ and for some $K \geq 2$. Then, the Musielak-Orlicz space $L^{\mathcal{H}_{\log}}(\Omega)$ is given by

$$L^{\mathcal{H}_{\log}}(\Omega) = \left\{u \in M(\Omega)\colon \varrho_{\mathcal{H}_{\log}}(u) < +\infty\right\},$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}_{\log}} := \inf\left\{\beta > 0\colon \varrho_{\mathcal{H}_{\log}}\left(\frac{u}{\beta}\right) \leq 1\right\},$$

where $\varrho_{\mathcal{H}_{\log}}(\cdot)$ denotes the associated modular defined by

$$\varrho_{\mathcal{H}_{\log}}(u) := \int_\Omega \mathcal{H}_{\log}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\log(e+|u|)\right)\mathrm{d}x.$$

Note that $L^{\mathcal{H}_{\log}}(\Omega)$ is a separable, reflexive Banach space.

Next, we can define the Musielak–Orlicz Sobolev space $W^{1,\mathcal{H}_{\log}}(\Omega)$ by

$$W^{1,\mathcal{H}_{\log}}(\Omega) = \left\{u \in L^{\mathcal{H}_{\log}}(\Omega)\colon |\nabla u| \in L^{\mathcal{H}_{\log}}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}_{\log}} := \|u\|_{\mathcal{H}_{\log}} + \|\nabla u\|_{\mathcal{H}_{\log}}, \tag{2.2}$$

where $\|\nabla u\|_{\mathcal{H}_{\log}} := \|\,|\nabla u|\,\|_{\mathcal{H}_{\log}}$. We know that $W^{1,\mathcal{H}_{\log}}(\Omega)$ is a separable, reflexive Banach space.

The following embedding results can be found in the paper by Arora–Crespo-Blanco–Winkert [2, Propositions 3.7 and 3.9].

Proposition 2.3 *Let hypotheses (H_1) be satisfied, then the following hold:*

(i) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega)$ *is continuous;*

(ii) if $p \in C_+(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, *then* $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ *is continuous;*

(iii) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *is compact for* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$;

(iv) if $p \in C_+(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ *for some* $\gamma > N$, *then* $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p_*(\cdot)}(\partial\Omega)$ *is continuous;*

(v) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *is compact for* $r \in C(\overline{\Omega})$ *with* $1 \le r(x) < p_*(x)$ *for all* $x \in \overline{\Omega}$;

(vi) $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{\mathcal{H}_{\log}}(\Omega)$ *is compact.*

We equip the space $W^{1,\mathcal{H}_{\log}}(\Omega)$ with the following equivalent norm (see Proposition 3.1 in Section 3)

$$\|u\| = \inf\left\{\lambda > 0\colon \int_\Omega \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\lambda}\right|^{q(x)} \log\left(e + \left|\frac{\nabla u}{\lambda}\right|\right)\right) \mathrm{d}x + \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} \mathrm{d}x + \int_{\partial\Omega} \left|\frac{u}{\lambda}\right|^{p(x)} \mathrm{d}\sigma \le 1\right\}, \tag{2.3}$$

induced by the modular

$$\varrho(u) = \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)} \log(e + |\nabla u|)\right) \mathrm{d}x + \int_\Omega |u|^{p(x)} \mathrm{d}x + \int_{\partial\Omega} |u|^{p(x)} \mathrm{d}\sigma, \tag{2.4}$$

for all $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$.

The modular $\varrho(\cdot)$ in (2.4) is closely related to the norm $\|\cdot\|$ in (2.3) as seen below. First, we recall the following important lemma, see Arora–Crespo-Blanco–Winkert [2, Lemma 3.1]. Note that a function $h\colon (0,\infty) \to \mathbb{R}$ is called almost increasing if there exists $a \ge 1$ such that $h(s) \le a h(t)$ for all $0 < s < t$.

Lemma 2.4 *The function* $f_\varepsilon\colon [0,+\infty) \to [0,+\infty)$ *given by*

$$f_\varepsilon(t) = \frac{t^\varepsilon}{\log(e+t)}$$

is increasing for $\varepsilon \ge \kappa$ *and almost increasing for* $0 < \varepsilon < \kappa$ *with constant* a_ε, *where* $\kappa = e/(e+t_0)$, *with* t_0 *being the only positive solution of* $t_0 = e\log(e+t_0)$.

Proposition 2.5 *Let hypotheses* (H_1) *be satisfied, then the following hold:*

(i) $\|u\| = \lambda$ *if and only if* $\varrho\left(\frac{u}{\lambda}\right) = 1$ *for* $u \ne 0$ *and* $\lambda > 0$;
(ii) $\|u\| < 1$ *(resp.* $= 1, > 1$*) if and only if* $\varrho(u) < 1$ *(resp.* $= 1, > 1$*)*;
(iii) $\min\left\{\|u\|^{p_-}, \|u\|^{q_+ + \kappa}\right\} \le \varrho(u) \le \max\left\{\|u\|^{p_-}, \|u\|^{q_+ + \kappa}\right\}$;
(iv)

$$\frac{1}{a_\varepsilon}\min\left\{\|u\|^{p_-}, \|u\|^{q_+ + \varepsilon}\right\} \le \varrho(u) \le a_\varepsilon \max\left\{\|u\|^{p_-}, \|u\|^{q_+ + \varepsilon}\right\}$$

for $0 < \varepsilon < \kappa$, *where* κ *and* a_ε *are the same as in Lemma 2.4*;
(v) $\|u\| \to 0$ *if and only if* $\varrho(u) \to 0$;
(vi) $\|u\| \to \infty$ *if and only if* $\varrho(u) \to \infty$.

As shown in [2], the space $W^{1,\mathcal{H}_{\log}}(\Omega)$ is closed under truncation.

Proposition 2.6 *Let hypotheses* (H_1) *be satisfied, then*

(i) If $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$, *then* $u^\pm \in W^{1,\mathcal{H}_{\log}}(\Omega)$ *with* $\nabla(\pm u) = \nabla u 1_{\{\pm u > 0\}}$;
(ii) if $u_n \to u$ *in* $W^{1,\mathcal{H}_{\log}}(\Omega)$, *then* $u_n^\pm \to u^\pm$ *in* $W^{1,\mathcal{H}_{\log}}(\Omega)$.

The following lemma will be used later.

Lemma 2.7 *Let $Q > 1$ and $h\colon [0,\infty) \to [0,\infty)$ given by $h(t) = \frac{t}{Q(e+t)\log(e+t)}$. Then h attains its maximum value at t_0 and the value is $\frac{\kappa}{Q}$, where t_0 and κ are the same as in Lemma 2.4.*

Now, let $A\colon W^{1,\mathcal{H}_{\log}}(\Omega) \to W^{1,\mathcal{H}_{\log}}(\Omega)^*$ be the nonlinear operator defined by

$$\begin{aligned}
&\langle A(u), v\rangle \\
&:= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\
&\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\
&\quad + \int_\Omega |u|^{p(x)-2}uv \,\mathrm{d}x + \int_{\partial\Omega} |u|^{p(x)-2}uv \,\mathrm{d}\sigma
\end{aligned} \tag{2.5}$$

for all $u, v \in W^{1,\mathcal{H}_{\log}}(\Omega)$. The following proposition is a direct consequence of Proposition 3.4, see also Arora–Crespo-Blanco–Winkert [2, Theorem 4.5].

Proposition 2.8 *Let hypothesis (H_1) be satisfied. Then, the operator A given in (2.5) is bounded (that is, it maps bounded sets into bounded sets), continuous, strictly monotone and satisfies the (S_+)-property, that is,*

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \le 0,$$

imply $u_n \to u$ in $W^{1,\mathcal{H}_{\log}}(\Omega)$.

We also recall some basic inequalities for the logarithmic. For $s, t \ge 0$ and $C \ge 1$, we have

$$\log(e+st) \le \log(e+s) + \log(e+t), \tag{2.6}$$

$$\log(e+Cs) \le C\log(e+s), \tag{2.7}$$

and for $s, t \ge 0$ and $q \ge 1$, one has

$$\begin{aligned}
(s+t)^q \log(e+s+t) &\le (2s)^q \log(e+2s) + (2t)^q \log(e+2t) \\
&\le 2^{q+1}s^q \log(e+s) + 2^{q+1}t^q \log(e+t).
\end{aligned} \tag{2.8}$$

Finally, we present the main tools which are needed for the existence proofs. Given a Banach space X, we say that a functional $\varphi\colon X \to \mathbb{R}$ satisfies the Cerami condition or C-condition if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{as } n \to \infty,$$

contains a strongly convergent subsequence. Furthermore, we say that φ satisfies the Cerami condition at the level $c \in \mathbb{R}$ or the C_c-condition if this compactness property holds for all the sequences such that $\varphi(u_n) \to c$ as $n \to \infty$ instead of for all the bounded sequences.

The following version of the mountain-pass theorem is taken from the book by Papageorgiou–Rădulescu–Repovš [39, Theorem 5.4.6].

Theorem 2.9 *[Mountain-pass theorem] Let X be a Banach space and suppose that $\varphi \in C^1(X)$, $u_0, u_1 \in X$ with $\|u_1 - u_0\| > \delta > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} \le \inf\{\varphi(u)\colon \|u - u_0\| = \delta\} = m_\delta,$$

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t)) \text{ with } \Gamma = \{\gamma \in C([0,1], X)\colon \gamma(0) = u_0, \gamma(1) = u_1\},$$

and that φ satisfies the C_c-condition. Then $c \geq m_\delta$ and c is a critical value of φ. Moreover, if $c = m_\delta$, then there exists $u \in \partial B_\delta(u_0)$ such that $\varphi'(u) = 0$.

The quantitative deformation lemma given in the next result can be found in the book by Willem [48, Lemma 2.3].

Lemma 2.10 *[Quantitative deformation lemma] Let X be a Banach space, $\varphi \in C^1(X;\mathbb{R})$, $\emptyset \neq S \subseteq X$, $c \in \mathbb{R}$, $\varepsilon, \delta > 0$ be such that for all $u \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$ there holds $\|\varphi'(u)\|_* \geq 8\varepsilon/\delta$, where $S_r = \{u \in X \colon d(u,S) = \inf_{u_0 \in S} \|u - u_0\| < r\}$ for any $r > 0$. Then there exists $\eta \in C([0,1] \times X; X)$ such that*

(i) $\eta(t,u) = u$, *if* $t = 0$ *or if* $u \notin \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$;
(ii) $\varphi(\eta(1,u)) \leq c - \varepsilon$ *for all* $u \in \varphi^{-1}((-\infty, c+\varepsilon]) \cap S$;
(iii) $\eta(t,\cdot)$ *is an homeomorphism of* X *for all* $t \in [0,1]$;
(iv) $\|\eta(t,u) - u\| \leq \delta$ *for all* $u \in X$ *and* $t \in [0,1]$;
(v) $\varphi(\eta(\cdot,u))$ *is decreasing for all* $u \in X$;
(vi) $\varphi(\eta(t,u)) < c$ *for all* $u \in \varphi^{-1}((-\infty, c]) \cap S_\delta$ *and* $t \in (0,1]$.

Finally, we mention the Poincaré–Miranda existence theorem, see Dinca–Mawhin [19, Corollary 2.2.15].

Theorem 2.11 *[Poincaré-Miranda existence theorem] Let $P = [-t_1, t_1] \times \cdots \times [-t_N, t_N]$ with $t_i > 0$ for $i \in \{1, \ldots, N\}$ and*

$$d \colon P \to \mathbb{R}^N, \quad a = (a_1, \ldots, a_N) \mapsto d(a) = (d_1(a), \ldots, d_N(a))$$

be continuous. If for each $i \in \{1, \ldots, N\}$ one has

$$\begin{aligned} d_i(a) &\leq 0 \quad \text{when } a \in P \text{ and } a_i = -t_i, \\ d_i(a) &\geq 0 \quad \text{when } a \in P \text{ and } a_i = t_i, \end{aligned}$$

then d has at least one zero in P.

3 A new equivalent norm

In this section we are going to prove the existence of a new equivalent norm in the space $W^{1,\mathcal{H}_{\log}}(\Omega)$. In order to do this, in addition to (H_1), we also need the following hypotheses:

(H_3) (i) $\zeta_1, \zeta_2 \in C(\overline{\Omega})$ with $1 \leq \zeta_1(x) \leq p^*(x)$ and $1 \leq \zeta_2(x) \leq p_*(x)$ for all $x \in \overline{\Omega}$, where

(a_1) $p \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, if $\zeta_1(x) = p^*(x)$ for some $x \in \overline{\Omega}$;
(a_2) $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$, if $\zeta_2(x) = p_*(x)$ for some $x \in \overline{\Omega}$;

(ii) $\omega_1 \in L^\infty(\Omega)$ with $\omega_1(x) \geq 0$ for a.a. $x \in \Omega$;
(iii) $\omega_2 \in L^\infty(\partial\Omega)$ with $\omega_2(x) \geq 0$ for a.a. $x \in \partial\Omega$;
(iv) $\omega_1 \not\equiv 0$ or $\omega_2 \not\equiv 0$.

Next, we define the seminormed spaces

$$\begin{aligned} L^{\zeta_1(\cdot)}_{\omega_1}(\Omega) &= \left\{ u \in M(\Omega) \colon \int_\Omega \omega_1(x)|u|^{\zeta_1(x)}\,\mathrm{d}x < \infty \right\}, \\ L^{\zeta_2(\cdot)}_{\omega_2}(\partial\Omega) &= \left\{ u \in M(\Omega) \colon \int_{\partial\Omega} \omega_2(x)|u|^{\zeta_2(x)}\,\mathrm{d}\sigma < \infty \right\}, \end{aligned}$$

equipped with the associated seminorms

$$\|u\|_{\zeta_1(\cdot),\omega_1} = \inf\left\{\lambda > 0\colon \int_\Omega \omega_1(x)\left|\frac{u}{\lambda}\right|^{\zeta_1(x)} \mathrm{d}x \le 1\right\},$$
$$\|u\|_{\zeta_2(\cdot),\omega_2,\partial\Omega} = \inf\left\{\lambda > 0\colon \int_{\partial\Omega} \omega_2(x)\left|\frac{u}{\lambda}\right|^{\zeta_2(x)} \mathrm{d}\sigma \le 1\right\},$$

respectively. We define

$$\|u\|^{\bullet}_{1,\mathcal{H}_{\log}} = \|\nabla u\|_{\mathcal{H}_{\log}} + \|u\|_{\zeta_1(\cdot),\omega_1} + \|u\|_{\zeta_2(\cdot),\omega_2,\partial\Omega}, \tag{3.1}$$

and

$$\begin{aligned}\|u\|^{\circ}_{1,\mathcal{H}_{\log}} &= \inf\left\{\lambda > 0\colon \int_\Omega \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + \mu(x)\left|\frac{\nabla u}{\lambda}\right|^{q(x)} \log\left(e + \frac{|\nabla u|}{\lambda}\right)\right) \mathrm{d}x \right.\\ &\left.\quad + \int_\Omega \omega_1(x)\left|\frac{u}{\lambda}\right|^{\zeta_1(x)} \mathrm{d}x + \int_{\partial\Omega} \omega_2(x)\left|\frac{u}{\lambda}\right|^{\zeta_2(x)} \mathrm{d}\sigma \le 1\right\},\end{aligned} \tag{3.2}$$

which are norms on $W^{1,\mathcal{H}_{\log}}(\Omega)$. We are going to show that theses norms are both equivalent to the usual one $\|\cdot\|_{1,\mathcal{H}_{\log}}$ given in (2.2).

Proposition 3.1 *Let hypotheses (H_1) and (H_3) be satisfied. Then, $\|\cdot\|^{\bullet}_{1,\mathcal{H}_{\log}}$ and $\|\cdot\|^{\circ}_{1,\mathcal{H}_{\log}}$ given in* (3.1) *and* (3.2)*, respectively, are both equivalent norms on $W^{1,\mathcal{H}_{\log}}(\Omega)$.*

Proof We will show the proof only in the case $\zeta_1(x) = p^*(x)$ and $\zeta_2(x) = p_*(x)$ for all $x \in \overline{\Omega}$. The remaining cases can be shown similarly. To this end, assume that $p \in C(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$ which implies that $p \in C(\overline{\Omega}) \cap C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$, see (2.1). Then, for $u \in W^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\}$ we have

$$\int_\Omega \omega_1(x)\left(\frac{|u|}{\|u\|_{p^*(\cdot)}}\right)^{p^*(x)} \mathrm{d}x \le \|\omega_1\|_\infty \varrho_{p^*(\cdot)}\left(\frac{u}{\|u\|_{p^*(\cdot)}}\right) = \|\omega_1\|_\infty,$$
$$\int_{\partial\Omega} \omega_2(x)\left(\frac{|u|}{\|u\|_{p_*(\cdot)}}\right)^{p_*(x)} \mathrm{d}\sigma \le \|\omega_2\|_{\infty,\partial\Omega} \varrho_{p_*(\cdot),\partial\Omega}\left(\frac{u}{\|u\|_{p_*(\cdot)}}\right) = \|\omega_2\|_{\infty,\partial\Omega}.$$

Therefore,

$$\|u\|_{p^*(\cdot),\omega_1} \le \max\{1, \|\omega_1\|_\infty\}\|u\|_{p^*(\cdot)},$$
$$\|u\|_{p_*(\cdot),\omega_2,\partial\Omega} \le \max\{1, \|\omega_2\|_{\infty,\partial\Omega}\}\,\|u\|_{p_*(\cdot),\partial\Omega}.$$

From these inequalities and Proposition 2.3 (ii), (iv), we obtain

$$\begin{aligned}\|u\|^{\bullet}_{1,\mathcal{H}_{\log}} &\le \|\nabla u\|_{\mathcal{H}_{\log}} + \max\{1, \|\omega_1\|_\infty\}\|u\|_{p^*(\cdot)}\\ &\quad + \max\{1, \|\omega_2\|_{\infty,\partial\Omega}\}\|u\|_{p_*(\cdot),\partial\Omega}\\ &\le \|\nabla u\|_{\mathcal{H}_{\log}} + C_1\|u\|_{1,\mathcal{H}_{\log}} + C_2\|u\|_{1,\mathcal{H}_{\log}}\\ &\le C_3\|u\|_{1,\mathcal{H}_{\log}},\end{aligned} \tag{3.3}$$

for all $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$ with positive constants C_1, C_2 and C_3.

Now we will show that

$$\|u\|_{\mathcal{H}_{\log}} \le C_4\|u\|^{\bullet}_{1,\mathcal{H}_{\log}}, \tag{3.4}$$

for some $C_4 > 0$. Arguing indirectly and supposing that (3.4) is not satisfied, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\|u_n\|_{\mathcal{H}_{\log}} > n\|u_n\|^{\bullet}_{1,\mathcal{H}_{\log}} \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Taking $y_n = \frac{u_n}{\|u_n\|_{\mathcal{H}_{\log}}}$, we have $\|y_n\|_{\mathcal{H}_{\log}} = 1$ and (3.5) can be rewritten as

$$\frac{1}{n} > \|y_n\|^{\bullet}_{1,\mathcal{H}_{\log}}. \tag{3.6}$$

The relations $\|y_n\|_{\mathcal{H}_{\log}} = 1$ and $\|\nabla y_n\|_{\mathcal{H}_{\log}} < 1$ (see (3.6)) imply that the sequence $\{y_n\}_{n\in\mathbb{N}} \subset W^{1,\mathcal{H}_{\log}}(\Omega)$ is bounded. Then, applying Proposition 2.3 (ii), (iv), we may assume, up to a subsequence if necessary, that

$$y_n \rightharpoonup y \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad y_n \rightharpoonup y \quad \text{in } L^{p^*(\cdot)}(\Omega) \text{ and } L^{p_*(\cdot)}(\partial\Omega). \tag{3.7}$$

Since $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{\mathcal{H}_{\log}}(\Omega)$ is compact (see Proposition 2.3 (vi)), from (3.7), we obtain $y_n \to y$ in $L^{\mathcal{H}_{\log}}(\Omega)$ and since $\|y_n\|_{\mathcal{H}_{\log}} = 1$, it follows that $y \neq 0$. Now using the weak lower semicontinuity of the norm $\|\nabla \cdot\|_{\mathcal{H}_{\log}}$ and of the seminorms $\|\cdot\|_{p^*(\cdot),\omega_1}$, $\|\cdot\|_{p_*(\cdot),\omega_2,\partial\Omega}$ together with the convergence properties in (3.7) , we get, by passing to the limit in (3.6) as $n \to \infty$, that

$$0 \geq \|y\|^{\bullet}_{1,\mathcal{H}_{\log}} = \|\nabla y\|_{\mathcal{H}_{\log}} + \|y\|_{p^*(\cdot),\omega_1} + \|y\|_{p_*(\cdot),\omega_2,\partial\Omega}.$$

From this we conclude that $y \equiv L \neq 0$ is a constant and so we have, by using (H$_3$) (iv), that

$$0 \geq |L|\|1\|_{p^*(\cdot),\omega_1} + |L|\|1\|_{p_*(\cdot),\omega_2,\partial\Omega} > 0,$$

which is a contradiction. Hence, (3.4) must hold, which implies

$$\|u\|_{1,\mathcal{H}_{\log}} \leq C_7 \, \|u\|^{\bullet}_{1,\mathcal{H}_{\log}}, \tag{3.8}$$

for some $C_7 > 0$. From (3.3) and (3.8) we see that $\|\cdot\|_{1,\mathcal{H}_{\log}}$ and $\|\cdot\|^{\bullet}_{1,\mathcal{H}_{\log}}$ are equivalent.

In the second part we will show that $\|\cdot\|^{\bullet}_{1,\mathcal{H}_{\log}}$ and $\|\cdot\|^{\circ}_{1,\mathcal{H}_{\log}}$ are equivalent norms in $W^{1,\mathcal{H}_{\log}}(\Omega)$. Indeed, for $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$, one has

$$\begin{aligned}
&\int_\Omega \left(\left(\frac{|\nabla u|}{\|u\|^{\bullet}_{1,\mathcal{H}_{\log}}} \right)^{p(x)} + \mu(x) \left(\frac{|\nabla u|}{\|u\|^{\bullet}_{1,\mathcal{H}_{\log}}} \right)^{q(x)} \log \left(e + \frac{|\nabla u|}{\|u\|^{\bullet}_{1,\mathcal{H}_{\log}}} \right) \right) \mathrm{d}x \\
&+ \int_\Omega \omega_1(x) \left(\frac{|u|}{\|u\|^{\bullet}_{1,\mathcal{H}_{\log}}} \right)^{p^*(x)} \mathrm{d}x + \int_{\partial\Omega} \omega_2(x) \left(\frac{|u|}{\|u\|^{\bullet}_{1,\mathcal{H}_{\log}}} \right)^{p_*(x)} \mathrm{d}\sigma \\
&\leq \varrho_{\mathcal{H}_{\log}} \left(\frac{\nabla u}{\|\nabla u\|_{\mathcal{H}_{\log}}} \right) + \int_\Omega \omega_1(x) \left(\frac{|u|}{\|u\|_{p^*(\cdot),\omega_1}} \right)^{p^*(x)} \mathrm{d}x \\
&\quad + \int_{\partial\Omega} \omega_2(x) \left(\frac{|u|}{\|u\|_{p_*(\cdot),\omega_2,\partial\Omega}} \right)^{p_*(x)} \mathrm{d}\sigma \\
&= 3.
\end{aligned}$$

Hence, we have

$$\|u\|^{\circ}_{1,\mathcal{H}_{\log}} \leq 3\|u\|^{\bullet}_{1,\mathcal{H}_{\log}}. \tag{3.9}$$

Next, we show the other direction. We have

$$\begin{aligned}&\int_\Omega \left(\left(\frac{|\nabla u|}{\|u\|^\circ_{1,\mathcal{H}_{\log}}}\right)^{p(x)} + \mu(x)\left(\frac{|\nabla u|}{\|u\|^\circ_{1,\mathcal{H}_{\log}}}\right)^{q(x)} \log\left(e+\frac{|\nabla u|}{\|u\|^\circ_{1,\mathcal{H}_{\log}}}\right)\right) \mathrm{d}x\\ &+\int_\Omega \omega_1(x)\left(\frac{|u|}{\|u\|^\circ_{1,\mathcal{H}_{\log}}}\right)^{p^*(x)} \mathrm{d}x + \int_{\partial\Omega} \omega_2(x)\left(\frac{|u|}{\|u\|^\circ_{1,\mathcal{H}_{\log}}}\right)^{p_*(x)} \mathrm{d}\sigma \\ &\le \varrho^\circ_{1,\mathcal{H}_{\log}}\left(\frac{u}{\|u\|^\circ_{1,\mathcal{H}_{\log}}}\right),\end{aligned} \tag{3.10}$$

where $\varrho^\circ_{1,\mathcal{H}_{\log}}$ is the related modular to the norm $\|\cdot\|^\circ_{1,\mathcal{H}_{\log}}$ defined by

$$\begin{aligned}\varrho^\circ_{1,\mathcal{H}_{\log}}(u) = &\int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\log(e+|\nabla u|)\right) \mathrm{d}x\\ &+\int_\Omega \omega_1(x)|u|^{p^*(x)}\,\mathrm{d}x + \int_{\partial\Omega}\omega_2(x)|u|^{p_*(x)}\,\mathrm{d}\sigma.\end{aligned} \tag{3.11}$$

It is clear that the function $\lambda \mapsto \varrho^\circ_{1,\mathcal{H}_{\log}}(\lambda u)$ is continuous, convex and even and it is strictly increasing when $\lambda \in [0,\infty)$, whereby $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$. Hence, from the definition, we have

$$\|u\|^\circ_{1,\mathcal{H}_{\log}} = \lambda \quad \text{if and only if} \quad \varrho^\circ_{1,\mathcal{H}_{\log}}\left(\frac{u}{\lambda}\right) = 1.$$

Therefore, from this fact and (3.10), it follows that

$$\|\nabla u\|_{\mathcal{H}_{\log}} \le \|u\|^\circ_{1,\mathcal{H}_{\log}}, \quad \|u\|_{p^*(\cdot),\omega_1} \le \|u\|^\circ_{1,\mathcal{H}_{\log}}, \quad \text{and} \quad \|u\|_{p_*(\cdot),\omega_2,\partial\Omega} \le \|u\|^\circ_{1,\mathcal{H}_{\log}},$$

which implies

$$\frac{1}{3}\|u\|^\bullet_{1,\mathcal{H}_{\log}} \le \|u\|^\circ_{1,\mathcal{H}_{\log}}. \tag{3.12}$$

From (3.9) and (3.12) we obtain the last assertion of the proposition. □

Now, we are going to state the relation between the modular $\varrho^\circ_{1,\mathcal{H}_{\log}}(\cdot)$ and the associated norm $\|\cdot\|^\circ_{1,\mathcal{H}_{\log}}$.

Proposition 3.2 *Let hypotheses* (H_1) *and* (H_3) *be satisfied,* $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$, $\lambda \in \mathbb{R}$, *and*

$$r_1 := \min\{p_-, (\zeta_1)_-, (\zeta_2)_-\} \quad \textit{as well as} \quad r_2 := \max\{q_+ + \varepsilon, (\zeta_1)_+, (\zeta_2)_+\},$$

for $0 < \varepsilon < \kappa$, *where* κ *and* a_ε *are the same as in Lemma 2.4. Then the following hold:*

(i) $\|u\|^\circ_{1,\mathcal{H}_{\log}} = \lambda$ *if and only if* $\varrho^\circ_{1,\mathcal{H}_{\log}}\left(\frac{u}{\lambda}\right) = 1$ *for* $u \ne 0$ *and* $\lambda > 0$*;*

(ii) $\|u\|^\circ_{1,\mathcal{H}_{\log}} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\varrho^\circ_{1,\mathcal{H}_{\log}}(u) < 1$ *(resp.* $= 1, > 1$*);*

(iii) $\min\left\{\left(\|u\|^\circ_{1,\mathcal{H}_{\log}}\right)^{r_1}, \left(\|u\|^\circ_{1,\mathcal{H}_{\log}}\right)^{r_2}\right\} \le \varrho^\circ_{1,\mathcal{H}_{\log}}(u) \le \max\left\{\left(\|u\|^\circ_{1,\mathcal{H}_{\log}}\right)^{r_1}, \left(\|u\|^\circ_{1,\mathcal{H}_{\log}}\right)^{r_2}\right\}$;

(iv)

$$\begin{aligned}&\frac{1}{a_\varepsilon}\min\left\{\left(\|u\|^\circ_{1,\mathcal{H}_{\log}}\right)^{r_1}, \left(\|u\|^\circ_{1,\mathcal{H}_{\log}}\right)^{r_2}\right\}\\ &\le \varrho^\circ_{1,\mathcal{H}_{\log}}(u) \le a_\varepsilon \max\left\{\left(\|u\|^\circ_{1,\mathcal{H}_{\log}}\right)^{r_1}, \left(\|u\|^\circ_{1,\mathcal{H}_{\log}}\right)^{r_2}\right\};\end{aligned}$$

(v) $\|u\|^\circ_{1,\mathcal{H}_{\log}} \to 0$ *if and only if* $\varrho^\circ_{1,\mathcal{H}_{\log}}(u) \to 0$*;*
(vi) $\|u\|^\circ_{1,\mathcal{H}_{\log}} \to \infty$ *if and only if* $\varrho^\circ_{1,\mathcal{H}_{\log}}(u) \to \infty$*.*

Proof The proof can be done similarly as the proofs of Propositions 3.4 and 3.6 as well as Lemma 3.3 by Arora–Crespo-Blanco–Winkert [2]. □

Next we introduce the operator $B\colon W^{1,\mathcal{H}_{\log}}(\Omega) \to W^{1,\mathcal{H}_{\log}}(\Omega)^*$ given by

$$\begin{aligned}
&\langle B(u), v\rangle = \\
&\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\
&\quad + \int_\Omega \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\
&\quad + \int_\Omega \omega_1(x)|u|^{\zeta_1(x)-2}uv \,\mathrm{d}x + \int_{\partial\Omega} \omega_2(x)|u|^{\zeta_2(x)-2}uv \,\mathrm{d}\sigma,
\end{aligned} \tag{3.13}$$

for all $u, v \in W^{1,\mathcal{H}_{\log}}(\Omega)$.

The next lemma is taken from Arora–Crespo-Blanco–Winkert [2, Lemma 4.3] which is needed to show the (S_+)-property of the operator B.

Lemma 3.3 *[Young's inequality for the product of a power-law and a logarithm] Let* $s, t \geq 0$, $r > 1$ *then*

$$st^{r-1}\left[\log(e+t) + \frac{t}{r(e+t)}\right] \leq \frac{s^r}{r}\log(e+s) + t^r\left[\frac{r-1}{r}\log(e+t) + \frac{t}{r(e+t)}\right].$$

We have the following properties for the operator B.

Proposition 3.4 *Let hypotheses* (H_1) *and* (H_3) *be satisfied. Then, the operator* $B\colon W^{1,\mathcal{H}_{\log}}(\Omega) \to W^{1,\mathcal{H}_{\log}}(\Omega)^*$ *given in* (3.13) *is bounded, continuous and strictly monotone. If, in addition,* $1 < \zeta_1(x), \zeta_2(x)$ *for all* $x \in \overline{\Omega}$*, then* B *is of type* (S_+)*.*

Proof As before, we only study the case when $\zeta_1(x) = p^*(x)$ and $\zeta_2(x) = p_*(x)$ for all $x \in \overline{\Omega}$. Analogously to the proof of Theorem 4.4 by Arora–Crespo-Blanco–Winkert [2], we are able to show that B is bounded, continuous and strictly monotone. We only need to prove that B fulfills the (S_+)-property. For this purpose, let $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}_{\log}}(\Omega)$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\, \langle B(u_n), u_n - u\rangle \leq 0. \tag{3.14}$$

Taking Proposition 2.3 (ii), (iv) into account, yields, up to a subsequence if necessary, that

$$u_n \rightharpoonup u \quad \text{in } L^{p^*(\cdot)}(\Omega) \quad \text{and} \quad u_n \rightharpoonup u \quad \text{in } L^{p_*(\cdot)}(\partial\Omega). \tag{3.15}$$

The strict monotonicity of B along with (3.14) and (3.15) imply that

$$\begin{aligned}
0 \leq \liminf_{n\to\infty}\, \langle B(u_n) - B(u), u_n - u\rangle &\leq \limsup_{n\to\infty}\, \langle B(u_n) - B(u), u_n - u\rangle \\
&= \limsup_{n\to\infty}\, \langle B(u_n), u_n - u\rangle \leq 0.
\end{aligned}$$

Therefore we have

$$\lim_{n\to\infty}\, \langle B(u_n) - B(u), u_n - u\rangle = 0. \tag{3.16}$$

Since all terms in (3.16) are nonnegative, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot(\nabla u_n-\nabla u)\,\mathrm{d}x = 0,\\
&\lim_{n\to\infty}\int_\Omega \omega_1(x)\left(|u_n|^{p^*(x)-2}u_n - |u|^{p^*(x)-2}u\right)(u_n-u)\,\mathrm{d}x = 0,\\
&\lim_{n\to\infty}\int_{\partial\Omega} \omega_2(x)\left(|u_n|^{p_*(x)-2}u_n - |u|^{p_*(x)-2}u\right)(u_n-u)\,\mathrm{d}\sigma = 0.
\end{aligned}
\tag{3.17}
$$

We now claim that

$$
\lim_{n\to\infty}\int_\Omega w_1(x)|u_n|^{p^*(x)}\,\mathrm{d}x = \int_\Omega w_1(x)|u|^{p^*(x)}\,\mathrm{d}x, \tag{3.18}
$$

$$
\lim_{n\to\infty}\int_{\partial\Omega} \omega_2(x)|u_n|^{p_*(x)}\,\mathrm{d}\sigma = \int_{\partial\Omega} \omega_2(x)|u|^{p_*(x)}\,\mathrm{d}\sigma. \tag{3.19}
$$

Indeed, due to Young's inequality, we have

$$
\begin{aligned}
&\int_\Omega \omega_1(x)p^*(x)|u_n|^{p^*(x)-2}u_n(u_n-u)\,\mathrm{d}x\\
&\ge \int_\Omega \omega_1(x)p^*(x)|u_n|^{p^*(x)}\,\mathrm{d}x - \int_\Omega \omega_1(x)p^*(x)|u_n|^{p^*(x)-1}|u|\,\mathrm{d}x\\
&\ge \int_\Omega \omega_1(x)p^*(x)|u_n|^{p^*(x)}\,\mathrm{d}x - \int_\Omega \omega_1(x)(p^*(x)-1)|u_n|^{p^*(x)}\,\mathrm{d}x\\
&\quad -\int_\Omega \omega_1(x)|u|^{p^*(x)}\,\mathrm{d}x\\
&= \int_\Omega \omega_1(x)|u_n|^{p^*(x)}\,\mathrm{d}x - \int_\Omega \omega_1(x)|u|^{p^*(x)}\,\mathrm{d}x,
\end{aligned}
$$

for all $n\in\mathbb{N}$. Consequently,

$$
\begin{aligned}
&\int_\Omega w_1(x)|u_n|^{p^*(x)}\,\mathrm{d}x - \int_\Omega w_1(x)|u|^{p^*(x)}\,\mathrm{d}x\\
&\le \int_\Omega w_1(x)p^*(x)|u_n|^{p^*(x)-2}u_n(u_n-u)\,\mathrm{d}x\\
&= \int_\Omega \omega_1(x)p^*(x)\left(|u_n|^{p^*(x)-2}u_n - |u|^{p^*(x)-2}u\right)(u_n-u)\,\mathrm{d}x\\
&\quad + \int_\Omega \omega_1(x)p^*(x)|u|^{p^*(x)-2}u(u_n-u)\,\mathrm{d}x \to 0,
\end{aligned}
$$

as $n\to\infty$, because of (3.15) and (3.17). On the other hand, we have

$$
\begin{aligned}
&\int_\Omega w_1(x)|u|^{p^*(x)}\,\mathrm{d}x - \int_\Omega w_1(x)|u_n|^{p^*(x)}\,\mathrm{d}x\\
&\le \int_\Omega w_1(x)p^*(x)|u|^{p^*(x)-2}u(u-u_n)\,\mathrm{d}x \to 0,
\end{aligned}
$$

as $n\to\infty$ due to (3.15). From these observations, we conclude that (3.18) holds. An analogous argument allows us to get that (3.19) holds. Now, as it was done in the Claim of Theorem 3.3 by Crespo-Blanco–Gasiński–Harjulehto–Winkert [16], using (3.17) and (3.15), we can show in a very similar way that

$$\begin{aligned} \nabla u_n &\to \nabla u && \text{in } L^{p(\cdot)}(\Omega), \\ u_n &\to u && \text{in } L^{p^*(\cdot)}_{\omega_1}(\Omega), \\ u_n &\to u && \text{in } L^{p_*(\cdot)}_{\omega_2}(\partial\Omega). \end{aligned} \tag{3.20}$$

From (3.20) we conclude that

$$\begin{aligned} \nabla u_n &\to \nabla u && \text{in measure in } \Omega, \\ \omega_1(x)^{\frac{1}{p^*(x)}} u_n &\to \omega_1(x)^{\frac{1}{p^*(x)}} u && \text{in measure in } \Omega, \\ \omega_2(x)^{\frac{1}{p_*(x)}} u_n &\to \omega_2(x)^{\frac{1}{p_*(x)}} u && \text{in measure on } \partial\Omega. \end{aligned} \tag{3.21}$$

Exactly as in the proof of Theorem 4.4 by Arora–Crespo-Blanco–Winkert [2] we have, by using Young's inequality and Lemma 3.3, that

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(u_n - u)\,\mathrm{d}x \\ &\quad + \int_\Omega \mu(x) \left[\log(e + |\nabla u_n|) + \frac{|\nabla u_n|}{q(x)(e + |\nabla u_n|)}\right] |\nabla u_n|^{q(x)-2} \nabla u_n \cdot \nabla(u_n - u)\,\mathrm{d}x \\ &\geq \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\mu(x)}{q(x)} |\nabla u_n|^{q(x)} \log(e + |\nabla u_n|)\,\mathrm{d}x - \int_\Omega \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \log(e + |\nabla u|)\,\mathrm{d}x. \end{aligned}$$

Similarly, applying again Young's inequality, we obtain

$$\begin{aligned} \int_\Omega |u_n|^{p^*(x)-2} u_n(u_n - u)\,\mathrm{d}x &\geq \int_\Omega \frac{1}{p^*(x)} |u_n|^{p^*(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{p^*(x)} |u|^{p^*(x)}\,\mathrm{d}x, \\ \int_{\partial\Omega} |u_n|^{p_*(x)-2} u_n(u_n - u)\,\mathrm{d}\sigma &\geq \int_{\partial\Omega} \frac{1}{p_*(x)} |u_n|^{p_*(x)}\,\mathrm{d}\sigma - \int_{\partial\Omega} \frac{1}{p_*(x)} |u|^{p_*(x)}\,\mathrm{d}\sigma. \end{aligned}$$

From the above considerations we obtain that

$$\begin{aligned} \langle B(u_n), u_n - u \rangle &\geq \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\mu(x)}{q(x)} |\nabla u_n|^{q(x)} \log(e + |\nabla u_n|)\,\mathrm{d}x \\ &\quad - \int_\Omega \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \log(e + |\nabla u|)\,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\omega_1(x)}{p^*(x)} |u_n|^{p^*(x)}\,\mathrm{d}x - \int_\Omega \frac{\omega_1(x)}{p^*(x)} |u|^{p^*(x)}\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \frac{\omega_2(x)}{p_*(x)} |u_n|^{p_*(x)}\,\mathrm{d}\sigma - \int_{\partial\Omega} \frac{\omega_2(x)}{p_*(x)} |u|^{p_*(x)}\,\mathrm{d}\sigma. \end{aligned}$$

Thus, using (3.14), (3.18) as well as (3.19), and letting $n \to \infty$, we get

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \,\mathrm{d}x + \int_\Omega \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \log(e + |\nabla u|)\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{\omega_1(x)}{p^*(x)} |u|^{p^*(x)} \,\mathrm{d}x + \int_{\partial\Omega} \frac{\omega_2(x)}{p_*(x)} |u|^{p_*(x)} \,\mathrm{d}\sigma. \\
&\geq \limsup_{n\to\infty} \left(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \,\mathrm{d}x + \int_\Omega \frac{\mu(x)}{q(x)} |\nabla u_n|^{q(x)} \log(e + |\nabla u_n|)\,\mathrm{d}x \right) \\
&\quad + \int_\Omega \frac{\omega_1(x)}{p^*(x)} |u|^{p^*(x)} \,\mathrm{d}x + \int_{\partial\Omega} \frac{\omega_2(x)}{p_*(x)} |u|^{p_*(x)} \,\mathrm{d}\sigma.
\end{aligned}$$

We have thus shown that

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \,\mathrm{d}x + \int_\Omega \frac{\mu(x)}{q(x)} |\nabla u|^{q(x)} \log(e + |\nabla u|)\,\mathrm{d}x \\
&\geq \limsup_{n\to\infty} \left(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \,\mathrm{d}x + \int_\Omega \frac{\mu(x)}{q(x)} |\nabla u_n|^{q(x)} \log(e + |\nabla u_n|)\,\mathrm{d}x \right).
\end{aligned}$$

But then, from Fatou's Lemma, we get that the limes inferior fulfills the opposite inequalities which gives at the end (see [16, (3.8), (3.9) and (3.10)] for a more detailed argument)

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \left(\frac{|\nabla u_n|^{p(x)}}{p(x)} + \mu(x) \frac{|\nabla u_n|^{q(x)}}{q(x)} \log(e + |\nabla u_n|) \right) \mathrm{d}x \\
&= \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x) \frac{|\nabla u|^{q(x)}}{q(x)} \log(e + |\nabla u|) \right) \mathrm{d}x.
\end{aligned} \tag{3.22}$$

From the convergence in measure in (3.21), it follows that the left-hand side of (3.22) converge in measure to the right-hand side. Applying the converse of Vitali's theorem implies the uniform integrability of the sequence of functions

$$\left\{ \frac{|\nabla u_n|^{p(x)}}{p(x)} + \mu(x) \frac{|\nabla u_n|^{q(x)}}{q(x)} \log(e + |\nabla u_n|) \right\}_{n\in\mathbb{N}}.$$

This implies that the sequence

$$A_n := \left\{ |\nabla u_n - \nabla u|^{p(x)} + \mu(x) |\nabla u_n - \nabla u|^{q(x)} \log(e + |\nabla u_n - \nabla u|) \right\}_{n\in\mathbb{N}},$$

is uniformly integrable (use (2.8) in order to see it). Therefore, one has

$$0 = \lim_{n\to\infty} \int_\Omega A_n \,\mathrm{d}x.$$

Hence, from (3.20), we get

$$\begin{aligned}
&\lim_{n\to\infty} \varrho^\circ_{1,\mathcal{H}_{\log}}(u_n - u) \\
&= \lim_{n\to\infty} \left(\int_\Omega \left(|\nabla u_n - \nabla u|^{p(x)} + \mu(x) |\nabla u_n - \nabla u|^{q(x)} \log(e + |\nabla u_n - \nabla u|) \right) \mathrm{d}x \right. \\
&\qquad \left. + \int_\Omega \omega_1(x) |u_n - u|^{p^*(x)} \,\mathrm{d}x + \int_{\partial\Omega} \omega_2(x) |u_n - u|^{p_*(x)} \,\mathrm{d}\sigma \right) = 0,
\end{aligned}$$

see the definition in (3.11). But from Proposition 3.2 (v) together with Proposition 3.1 we know that this is equivalent to $\|u_n - u\|_{1,\mathcal{H}_{\log}} \to 0$. Therefore, $u_n \to u$ in $W^{1,\mathcal{H}_{\log}}(\Omega)$. □

4 A priori bounds

This section is devoted to the boundedness of weak solutions of problem (1.11). We give the result for more general problems than the one in (1.11) and consider the equation

$$
\begin{aligned}
-\operatorname{div}\mathcal{K}(u) &= \mathcal{B}(x,u,\nabla u) \qquad && \text{in } \Omega,\\
\mathcal{K}(u)\cdot \nu &= \mathcal{C}(x,u) && \text{on } \partial\Omega,
\end{aligned} \tag{4.1}
$$

where $\operatorname{div}\mathcal{K}$ is the logarithmic double phase operator given in (1.3), and $\mathcal{B}\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ as well as $\mathcal{C}\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions fulfilling general structure conditions, see below. We say that $u\in W^{1,\mathcal{H}_{\log}}(\Omega)$ is a weak solution of (4.1) if

$$
\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)\left[\log(e+|\nabla u|) + \frac{|\nabla u|}{q(x)(e+|\nabla u|)}\right]|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla v\,\mathrm{d}x\\
&= \int_\Omega \mathcal{B}(x,u,\nabla u)v\,\mathrm{d}x + \int_{\partial\Omega}\mathcal{C}(x,u)v\,\mathrm{d}\sigma.
\end{aligned}
$$

is satisfied for all $v\in W^{1,\mathcal{H}_{\log}}(\Omega)$.

We study first the subcritical case and assume the following hypotheses.

(H_4) $p,q\in C_+(\overline{\Omega})$ with $p(x)\le q(x)<p_*(x)$ for all $x\in\overline{\Omega}$, $0\le\mu(\cdot)\in L^\infty(\Omega)$ while $\mathcal{B}\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $\mathcal{C}\colon\partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions such that there exists $r,\ell\in C(\overline{\Omega})$ with $q(x)<r(x)<p^*(x)$, $q(x)<l(x)<p_*(x)$ for all $x\in C(\overline{\Omega})$ and

$$
\begin{aligned}
|\mathcal{B}(x,t,\xi)| &\le b\left[|\xi|^{\frac{p(x)}{r'(x)}} + |t|^{r(x)-1} + 1\right], \quad && \text{for a.a. } x\in\Omega,\\
|\mathcal{C}(x,t)| &\le c\left[|t|^{\ell(x)-1}+1\right] && \text{for a.a. } x\in\partial\Omega,
\end{aligned}
$$

for all $t\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^N$ with positive constants b,c.

Under (H_4) along with Proposition 2.3 (iii), (v), it is clear that the definition of a weak solution of (4.1) given above is well-defined. We have the following result.

Theorem 4.1 *Let hypotheses (H_4) be satisfied and let $u\in W^{1,\mathcal{H}_{\log}}(\Omega)$ be a weak solution of problem (4.1). Then, $u\in L^\infty(\Omega)\cap L^\infty(\partial\Omega)$ and*

$$
\|u\|_\infty + \|u\|_{\infty,\partial\Omega} \le C\left[1+\int_\Omega |u|^{r(x)}\,\mathrm{d}x + \int_{\partial\Omega}|u|^{\ell(x)}\,\mathrm{d}\sigma\right]^\alpha,
$$

where $C,\alpha>0$ are independent of u.

Proof Since $W^{1,\mathcal{H}_{\log}}(\Omega)\hookrightarrow W^{1,p(\cdot)}(\Omega)$ continuously by Proposition 2.3 (i) and because

$$
\mathcal{K}(\xi)\cdot\xi \ge |\xi|^{p(x)}
$$

for a.a. $x\in\Omega$ and for all $\xi\in\mathbb{R}^N$, the result follows from Winkert–Zacher [49, 50], see also Ho–Winkert [32]. □

For the critical case we have to redefine the critical variable exponents to $p\in C_+(\overline{\Omega})$ as

$$
\hat{p}^*(x) = \begin{cases} p^*(x) & \text{if } p^+ < N,\\ q_1(x) & \text{if } N\le p^+,\end{cases} \qquad \text{for all } x\in\overline{\Omega}
$$

and

$$\hat{p}_*(x) = \begin{cases} p_*(x) & \text{if } p^+ < N, \\ q_2(x) & \text{if } N \le p^+, \end{cases} \quad \text{for all } x \in \overline{\Omega},$$

where $q_1, q_2 \in C(\overline{\Omega})$ are arbitrarily chosen such that $p(x) < q_1(x) \le p^*(x)$ and $p(x) < q_2(x) \le p_*(x)$ for all $x \in \overline{\Omega}$. We suppose the following assumptions.

(H$_5$) $p \in C_+(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma > N$, $q \in C_+(\overline{\Omega})$ with $p(x) \le q(x) < \hat{p}^*(x)$ for all $x \in \overline{\Omega}$, $0 \le \mu(\cdot) \in L^\infty(\Omega)$ while $\mathcal{B}\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{C}\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that

$$|\mathcal{B}(x,t,\xi)| \le b\left[|\xi|^{p(x)\frac{\hat{p}^*(x)}{\hat{p}^*(x)-1}} + |t|^{\hat{p}^*(x)-1} + 1\right], \quad \text{for a.a. } x \in \Omega,$$
$$|\mathcal{C}(x,t)| \le c\left[|t|^{\hat{p}_*(x)-1} + 1\right] \quad \text{for a.a. } x \in \partial\Omega,$$

for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$ with positive constants b, c.

Theorem 4.2 *Let hypotheses (H$_5$) be satisfied. Then any weak solution of problem (4.1) is of class $u \in L^\infty(\Omega) \cap L^\infty(\partial\Omega)$.*

Proof From (H$_5$) and (2.1), we see that all embeddings in Proposition 2.3 hold true. Therefore, with the same arguments as in the proof of Theorem 4.1, we obtain the required assertion from the paper by Ho–Kim–Winkert–Zhang [31, Theorem 4.1]. □

5 Constant sign solutions

In this section we are concerned with the existence of constant sign solutions to problem (1.11). First, we introduce the energy functional $\varphi\colon W^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ related to (1.11) given by

$$\varphi(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)} \log(e + |\nabla u|) \right) \mathrm{d}x + \int_\Omega \frac{|u|^{p(x)}}{p(x)}\,\mathrm{d}x$$
$$+ \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)}\,\mathrm{d}\sigma - \int_\Omega F(x,u)\,\mathrm{d}x - \int_{\partial\Omega} G(x,u)\,\mathrm{d}\sigma.$$

We also need the following truncated energy functionals $\varphi_\pm\colon W^{1,\mathcal{H}_{\log}}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)} \log(e + |\nabla u|) \right) \mathrm{d}x + \int_\Omega \frac{|u|^{p(x)}}{p(x)}\,\mathrm{d}x$$
$$+ \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)}\,\mathrm{d}\sigma - \int_\Omega F(x,\pm u^\pm)\,\mathrm{d}x - \int_{\partial\Omega} G(x,\pm u^\pm)\,\mathrm{d}\sigma,$$

where $F(x,\pm t^\pm) = \int_0^t f(x,\pm s^\pm)\,\mathrm{d}s$ and $G(x,\pm t^\pm) = \int_0^t g(x,\pm s^\pm)\,\mathrm{d}s$. Note that the functionals φ and $\varphi_\pm$ are of class C^1.

We start by showing that the truncated functionals $\varphi_\pm$ satisfy the C-condition.

Proposition 5.1 *Let hypotheses (H$_1$) and (H$_2$) be satisfied. Then the functionals $\varphi_\pm$ satisfy the C-condition.*

Proof We only prove the assertion for φ_+, the proof for φ_- is very similar. To this end, let $M_1 > 0$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}_{\log}}(\Omega)$ be a sequence such that

$$|\varphi_+(u_n)| \le M_1 \quad \text{for all } n \in \mathbb{N}, \tag{5.1}$$

$$(1+\|u_n\|)\varphi_+'(u_n) \to 0 \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega)^*. \tag{5.2}$$

From (5.2) we can find a sequence $\varepsilon_n \to 0^+$ such that for all $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$

$$\begin{aligned}
&\left| \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla v \,\mathrm{d}x \right.\\
&+ \int_\Omega \mu(x)\left[\log(e+|\nabla u_n|) + \frac{|\nabla u_n|}{q(x)(e+|\nabla u_n|)}\right]|\nabla u_n|^{q(x)-2}\nabla u_n\cdot\nabla v\,\mathrm{d}x\\
&+ \int_\Omega |u_n|^{p(x)-2}u_n v\,\mathrm{d}x + \int_{\partial\Omega}|u_n|^{p(x)-2}u_n v\,\mathrm{d}\sigma - \int_\Omega f(x,u_n^+)v\,\mathrm{d}x\\
&\left. - \int_{\partial\Omega} g(x,u_n^+)v\,\mathrm{d}\sigma\right| \le \frac{\varepsilon_n\|v\|}{1+\|u_n\|} \quad \text{for all } n\in\mathbb{N}.
\end{aligned} \tag{5.3}$$

With view to Proposition 2.6 we know that $v^\pm \in W^{1,\mathcal{H}_{\log}}(\Omega)$ whenever $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$. Taking $v = -u_n^- \in W^{1,\mathcal{H}_{\log}}(\Omega)$ as test function in (5.3) and using $f(x,u_n^+)u_n^- = 0$ for a.a. $x \in \Omega$ as well as $g(x,u_n^+)u_n^- = 0$ for a.a. $x \in \partial\Omega$ we have

$$\begin{aligned}
&\varrho(u_n^-)\\
&\le \int_\Omega \left(\left|\nabla u_n^-\right|^{p(x)} + \mu(x)\left[\log\left(e+\left|\nabla u_n^-\right|\right) + \frac{\left|\nabla u_n^-\right|}{q(x)(e+\left|\nabla u_n^-\right|)}\right]\left|\nabla u_n^-\right|^{q(x)}\right)\mathrm{d}x\\
&\quad + \int_\Omega |u_n^-|^{p(x)}\,\mathrm{d}x + \int_{\partial\Omega}|u_n^-|^{p(x)}\,\mathrm{d}\sigma\\
&\le \varepsilon_n \quad \text{for all } n\in\mathbb{N},
\end{aligned}$$

where we have used that $\frac{|\nabla u_n^-|}{q(x)(e+|\nabla u_n^-|)}$ is nonnegative. This implies that

$$-u_n^- \to 0 \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega), \tag{5.4}$$

see Proposition 2.5 (v).

Claim 1: $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $L^{\alpha_-}(\Omega)$ and in $L^{\zeta_-}(\partial\Omega)$.

We take $v = u_n^+ \in W^{1,\mathcal{H}_{\log}}(\Omega)$ in (5.3) and obtain

$$\begin{aligned}
&-\left(1+\frac{\kappa}{q_-}\right)\varrho(u_n^+) + \int_\Omega f(x,u_n^+)u_n^+\,\mathrm{d}x + \int_{\partial\Omega} g(x,u_n^+)u_n^+\,\mathrm{d}\sigma\\
&\le -\int_\Omega \left|\nabla u_n^+\right|^{p(x)}\,\mathrm{d}x\\
&\quad - \int_\Omega \mu(x)\left[\log\left(e+\left|\nabla u_n^+\right|\right) + \frac{\left|\nabla u_n^+\right|}{q(x)(e+\left|\nabla u_n^+\right|)}\right]\left|\nabla u_n^+\right|^{q(x)}\,\mathrm{d}x\\
&\quad - \int_\Omega |u_n^+|^{p(x)}\,\mathrm{d}x - \int_{\partial\Omega}|u_n^+|^{p(x)}\,\mathrm{d}\sigma\\
&\quad + \int_\Omega f(x,u_n^+)u_n^+\,\mathrm{d}x + \int_{\partial\Omega} g(x,u_n^+)u_n^+\,\mathrm{d}\sigma \le \varepsilon_n \quad \text{for all } n\in\mathbb{N}.
\end{aligned} \tag{5.5}$$

To see this we first use Lemma 2.7 to estimate

$$\begin{aligned}
&\int_\Omega \mu(x)\left[\log(e+\left|\nabla u_n^+\right|)+\frac{\left|\nabla u_n^+\right|}{q(x)(e+\left|\nabla u_n^+\right|)}\right]\left|\nabla u_n^+\right|^{q(x)}\,\mathrm{d}x\\
&=\int_\Omega \mu(x)\left[1+\frac{\left|\nabla u_n^+\right|}{q(x)(e+\left|\nabla u_n^+\right|)\log(e+\left|\nabla u_n^+\right|)}\right]\left|\nabla u_n^+\right|^{q(x)}\log(e+\left|\nabla u_n^+\right|)\,\mathrm{d}x\\
&\le\left(1+\frac{\kappa}{q_-}\right)\int_\Omega \mu(x)\left|\nabla u_n^+\right|^{q(x)}\log(e+\left|\nabla u_n^+\right|)\,\mathrm{d}x.
\end{aligned}$$

Then, since $\left(1+\frac{\kappa}{q_-}\right)>1$, we have

$$\begin{aligned}
&\left(1+\frac{\kappa}{q_-}\right)\varrho(u_n^+)\\
&\ge\int_\Omega |\nabla u_n^+|^{p(x)}\,\mathrm{d}x+\left(1+\frac{\kappa}{q_-}\right)\int_\Omega \mu(x)|\nabla u_n^+|^{q(x)}\log(e+|\nabla u_n^+|)\,\mathrm{d}x\\
&\quad+\int_\Omega |u_n^+|^{p(x)}\,\mathrm{d}x+\int_{\partial\Omega}|u_n^+|^{p(x)}\,\mathrm{d}\sigma\\
&\ge\int_\Omega |\nabla u_n^+|^{p(x)}\,\mathrm{d}x\\
&\quad+\int_\Omega \mu(x)\left(\log(e+|\nabla u|^{q(x)})+\frac{|\nabla u_n^+|}{q(x)(e+|\nabla u_n^+|)}\right)|\nabla u_n^+|^{q(x)}\,\mathrm{d}x\\
&\quad+\int_\Omega |u_n^+|^{p(x)}\,\mathrm{d}x+\int_{\partial\Omega}|u_n^+|^{p(x)}\,\mathrm{d}\sigma.
\end{aligned}$$

Hence, we get (5.5).

On the other hand, from (5.1) and (5.4), it holds

$$\begin{aligned}
M_2&\ge q_+\left(1+\frac{\kappa}{q_-}\right)\varphi_+(u_n^+)\\
&\ge\left(1+\frac{\kappa}{q_-}\right)\varrho(u_n^+)-\int_\Omega q_+\left(1+\frac{\kappa}{q_-}\right)F(x,u_n^+)\,\mathrm{d}x\\
&\quad-\int_{\partial\Omega} q_+\left(1+\frac{\kappa}{q_-}\right)G(x,u_n^+)\,\mathrm{d}\sigma
\end{aligned}\tag{5.6}$$

for all $n\in\mathbb{N}$ and for some $M_2>0$. Adding (5.5) and (5.6) leads to

$$\begin{aligned}
&\int_\Omega\left(f(x,u_n^+)u_n^+-q_+\left(1+\frac{\kappa}{q_-}\right)F(x,u_n^+)\right)\mathrm{d}x\\
&+\int_{\partial\Omega}\left(g(x,u_n^+)u_n^+-q_+\left(1+\frac{\kappa}{q_-}\right)G(x,u_n^+)\right)\mathrm{d}\sigma\le M_3,
\end{aligned}\tag{5.7}$$

for all $n\in\mathbb{N}$ and for some $M_3>0$. Note that we can assume in (H_2)(iv), without any loss of generality, that $\alpha_-\le\beta_-$ and $\zeta_-\le\theta_-$. Therefore, applying hypothesis (H_2)(iv), we can find numbers $\hat{K}_3, \tilde{K}_3, \hat{K}_4, \tilde{K}_4 > 0$ such that

$$f(x,t)t - q_+\left(1+\frac{\kappa}{q_-}\right)F(x,t) \geq \hat{K}_3|t|^{\alpha_-} - \tilde{K}_3 \quad \text{for a.a. } x \in \Omega,$$
$$g(x,t)t - q_+\left(1+\frac{\kappa}{q_-}\right)G(x,t) \geq \hat{K}_4|t|^{\zeta_-} - \tilde{K}_4 \quad \text{for a.a. } x \in \partial\Omega,$$

for all $t \in \mathbb{R}$. Using this in (5.7) gives

$$\hat{K}_3 \left\|u_n^+\right\|_{\alpha_-}^{\alpha_-} + \hat{K}_4 \left\|u_n^+\right\|_{\zeta_-,\partial\Omega}^{\zeta_-} \leq M_4,$$

and so

$$\left\|u_n^+\right\|_{\alpha_-} \leq M_5 \quad \text{and} \quad \left\|u_n^+\right\|_{\zeta_-,\partial\Omega} \leq \tilde{M}_5 \quad \text{for all } n \in \mathbb{N}, \tag{5.8}$$

and for some $M_5, \tilde{M}_5 > 0$. This shows Claim 1.

Claim 2: $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}_{\log}}(\Omega)$.

First, from hypotheses (H_2)(i), (iv), it is clear that

$$\alpha_- < r_+ < (p_-)^* \quad \text{and} \quad \zeta_- < \ell_+ < (p_-)_*.$$

Therefore, we can find numbers $s, \tau \in (0,1)$ such that

$$\frac{1}{r_+} = \frac{s}{(p_-)^*} + \frac{1-s}{\alpha_-} \quad \text{and} \quad \frac{1}{\ell_+} = \frac{\tau}{(p_-)_*} + \frac{1-\tau}{\zeta_-}. \tag{5.9}$$

Applying the interpolation inequality, see Papageorgiou–Winkert [43, Proposition 2.3.17], it follows that

$$\left\|u_n^+\right\|_{r_+} \leq \left\|u_n^+\right\|_{(p_-)^*}^{s} \left\|u_n^+\right\|_{\alpha_-}^{1-s},$$
$$\left\|u_n^+\right\|_{\ell_+,\partial\Omega} \leq \left\|u_n^+\right\|_{(p_-)_*,\partial\Omega}^{\tau} \left\|u_n^+\right\|_{\zeta_-,\partial\Omega}^{1-\tau} \quad \text{for all } n \in \mathbb{N}.$$

Now we can use (5.8) and obtain

$$\left\|u_n^+\right\|_{r_+} \leq M_6 \left\|u_n^+\right\|_{(p_-)^*}^{s} \quad \text{as well as} \quad \left\|u_n^+\right\|_{\ell_+,\partial\Omega} \leq \tilde{M}_6 \left\|u_n^+\right\|_{(p_-)_*,\partial\Omega}^{\tau}, \tag{5.10}$$

for all $n \in \mathbb{N}$ and for some $M_6, \tilde{M}_6 > 0$. For simplicity, we can assume that $\|u_n^+\| \geq 1$ for all $n \in \mathbb{N}$. We now use Proposition 2.5 (iii) followed by (5.3) with $v = u_n^+ \in W^{1,\mathcal{H}_{\log}}(\Omega)$ together with the growth in (H_2)(i) as well as the embeddings

$$W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow L^{(p_-)^*}(\Omega), \qquad L^{r_+}(\Omega) \hookrightarrow L^1(\Omega),$$
$$W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow L^{(p_-)_*}(\partial\Omega), \qquad L^{\ell_+}(\partial\Omega) \hookrightarrow L^1(\partial\Omega),$$

and (5.10) to obtain

$$\begin{aligned}
\|u_n^+\|^{p_-} &\leq \varrho(u_n^+) \\
&\leq \varepsilon_n + K_1(\|u_n^+\|_1 + \|u_n^+\|_{r_+}^{r_+}) + K_2(\|u_n^+\|_{1,\partial\Omega} + \|u_n^+\|_{\ell_+,\partial\Omega}^{\ell_+}) \\
&\leq \varepsilon_n + M_7\left(1 + \left\|u_n^+\right\|_{(p_-)^*}^{sr_+}\right) + \tilde{M}_7\left(1 + \left\|u_n^+\right\|_{(p_-)_*,\partial\Omega}^{\tau\ell_+}\right) \\
&\leq M_8\left(1 + \|u_n^+\|^{sr_+} + \|u_n^+\|^{\tau\ell_+}\right),
\end{aligned} \tag{5.11}$$

for all $n \in \mathbb{N}$ and for some $M_7, \tilde{M}_7, M_8 > 0$. From (5.9), the definition of $(p_-)^*$ and (H$_2$)(iv) we obtain

$$\begin{aligned} sr_+ &= \frac{(p_-)^*(r_+ - \alpha_-)}{(p_-)^* - \alpha_-} = \frac{Np_-(r_+ - \alpha_-)}{Np_- - N\alpha_- + p_-\alpha_-} \\ &< \frac{Np_-(r_+ - \alpha_-)}{Np_- - N\alpha_- + p_-(r_+ - p_-)\frac{N}{p_-}} = p_-. \end{aligned} \tag{5.12}$$

Next, we see from (H$_2$)(iv) that

$$\zeta_- > \frac{\zeta_-}{p_-} + (\ell_+ - p_-)\frac{N-1}{p_-}.$$

This along with the definition of $(p_-)_*$ and (5.9) implies that

$$\begin{aligned} \tau\ell_+ &= \frac{(p_-)_*(\ell_+ - \zeta_-)}{(p_-)_* - \zeta_-} = \frac{(N-1)p_-(\ell_+ - \zeta_-)}{(N-1)p_- - N\zeta_- + p_-\zeta_-} \\ &< \frac{(N-1)p_-(\ell_+ - \zeta_-)}{(N-1)p_- - N\zeta_- + p_-\left(\frac{\zeta_-}{p_-} + (\ell_+ - p_-)\frac{N-1}{p_-}\right)} = p_-. \end{aligned} \tag{5.13}$$

Therefore, combining (5.11), (5.12) and (5.13) we see that $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}_{\log}}(\Omega)$. This proves Claim 2.

Claim 3: $u_n \to u$ in $W^{1,\mathcal{H}_{\log}}(\Omega)$ up to a subsequence.

From (5.4) and Claim 2, we see that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\mathcal{H}_{\log}}(\Omega)$. Hence, we can find a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, not relabeled, such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega). \tag{5.14}$$

Choosing $v = u_n - u \in W^{1,\mathcal{H}_{\log}}(\Omega)$ in (5.3) gives

$$\lim_{n\to\infty} \langle \varphi_+'(u_n), u_n - u\rangle = 0.$$

Moreover, we have

$$\lim_{n\to\infty} \int_\Omega f(x, u_n^+)(u_n - u)\,\mathrm{d}x = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\partial\Omega} g(x, u_n^+)(u_n - u)\,\mathrm{d}\sigma = 0.$$

From this we obtain that

$$\lim_{k\to\infty} \langle A(u_{n_k}), u_{n_k} - u\rangle = 0. \tag{5.15}$$

Then, the weak convergence in (5.14) along with (5.15) and the (S$_+$)-property of the operator A (see Proposition 2.8) imply that

$$u_n \to u \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega).$$

This finishes the proof. □

The next result is needed in order to show the mountain-pass geometry.

Proposition 5.2 *Let hypotheses (H$_1$) and (H$_2$) be satisfied. Then there exist constants $C_i > 0$, $i \in \{1, \ldots, 5\}$ such that for all $\varepsilon > 0$*

$$\varphi(u), \varphi_\pm(u) \geq \begin{cases} C_1 a_\varepsilon^{-1}\|u\|^{q_+ + \varepsilon} - C_2\|u\|^{r_-} - C_3\|u\|^{\ell_-}, & \text{if } \|u\| \leq \min\{1, C_4, C_5\}, \\ C_1\|u\|^{p_-} - C_2\|u\|^{r_+} - C_3\|u\|^{\ell_+}, & \text{if } \|u\| \geq \max\{1, C_4, C_5\}, \end{cases}$$

where a_ε is the same as in Lemma 2.4.

Proof We only show the assertion for φ, the proofs for $\varphi_\pm$ are similar. From (H_2)(i), (ii) we know that for each $\varepsilon > 0$ there exists $c_\varepsilon, \tilde{c}_\varepsilon > 0$ such that

$$\begin{aligned} |F(x,t)| &\le \frac{\varepsilon}{p(x)}|t|^{p(x)} + c_\varepsilon |t|^{r(x)} \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R},\\ |G(x,t)| &\le \frac{\varepsilon}{p(x)}|t|^{p(x)} + \tilde{c}_\varepsilon |t|^{\ell(x)} \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } t \in \mathbb{R}. \end{aligned} \tag{5.16}$$

Applying (5.16), Proposition 2.1 (iii), (iv) and Proposition 2.2 (iii), (iv) as well as the embeddings $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ with constant $C_{\mathcal{H}_{\log}}$ and $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{\ell(\cdot)}(\partial\Omega)$ with constant $C_{\mathcal{H}_{\log},\partial\Omega}$ (see Proposition 2.3 (iii) and (v)), we have for $u \in W^{1,\mathcal{H}_{\log}}(\Omega)$

$$\begin{aligned} \varphi(u) &\ge \frac{1}{q_+}\varrho_{\mathcal{H}_{\log}}(\nabla u) + \frac{1}{p_+}\varrho_{p(\cdot)}(u) + \frac{1}{p_+}\varrho_{p(\cdot),\partial\Omega}(u)\\ &\quad - \frac{\varepsilon}{p_-}\varrho_{p(\cdot)}(u) - c_\varepsilon \varrho_{r(\cdot)}(u) - \frac{\varepsilon}{p_-}\varrho_{p(\cdot),\partial\Omega}(u) - \tilde{c}_\varepsilon \varrho_{\ell(\cdot),\partial\Omega}(u)\\ &= \frac{1}{q_+}\varrho_{\mathcal{H}_{\log}}(\nabla u) + \left(\frac{1}{p_+} - \frac{\varepsilon}{p_-}\right)\varrho_{p(\cdot)}(u) + \left(\frac{1}{p_+} - \frac{\varepsilon}{p_-}\right)\varrho_{p(\cdot),\partial\Omega}(u)\\ &\quad - c_\varepsilon \varrho_{r(\cdot)}(u) - \tilde{c}_\varepsilon \varrho_{\ell(\cdot),\partial\Omega}(u)\\ &\ge \min\left\{\frac{1}{q_+}, \frac{1}{p_+} - \frac{\varepsilon}{p_-}\right\}\varrho(u)\\ &\quad - c_\varepsilon \max\left\{\|u\|_{r(\cdot)}^{r_-}, \|u\|_{r(\cdot)}^{r_+}\right\} - \tilde{c}_\varepsilon \max\left\{\|u\|_{\ell(\cdot),\partial\Omega}^{\ell_-}, \|u\|_{\ell(\cdot),\partial\Omega}^{\ell_+}\right\}\\ &\ge \min\left\{\frac{1}{q_+}, \frac{1}{p_+} - \frac{\varepsilon}{p_-}\right\}\varrho(u)\\ &\quad - c_\varepsilon \max\left\{C_{\mathcal{H}_{\log}}^{r_-}\|u\|^{r_-}, C_{\mathcal{H}_{\log}}^{r_+}\|u\|^{r_+}\right\}\\ &\quad - \tilde{c}_\varepsilon \max\left\{C_{\mathcal{H}_{\log},\partial\Omega}^{\ell_-}\|u\|^{\ell_-}, C_{\mathcal{H}_{\log},\partial\Omega}^{\ell_+}\|u\|^{\ell_+}\right\}. \end{aligned}$$

Next, we choose $\varepsilon \in \left(0, \frac{(q_+ - p_+)p_-}{p_+ q_+}\right)$ which implies $\frac{1}{q_+} < \frac{1}{p_+} - \frac{\varepsilon}{p_-}$. Taking

$$C_1 = \frac{1}{q_+}, \quad C_4 = \frac{1}{C_{\mathcal{H}_{\log}}} \quad \text{and} \quad C_5 = \frac{1}{C_{\mathcal{H}_{\log},\partial\Omega}},$$

the assertion of the proposition follows from Proposition 2.5 (iii), (iv) by setting

$$\begin{aligned} C_2 &= c_\varepsilon C_{\mathcal{H}_{\log}}^{r_-} \quad \text{and} \quad C_3 = \tilde{c}_\varepsilon C_{\mathcal{H}_{\log},\partial\Omega}^{\ell_-} \quad \text{if } \|u\| \le \min\{1, C_4, C_5\},\\ C_2 &= c_\varepsilon C_{\mathcal{H}_{\log}}^{r_+} \quad \text{and} \quad C_3 = \tilde{c}_\varepsilon C_{\mathcal{H}_{\log},\partial\Omega}^{\ell_+} \quad \text{if } \|u\| \ge \max\{1, C_4, C_5\}. \end{aligned}$$

□

A direct consequence of Proposition 5.2 is the following result.

Proposition 5.3 *Let hypotheses* (H_1) *and* (H_2) *be satisfied. Then there exist* $\delta > 0$ *such that*

$$\inf_{\|u\|=\delta} \varphi(u) > 0 \quad \text{and} \quad \inf_{\|u\|=\delta} \varphi_\pm(u) > 0.$$

Alternatively, there exists $\lambda > 0$ *such that* $\varphi(u) > 0$ *for* $0 < \|u\| < \lambda$.

Proposition 5.4 *Let hypotheses* (H_1) *and* (H_2) *be satisfied. Then* $\varphi(tu) \xrightarrow{t\to\pm\infty} -\infty$ *whenever* $0 \neq u \in W^{1,\mathcal{H}_{\log}}(\Omega)$*. Moreover,* $\varphi_\pm(tu) \xrightarrow{t\to\pm\infty} -\infty$ *for all* $0 \neq u \in W^{1,\mathcal{H}_{\log}}(\Omega)$ *such that* $u \geq 0$ *a.e. in* Ω*.*

Proof We show the proof only for φ, it can be done similarly for $\varphi_\pm$ since if $u \geq 0$ a.e. in Ω, then $\varphi_\pm(tu) = \varphi(tu)$ for $\pm t > 0$. To this end, fix any $0 \neq u \in W^{1,\mathcal{H}_{\log}}(\Omega)$ and let $t, \varepsilon \in \mathbb{R}$ such that $|t|, \varepsilon \geq 1$. From (H_2)(i), (ii), we have

$$
\begin{aligned}
F(x,t) &\geq \frac{\varepsilon}{q_+}|t|^{q_+}\log(e+|t|) - c_\varepsilon \quad \text{for a.a. } x \in \Omega,\\
G(x,t) &\geq \frac{\varepsilon}{q_+}|t|^{q_+}\log(e+|t|) - c_\varepsilon \quad \text{for a.a. } x \in \partial\Omega,
\end{aligned}
\tag{5.17}
$$

for some $c_\varepsilon > 0$. From hypotheses (H_2)(i), (ii), we know that

$$
q_+ < r_+ < (p_-)^* \quad \text{and} \quad q_+ < \ell_+ < (p_-)_*.
$$

Using this and Proposition 2.3 (iii), (v) yields

$$
\|u\|_{q_+} < \infty \quad \text{and} \quad \|u\|_{q_+,\partial\Omega} < \infty. \tag{5.18}
$$

Now from (2.6), (5.17), and (5.18) it follows

$$
\begin{aligned}
\varphi(tu) \leq{}& \frac{|t|^{p_+}}{p_-}\Big(\varrho_{p(\cdot)}(\nabla u) + \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot),\partial\Omega}(u)\Big)\\
&+ \frac{|t|^{q_+}}{q_-}\log(e+|t|)\int_\Omega \mu(x)\,|\nabla u|^{q(x)}\,\mathrm{d}x\\
&+ \frac{|t|^{q_+}}{q_-}\int_\Omega \mu(x)\,|\nabla u|^{q(x)}\log(e+|\nabla u|)\,\mathrm{d}x\\
&- \frac{\varepsilon|t|^{q_+}}{q_+}\int_\Omega |u|^{q_+}\log(e+t|u|)\,\mathrm{d}x + c_\varepsilon|\Omega|\\
&- \frac{\varepsilon|t|^{q_+}}{q_+}\int_{\partial\Omega} |u|^{q_+}\log(e+t|u|)\,\mathrm{d}\sigma + c_\varepsilon|\partial\Omega|.
\end{aligned}
\tag{5.19}
$$

By the monotonicity of the logarithm function and (5.18) we have

$$
\begin{aligned}
&\int_{\{x\in\Omega:\, u(x)\geq 1\}} |u|^{q_+}\log(e+t|u|)\,\mathrm{d}x\\
&\geq \log(e+t)\int_{\{x\in\Omega:\, u(x)\geq 1\}} |u|^{q_+}\,\mathrm{d}x,\\
&\int_{\{x\in\partial\Omega:\, u(x)\geq 1\}} |u|^{q_+}\log(e+t|u|)\,\mathrm{d}\sigma\\
&\geq \log(e+t)\int_{\{x\in\partial\Omega:\, u(x)\geq 1\}} |u|^{q_+}\,\mathrm{d}\sigma.
\end{aligned}
\tag{5.20}
$$

On the other hand, using (2.7), we obtain

$$
\begin{aligned}
&\int_{\{x\in\Omega:\,0<u(x)<1\}} |u|^{q_+}\frac{1/|u|}{1/|u|}\log(e+t|u|)\,\mathrm{d}x\\
&\geq \log(e+t)\int_{\{x\in\Omega:\,0<u(x)<1\}} |u|^{q_++1}\,\mathrm{d}x,\\
&\int_{\{x\in\partial\Omega:\,0<u(x)<1\}} |u|^{q_+}\frac{1/|u|}{1/|u|}\log(e+t|u|)\,\mathrm{d}\sigma\\
&\geq \log(e+t)\int_{\{x\in\partial\Omega:\,0<u(x)<1\}} |u|^{q_++1}\,\mathrm{d}\sigma.
\end{aligned}
\tag{5.21}
$$

Combining (5.19), (5.20), and (5.21) results in

$$
\begin{aligned}
\varphi(tu) \leq\ & \frac{|t|^{p_+}}{p_-}\Big(\varrho_{p(\cdot)}(\nabla u)+\varrho_{p(\cdot)}(u)+\varrho_{p(\cdot),\partial\Omega}(u)\Big)\\
&+|t|^{q_+}\log(e+|t|)\Bigg[\frac{1}{q_-}\int_\Omega \mu(x)\,|\nabla u|^{q(x)}\,\mathrm{d}x\\
&-\frac{\varepsilon}{q_+}\Bigg(\int_{\{x\in\Omega:\,u(x)\geq 1\}}|u|^{q_+}\,\mathrm{d}x+\int_{\{x\in\Omega:\,0<u(x)<1\}}|u|^{q_++1}\,\mathrm{d}x\\
&\quad+\int_{\{x\in\partial\Omega:\,u(x)\geq 1\}}|u|^{q_+}\,\mathrm{d}\sigma+\int_{\{x\in\partial\Omega:\,0<u(x)<1\}}|u|^{q_++1}\,\mathrm{d}\sigma\Bigg)\Bigg]\\
&+\frac{|t|^{q_+}}{q_-}\int_\Omega \mu(x)\,|\nabla u|^{q(x)}\log(e+|\nabla u|)\,\mathrm{d}x+c_\varepsilon(|\Omega|+|\partial\Omega|).
\end{aligned}
$$

Taking ε sufficiently large, the second term becomes negative which implies that $\varphi(tu)\xrightarrow{t\to\pm\infty}-\infty$. □

Now we can prove the existence of constant sign solutions of problem (1.11).

Proof of Theorem 1.1 From Propositions 5.1, 5.3 and 5.4 we see that we can apply the mountain-pass theorem given in Theorem 2.9 to both functionals $\varphi_\pm$. Hence, we can find $u_0, v_0 \in W^{1,\mathcal{H}_{\log}}(\Omega)$ such that $\varphi_+'(u_0)=0$, $\varphi_-'(v_0)=0$, and

$$
\varphi_+(u_0), \varphi_-(v_0) \geq \inf_{\|u\|=\delta}\varphi_\pm(u) > 0 = \varphi_+(0).
$$

This shows that $u_0\neq 0\neq v_0$. Then, testing $\varphi_+'(u_0)=0$ with $-u_0^-$, we obtain $\varrho(u_0^-)=0$. So, Proposition 2.5 gives us $-u_0^-=0$ a.e. in Ω which implies that $u_0=u_0^+\geq 0$ a.e. in Ω. In the same way, testing $\varphi_-'(v_0)=0$ with v_0^+, shows that $v_0\leq 0$ a.e. in Ω. Finally, from Theorem 4.1, we get the assertion. □

6 Sign-changing solution

In this section we are going to prove the existence of a sign-changing solution which turns out to be a least energy sign-changing solution of problem (1.11). As already mentioned in the Introduction, we have to strengthen the hypotheses supposing now (H_1') and (H_2') instead of (H_1) and (H_2), respectively.

Remark 6.1 Note that a necessary assumption for (H$_1$') to be fulfilled is the inequality $p_+ + 1 \leq (p_-)_*$. In the case of constant exponents this inequality is equivalent to $\sqrt{N} \leq p$. This strong assumption is required for (H$_2$')(ii').

First, it is easy to see that (H$_2$')(i'), (ii') imply $q_+ + 1 \leq r_-$ and since $q_+ + 1 < (p_-)_*$, we are able to find $r, \ell \in C_+(\overline{\Omega})$ such that $q_+ + 1 \leq r_- \leq r_+ < (p_-)^*$ and $q_+ + 1 \leq \ell_- \leq \ell_+ < (p_-)_*$. Also, if (H$_2$')(ii') is satisfied, then for any $\varepsilon > 0$

$$\begin{aligned} \lim_{s\to\pm\infty} \frac{F(x,s)}{|s|^{q_+ +1-\varepsilon}} &= +\infty \quad \text{uniformly for a.a.}\, x \in \Omega, \\ \lim_{s\to\pm\infty} \frac{G(x,s)}{|s|^{q_+ +1-\varepsilon}} &= +\infty \quad \text{uniformly for a.a.}\, x \in \partial\Omega. \end{aligned} \tag{6.1}$$

Especially (H$_2$')(ii') implies (H$_2$)(ii).

In order to find a sign-changing solution of problem (1.11), we need the Nehari manifold associated to (1.11) which is defined by

$$\mathcal{N} = \left\{ u \in W^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\} \colon \langle \varphi'(u), u \rangle = 0 \right\}.$$

It is easy to see that all weak solutions of (1.11) belong to $\mathcal{N}$. Moreover, the corresponding nodal Nehari manifold of (1.11) is given by

$$\mathcal{N}_0 = \left\{ u \in W^{1,\mathcal{H}_{\log}}(\Omega) \colon \pm u^{\pm} \in \mathcal{N} \right\}.$$

Obviously, $\mathcal{N}_0$ contains all sign-changing solutions of (1.11).

We start by establishing some structure on $\mathcal{N}$ which will be used for the study on the set $\mathcal{N}_0$. First, we mention the following lemma which is needed in the next proposition. The proof is straightforward, so we will omit it.

Lemma 6.2 *Let $b > 0$ and $Q > 1$. Then the mapping $t \mapsto \frac{t^{1-\varepsilon} b}{Q(e+tb)}$, $t > 0$, is decreasing only for $\varepsilon \geq 1$.*

Proposition 6.3 *Let hypotheses (H$_1$') and (H$_2$') be satisfied. Then, for any $u \in W^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover,*

$$\begin{aligned} \varphi(t_u u) > 0, \quad & \frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu) = 0 \quad \textit{for } t = t_u, \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu) > 0 \quad \textit{for } 0 < t < t_u, \quad & \frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu) < 0 \quad \textit{for } t > t_u. \end{aligned}$$

In particular, $\varphi(tu) < \varphi(t_u u)$ for all $0 < t \neq t_u$.

Proof Let $u \in W^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\}$. We define the associated fibering map by

$$\Lambda_u \colon [0,\infty) \to \mathbb{R}, \quad \Lambda_u(t) = \varphi(tu).$$

Clearly, we have $\Lambda_u \in C([0,\infty))$, $\Lambda_u \in C^1((0,\infty))$ and $\Lambda_u(0) = 0$. Taking Propositions 5.3 and 5.4 into account we can find $K_1, K_2 > 0$ such that

$$\Lambda_u(t) > 0 \quad \text{for } 0 < t < K_1 \quad \text{and} \quad \Lambda_u(t) < 0 \quad \text{for } t > K_2. \tag{6.2}$$

Therefore, using the extreme value theorem, we know that the global maximum of Λ_u is achieved at a point $t_u \in (0, K_2]$. Clearly, this point is a critical point of Λ_u which by the chain rule implies that

$$0 = \Lambda_u'(t_u) = \langle \varphi'(t_u u), u \rangle.$$

Thus, it holds $t_u u \in \mathcal{N}$.

Next, we are going to show that t_u is indeed unique. For this purpose, we first make the following observations:

$$\begin{aligned}
t &\mapsto \frac{f(x,tu)}{t^{q_+}|u|^{q_+}} \text{ increasing } \Rightarrow t \mapsto \frac{f(x,tu)u}{t^{q_+}} \text{ increasing in } \{x \in \Omega\colon u(x) > 0\},\\
t &\mapsto \frac{f(x,tu)}{t^{q_+}|u|^{q_+}} \text{ decreasing } \Rightarrow t \mapsto \frac{f(x,tu)u}{t^{q_+}} \text{ increasing in } \{x \in \Omega\colon u(x) < 0\},\\
t &\mapsto \frac{g(x,tu)}{t^{q_+}|u|^{q_+}} \text{ increasing } \Rightarrow t \mapsto \frac{g(x,tu)u}{t^{q_+}} \text{ increasing in } \{x \in \partial\Omega\colon u(x) > 0\},\\
t &\mapsto \frac{g(x,tu)}{t^{q_+}|u|^{q_+}} \text{ decreasing } \Rightarrow t \mapsto \frac{g(x,tu)u}{t^{q_+}} \text{ increasing in } \{x \in \partial\Omega\colon u(x) < 0\}.
\end{aligned}$$

For $t > 0$ and for any $u \in W^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\}$, we have

$$\begin{aligned}
\frac{1}{t^{q_+}}\Lambda_u'(t) &= \int_\Omega \Bigg(\frac{1}{t^{q_+ + 1 - p(x)}} |\nabla u|^{p(x)} \\
&\quad + \frac{1}{t^{q_+ - q(x)}} \mu(x) |\nabla u|^{q(x)} \left[\frac{\log(e + t|\nabla u|)}{t} + \frac{|\nabla u|}{q(x)(e + t|\nabla u|)} \right] \Bigg) \mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{t^{q_+ + 1 - p(x)}} |u|^{p(x)} \,\mathrm{d}x + \int_{\partial\Omega} \frac{1}{t^{q_+ + 1 - p(x)}} |u|^{p(x)} \,\mathrm{d}\sigma \\
&\quad - \int_\Omega \frac{f(x,tu)u}{t^{q_+}} \,\mathrm{d}x - \int_{\partial\Omega} \frac{g(x,tu)u}{t^{q_+}} \,\mathrm{d}\sigma.
\end{aligned}$$

Let us study all the terms on the right-hand side. The first term is strictly decreasing for $\nabla u \neq 0$ since $p_+ < q_+ + 1$, the third and the fourth term are also strictly decreasing, again because of $p_+ < q_+ + 1$. For the second term we can use Lemmas 2.4 and 6.2 which imply that it is decreasing. The terms with f and g are decreasing due to the observations above. Since $u \neq 0$, the right-hand side of the equation above is strictly decreasing as a function in the variable t. Hence, there can be at most one value $t_u > 0$ such that $\Lambda_u'(t_u) = 0$, that is, $t_u u \in \mathcal{N}$. This shows the uniqueness of $t_u > 0$.

Finally, $\Lambda_u'(t)$ cannot take the value 0 anywhere else, so it has constant sign on $(0, t_u)$ and (t_u, ∞). Due to (6.2), they must be positive and negative, respectively. □

Next, we will show that φ is sequentially coercive restricted to the Nehari manifold $\mathcal{N}$, that is, for any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$ such that $\|u_n\| \xrightarrow{n\to\infty} +\infty$ it follows that $\varphi(u_n) \xrightarrow{n\to\infty} +\infty$.

Proposition 6.4 *Let hypotheses (H$_1$') and (H$_2$') be satisfied. Then the functional $\varphi\big|_{\mathcal{N}}$ is sequentially coercive.*

Proof Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$ be a sequence such that $\|u_n\| \xrightarrow{n\to\infty} +\infty$ and $y_n = u_n/\|u_n\|$. Then $\{y_n\}_{n\in\mathbb{N}} \subseteq W^{1,\mathcal{H}_{\log}}(\Omega)$ is bounded and so we can find a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ and $y \in W^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\begin{aligned}
&y_{n_k} \rightharpoonup y \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega),\\
&y_{n_k} \to y \quad \text{in } L^{r(\cdot)}(\Omega) \text{ and pointwisely a.e. in } \Omega,\\
&y_{n_k} \to y \quad \text{in } L^{\ell(\cdot)}(\partial\Omega) \text{ and pointwisely a.e. in } \partial\Omega.
\end{aligned} \tag{6.3}$$

We claim that $y = 0$. Suppose by contradiction that $y \neq 0$. Let $0 < \varepsilon < \kappa$ and, without any loss of generality, we can assume that there exists $k_0 \in \mathbb{N}$ such that $\|u_{n_k}\| \geq 1$ for all $k \geq k_0$. Applying Proposition 2.5 (iv) gives

$$\begin{aligned}\varphi(u_{n_k}) &\leq \frac{1}{p_-}\varrho(u_{n_k}) - \int_\Omega F(x, u_{n_k})\,\mathrm{d}x - \int_{\partial\Omega} G(x, u_{n_k})\,\mathrm{d}\sigma \\ &\leq \frac{a_\varepsilon}{p_-}\|u_{n_k}\|^{q_+ + \varepsilon} - \int_\Omega F(x, u_{n_k})\,\mathrm{d}x - \int_{\partial\Omega} G(x, u_{n_k})\,\mathrm{d}\sigma.\end{aligned}$$

Next, we divide the last inequality by $\|u_{n_k}\|^{q_+ + \varepsilon}$ and use the representation $y_n = u_n/\|u_n\|$ to obtain

$$\frac{\varphi(u_{n_k})}{\|u_{n_k}\|^{q_+ + \varepsilon}} \leq \frac{a_\varepsilon}{p_-} - \int_\Omega \frac{F(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon}\,\mathrm{d}x - \int_{\partial\Omega} \frac{G(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon}\,\mathrm{d}\sigma. \tag{6.4}$$

Note that from (6.1) we have

$$\begin{aligned}&\lim_{k\to\infty} \frac{F(x, u_{n_k})}{\|u_{n_k}\|^{q_+ + \varepsilon}} = \lim_{k\to\infty} \frac{F(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon} = \infty, \quad x \in \Omega \text{ with } y(x) \neq 0, \\ &\lim_{k\to\infty} \frac{G(x, u_{n_k})}{\|u_{n_k}\|^{q_+ + \varepsilon}} = \lim_{k\to\infty} \frac{G(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon} = \infty, \quad x \in \partial\Omega \text{ with } y(x) \neq 0.\end{aligned} \tag{6.5}$$

Now, due to (H$_2$')(i'), (ii'), we can find constants $M_9, M_{10} > 0$ such that

$$\begin{aligned}&F(x, t) > -M_9 \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R}, \\ &G(x, t) > -M_{10} \quad \text{for a.a. } x \in \partial\Omega \text{ and for all } t \in \mathbb{R}.\end{aligned} \tag{6.6}$$

Setting $\Omega_0 = \{x \in \Omega\colon y(x) = 0\}$, from (6.3), Fatou's Lemma, (6.6), and (6.5), it follows that

$$\begin{aligned}&\lim_{k\to\infty} \int_\Omega \frac{F(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon}\,\mathrm{d}x \\ &= \lim_{k\to\infty} \left(\int_{\Omega\setminus\Omega_0} \frac{F(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon}\,\mathrm{d}x + \int_{\Omega_0} \frac{F(x, u_{n_k})}{\left\|u_{n_k}\right\|^{q_+ + \varepsilon}}\,\mathrm{d}x\right) \\ &\geq \int_{\Omega\setminus\Omega_0} \left(\lim_{k\to\infty} \frac{F(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon}\right)\mathrm{d}x - \lim_{k\to\infty} \frac{M_9|\Omega_0|}{\|u_{n_k}\|^{q_+ + \varepsilon}} \\ &= \infty.\end{aligned} \tag{6.7}$$

Similarly, using $\Sigma_0 = \{x \in \partial\Omega\colon y(x) = 0\}$, we have

$$\begin{aligned}&\lim_{k\to\infty} \int_{\partial\Omega} \frac{G(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon}\,\mathrm{d}\sigma \\ &= \lim_{k\to\infty} \left(\int_{\partial\Omega\setminus\Sigma_0} \frac{G(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon}\,\mathrm{d}\sigma + \int_{\Sigma_0} \frac{G(x, u_{n_k})}{\left\|u_{n_k}\right\|^{q_+ + \varepsilon}}\,\mathrm{d}\sigma\right) \\ &\geq \int_{\partial\Omega\setminus\Sigma_0} \left(\lim_{k\to\infty} \frac{G(x, u_{n_k})}{|u_{n_k}|^{q_+ + \varepsilon}}|y_{n_k}|^{q_+ + \varepsilon}\right)\mathrm{d}\sigma - \lim_{k\to\infty} \frac{M_9|\Omega_0|}{\|u_{n_k}\|^{q_+ + \varepsilon}} \\ &= \infty.\end{aligned} \tag{6.8}$$

Now, passing to the limit in (6.4) as $k \to \infty$ and using (6.7) as well as (6.8), yields

$$\lim_{k\to\infty} \frac{\varphi(u_{n_k})}{\|u_{n_k}\|^{q_+ + \varepsilon}} = -\infty.$$

But this is a contradiction since $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$ and so $\varphi(u_n) > 0$ for all $n \in \mathbb{N}$ by Proposition 6.3. This shows that $y = 0$.

Because $u_{n_k} \in \mathcal{N}$ for all $k \in \mathbb{N}$, from Proposition 6.3, we know that $\varphi(u_{n_k}) \geq \varphi(Ly_{n_k})$ for all $k \in \mathbb{N}$ and for any $L > 1$. Using this and Proposition 2.5 (iii) we have for all $k \in \mathbb{N}$

$$\begin{aligned}\varphi(u_{n_k}) &\geq \varphi(Ly_{n_k})\\ &\geq \frac{1}{q_+}\varrho(Ly_{n_k}) - \int_\Omega F(x, Ly_{n_k})\,\mathrm{d}x - \int_{\partial\Omega} G(x, Ly_{n_k})\,\mathrm{d}\sigma\\ &\geq \frac{1}{q_+}\|Ly_{n_k}\|^{p_-} - \int_\Omega F(x, Ly_{n_k})\,\mathrm{d}x - \int_{\partial\Omega} G(x, Ly_{n_k})\,\mathrm{d}\sigma\\ &= \frac{L^{p_-}}{q_+} - \int_\Omega F(x, Ly_{n_k})\,\mathrm{d}x - \int_{\partial\Omega} G(x, Ly_{n_k})\,\mathrm{d}\sigma,\end{aligned} \tag{6.9}$$

since $\|y_{n_k}\| = 1$. Note that the two integrals on the right-hand side of (6.9) are strongly continuous due to (H_2)(i), see [2, Lemma 5.1 (v)]. Combining this fact with $Ly_{n_k} \rightharpoonup 0$ in $W^{1,\mathcal{H}_{\log}}(\Omega)$ (see (6.3) and the fact that $y = 0$), there exists a number $k_1 \in \mathbb{N}$ such that

$$\varphi(u_{n_k}) \geq \frac{L^{p_-}}{q_+} - 1 \quad \text{for all } k \geq k_1.$$

Since $L > 1$ was arbitrary chosen, we get $\varphi(u_{n_k}) \to +\infty$ as $k \to \infty$, which by the subsequence principle implies that $\varphi(u_n) \to +\infty$ as $n \to \infty$. □

Now we are able to prove that the infimum of φ over $\mathcal{N}$ and $\mathcal{N}_0$, respectively, is always positive.

Proposition 6.5 *Let hypotheses (H_1') and (H_2') be satisfied. Then*

$$\inf_{u\in\mathcal{N}} \varphi(u) > 0 \quad \textit{and} \quad \inf_{u\in\mathcal{N}_0} \varphi(u) > 0.$$

Proof The first part follows from Proposition 6.3 and Proposition 5.3, which imply that

$$\varphi(u) \geq \varphi\left(\frac{\delta}{\|u\|}u\right) \geq \inf_{\|u\|=\delta} \varphi(u) > 0 \quad \text{for all } u \in \mathcal{N},$$

with $\delta > 0$ is given by Proposition 5.3. Since $\varphi(u) = \varphi(u^+) + \varphi(-u^-)$ and $u^+, -u^- \in \mathcal{N}$ for $u \in \mathcal{N}_0$, the second assertion follows. □

Next we are going to prove that the infimum of φ restricted to the nodal Nehari manifold $\mathcal{N}_0$ is achieved.

Proposition 6.6 *Let hypotheses (H_1') and (H_2') be satisfied. Then there exists $w_0 \in \mathcal{N}_0$ such that $\varphi(w_0) = \inf_{u\in\mathcal{N}_0} \varphi(u)$.*

Proof Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_0$ be a minimizing sequence, that is, $\varphi(u_n) \searrow \inf_{u\in\mathcal{N}_0} \varphi(u)$. From Proposition 2.6 we know that $u_n^+, -u_n^- \in W^{1,\mathcal{H}_{\log}}(\Omega)$ for all $n \in \mathbb{N}$. Therefore, $\{\varphi(u_n^+)\}_{n\in\mathbb{N}}$ and $\{\varphi(-u_n^-)\}_{n\in\mathbb{N}}$ are bounded in $\mathbb{R}$ because $\varphi(u_n) = \varphi(u_n^+) + \varphi(-u_n^-)$ for all $n \in \mathbb{N}$ and since $\varphi(u_n^+) > 0$ as well as $\varphi(-u_n^-) > 0$ for all $n \in \mathbb{N}$ by Proposition 6.3. Hence, from Proposition 6.4 we know that $\{u_n^+\}_{n\in\mathbb{N}}$ and $\{-u_n^-\}_{n\in\mathbb{N}}$ are bounded in $W^{1,\mathcal{H}_{\log}}(\Omega)$. Taking Proposition 2.3 (iii), (v) into account, we can find subsequences $\{u_{n_k}^+\}_{k\in\mathbb{N}}$ and $\{-u_{n_k}^-\}_{k\in\mathbb{N}}$

and $v_1, v_2 \in W^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\begin{aligned}
&u_{n_k}^+ \rightharpoonup v_1, \quad u_{n_k}^- \rightharpoonup v_2 \quad \text{in } W^{1,\mathcal{H}_{\log}}(\Omega),\\
&u_{n_k}^+ \to v_1, \quad u_{n_k}^- \to v_2 \quad \text{in } L^{r(\cdot)}(\Omega) \text{ and pointwisely a.e. in } \Omega,\\
&u_{n_k}^+ \to v_1, \quad u_{n_k}^- \to v_2 \quad \text{in } L^{\ell(\cdot)}(\partial\Omega) \text{ and pointwisely a.e. in } \partial\Omega,\\
&\text{with } v_1 \geq 0, \quad v_2 \geq 0, \quad \text{and} \quad v_1 v_2 = 0.
\end{aligned} \tag{6.10}$$

Claim: $v_1, v_2 \neq 0$
Suppose by contradiction that $v_1 = 0$. Since $u_{n_k}^+ \in \mathcal{N}$, we have

$$\begin{aligned}
0 &= \langle \varphi'(u_{n_k}^+), u_{n_k}^+ \rangle\\
&= \varrho(u_{n_k}^+) + \int_\Omega \frac{|\nabla u_{n_k}^+|}{q(x)(e + |\nabla u_{n_k}^+|)} |\nabla u_{n_k}^+|^{q(x)} \,\mathrm{d}x\\
&\quad - \int_\Omega f(x, u_{n_k}^+) u_{n_k}^+ \,\mathrm{d}x - \int_{\partial\Omega} g(x, u_{n_k}^+) u_{n_k}^+ \,\mathrm{d}\sigma\\
&\geq \varrho(u_{n_k}^+) - \int_\Omega f(x, u_{n_k}^+) u_{n_k}^+ \,\mathrm{d}x - \int_{\partial\Omega} g(x, u_{n_k}^+) u_{n_k}^+ \,\mathrm{d}\sigma.
\end{aligned}$$

Passing to the limit as $k \to +\infty$ and using the convergence properties in (6.10) implies that $\varrho(u_{n_k}^+) \to 0$ which is by Proposition 2.5 (v) equivalent to $u_{n_k}^+ \to 0$ in $W^{1,\mathcal{H}_{\log}}(\Omega)$. Then, using the continuity of φ along with Proposition 6.5 yields

$$0 < \inf_{u \in \mathcal{N}} \varphi(u) \leq \varphi(u_{n_k}^+) \longrightarrow \varphi(0) = 0 \quad \text{as } k \to \infty,$$

which is a contradiction. Thus, $v_1 \neq 0$. In a similar way one shows that $v_2 \neq 0$ and so the Claim is proved.

From Proposition 6.3 and the Claim, we can find numbers $t_1, t_2 > 0$ such that $t_1 v_1, t_2 v_2 \in \mathcal{N}$. Next, we define

$$w_0 = t_1 v_1 + t_2 v_2.$$

Taking (6.10) into account, we see that $w_0^+ = t_1 v_1$ and $-w_0^- = t_2 v_2$ which implies $w_0 \in \mathcal{N}_0$. It should be noted that the positive terms in $\varphi(\cdot)$ are convex and continuous, so sequentially weakly lower semicontinuous. Since the terms with the functions F and G are strongly continuous, these are also sequentially weakly lower semicontinuous. Therefore, the functional $\varphi(\cdot)$ is sequentially weakly lower semicontinuous. Using this fact and Proposition 6.3 leads to

$$\begin{aligned}
\inf_{u \in \mathcal{N}_0} \varphi(u) = \lim_{k \to \infty} \varphi(u_{n_k}) &= \lim_{k \to \infty} \varphi(u_{n_k}^+) + \varphi(-u_{n_k}^-)\\
&\geq \liminf_{k \to \infty} \varphi(t_1 u_{n_k}^+) + \varphi(-t_2 u_{n_k}^-)\\
&\geq \varphi(t_1 v_1) + \varphi(t_2 v_2)\\
&= \varphi(w_0^+) + \varphi(-w_0^-) = \varphi(w_0) \geq \inf_{u \in \mathcal{N}_0} \varphi(u).
\end{aligned}$$

This finishes the proof. □

Finally, we have to show that the minimizer obtained in Proposition 6.6 is indeed a critical point of $\varphi(\cdot)$ and so a least energy sign-changing solution of (1.11).

Proposition 6.7 *Let hypotheses (H_1') and (H_2') be satisfied and $w_0 \in \mathcal{N}_0$ such that $\varphi(w_0) = \inf_{u\in\mathcal{N}_0} \varphi(u)$. Then w_0 is a critical point of φ.*

Proof Assume by contradiction that $\varphi'(w_0) \neq 0$. Then we can find numbers $\alpha, \beta_0 > 0$ such that

$$\|\varphi'(u)\|_* \geq \alpha, \quad \text{for all } u \in W^{1,\mathcal{H}_{\log}}(\Omega) \text{ with } \|u - w_0\| < 3\beta_0.$$

Let $\hat{C}_{\mathcal{H}_{\log}}$ be the embedding constant of $W^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p_-}(\Omega)$ (see Proposition 2.3 (iii)). Since $w_0^+ \neq 0 \neq w_0^-$, we have for any $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$ that

$$\|w_0 - v\| \geq \hat{C}_{\mathcal{H}_{\log}}^{-1} \|w_0 - v\|_{p_-} \geq \begin{cases} \hat{C}_{\mathcal{H}_{\log}}^{-1} \left\|w_0^-\right\|_{p_-}, & \text{if } v^- = 0, \\ \hat{C}_{\mathcal{H}_{\log}}^{-1} \left\|w_0^+\right\|_{p_-}, & \text{if } v^+ = 0. \end{cases}$$

Now we take a number β_1 such that

$$\beta_1 \in \left(0, \min\left\{\hat{C}_{\mathcal{H}_{\log}}^{-1} \left\|w_0^-\right\|_{p_-}, \hat{C}_{\mathcal{H}_{\log}}^{-1} \left\|w_0^+\right\|_{p_-}\right\}\right).$$

This implies that for any $v \in W^{1,\mathcal{H}_{\log}}(\Omega)$ with $\|w_0 - v\| < \beta_1$ it follows that $v^+ \neq 0 \neq v^-$.

Let $\beta = \min\{\beta_0, \beta_1/2\}$. Due to the continuity of the mapping $(s,t) \mapsto sw_0^+ - tw_0^-$ from $[0,\infty)^2$ into $W^{1,\mathcal{H}_{\log}}(\Omega)$, there exists $\delta \in (0,1)$ such that for all $s,t \geq 0$ with $\max\{|s - 1|, |t-1|\} < \delta$ it holds

$$\|sw_0^+ - tw_0^- - w_0\| < \beta. \tag{6.11}$$

Let $D = (1-\delta, 1+\delta)^2$. From Proposition 6.3 we have for $s,t \geq 0$ with $s \neq 1 \neq t$

$$\begin{aligned} \varphi(sw_0^+ - tw_0^-) &= \varphi(sw_0^+) + \varphi(-tw_0^-) \\ &< \varphi(w_0^+) + \varphi(-w_0^-) = \varphi(w_0) = \inf_{u\in\mathcal{N}_0} \varphi(u). \end{aligned} \tag{6.12}$$

From this it follows, in particular, that

$$\xi = \max_{(s,t)\in\partial D} \varphi(sw_0^+ - tw_0^-) < \varphi(w_0) = \inf_{u\in\mathcal{N}_0} \varphi(u).$$

Now we are able to apply quantitative deformation lemma given in Lemma 2.10 with

$$S = B(w_0, \beta), \quad c = \inf_{u\in\mathcal{N}_0} \varphi(u), \quad \varepsilon = \min\left\{\frac{c-\xi}{4}, \frac{\alpha\beta}{8}\right\}$$

with β as defined above. Since $S_{2\beta} = B(w_0, 3\beta)$ and with the choice of ε, we see that the conditions in Lemma 2.10 are satisfied. Hence, we can find a mapping η with the properties stated in the lemma. Moreover, due to the choice of ε, we have

$$\varphi(sw_0^+ - tw_0^-) \leq \xi + c - c < c - \left(\frac{c-\xi}{2}\right) \leq c - 2\varepsilon \tag{6.13}$$

for all $(s,t) \in \partial D$. Next, we define mappings $\Upsilon\colon [0,\infty)^2 \to W^{1,\mathcal{H}_{\log}}(\Omega)$ and $\Pi\colon (0,\infty)^2 \to \mathbb{R}^2$ by

$$\begin{aligned} \Upsilon(s,t) &= \eta(1, sw_0^+ - tw_0^-) \\ \Pi(s,t) &= \left(\frac{1}{s}\langle\varphi'(\Upsilon^+(s,t)), \Upsilon^+(s,t)\rangle, \frac{1}{t}\langle\varphi'(-\Upsilon^-(s,t)), -\Upsilon^-(s,t)\rangle\right). \end{aligned}$$

The continuity of η implies the continuity of Υ and since φ is C^1, Π is continuous as well. Applying Lemma 2.10 (i) and (6.13), we have, for all $(s,t) \in \partial D$, that $\Upsilon(s,t) = sw_0^+ - tw_0^-$ and

$$\Pi(s,t) = \left(\langle \varphi'(sw_0^+), w_0^+ \rangle \,,\ \langle \varphi'(-tw_0^-), -w_0^- \rangle \right).$$

Taking Proposition 6.3 into account, yields the componentwise inequalities

$$\begin{aligned} &\Pi_1(1-\delta,t) > 0 > \Pi_1(1+\delta,t),\\ &\Pi_2(t,1-\delta) > 0 > \Pi_2(t,1+\delta) \quad \text{for all } t \in (1-\delta,1+\delta), \end{aligned}$$

where $\Pi = (\Pi_1, \Pi_2)$. Then, by the Poincaré-Miranda existence theorem given in Theorem 2.11 applied to $d(s,t) = -\Pi(1+s,1+t)$, there exists a pair $(s_0,t_0) \in D$ such that $\Pi(s_0,t_0) = 0$ which can be equivalently written as

$$\langle \varphi'(\Upsilon^+(s_0,t_0)), \Upsilon^+(s_0,t_0)\rangle = 0 = \langle \varphi'(-\Upsilon^-(s_0,t_0)), -\Upsilon^-(s_0,t_0)\rangle.$$

Now, applying Lemma 2.10 (iv), (6.11), and the choice of δ gives

$$\begin{aligned} \|\Upsilon(s_0,t_0) - w_0\|_X &\le \|\Upsilon(s_0,t_0) - (t_0w_0^+ - t_0w_0^-)\| + \|t_0w_0^+ - t_0w_0^- - w_0\|\\ &\le \beta + \beta \le 2\beta \le \beta_1. \end{aligned}$$

Then, by the choice of β_1, we have

$$\Upsilon^+(s_0,t_0) \ne 0 \ne -\Upsilon^-(s_0,t_0),$$

which implies $\Upsilon(s_0,t_0) \in \mathcal{N}_0$. But, from Lemma 2.10 (ii), the choice of δ and (6.12), it follows that $\varphi(\Upsilon(s_0,t_0)) \le c - \varepsilon$. This is a contradiction and so w_0 is a critical point of φ. □

The proof of Theorem 1.3 follows now from Propositions 6.6 and 6.7 along with Theorem 4.1. We end this section with an example.

Example 6.8 Let (H$_1$) be satisfied. In addition, for simplification, we also assume that $q_+\kappa/q_- < 1$ which implies $q_+(1+\kappa/q_-) < q_+ + 1 < p_-^+$. Let $\hat{\kappa}_i, \tilde{\kappa}_i, \kappa_i \in C_+(\overline{\Omega})$, $i = 1,2$ such that

$$\begin{aligned} &q_+ + 1 \le \min\{(\hat{\kappa}_1)_-, (\tilde{\kappa}_1)_-, (\kappa_1)_-\}, \quad \max\{(\hat{\kappa}_1)_+, (\tilde{\kappa}_1)_+\} < (p_-)^*,\\ &q_+ + 1 \le \min\{(\hat{\kappa}_2)_-, (\tilde{\kappa}_2)_-, (\kappa_2)_-\}, \quad \max\{(\hat{\kappa}_2)_+, (\tilde{\kappa}_2)_+\} < (p_-)_* \end{aligned}$$

and

$$\begin{aligned} &\frac{\max\{(\hat{\kappa}_1)_+, (\tilde{\kappa}_1)_+\}}{p_-} - \frac{\min\{(\hat{\kappa}_1)_-, (\tilde{\kappa}_1)_-\}}{N} < 1,\\ &\frac{\max\{(\hat{\kappa}_2)_+, (\tilde{\kappa}_2)_+\}}{p_-} - \frac{\min\{(\hat{\kappa}_2)_-, (\tilde{\kappa}_2)_-\}}{N} < 1. \end{aligned}$$

Then, the functions

$$f(x,t) = \begin{cases} |t|^{\tilde{\kappa}_1(x)-2}t[1+\log(-t)], & \text{if} \quad t \le -1,\\ |t|^{\kappa_1(x)-2}t, & \text{if} \ -1 < t < 1, \\ |t|^{\hat{\kappa}_1(x)-2}t[1+\log(t)], & \text{if} \quad 1 \le t, \end{cases},$$

$$g(x,t) = \begin{cases} |t|^{\tilde{\kappa}_2(x)-2}t[1+\log(-t)], & \text{if} \quad t \le -1,\\ |t|^{\kappa_2(x)-2}t, & \text{if} \ -1 < t < 1,\\ |t|^{\hat{\kappa}_2(x)-2}t[1+\log(t)], & \text{if} \quad 1 \le t, \end{cases}$$

satisfy hypotheses (H$_2$').

Acknowledgements L. Gasiński thanks the University of Technology Berlin for the kind hospitality during a research stay in October 2024. M.F. Stapenhorst has been supported by FAPESP 2021/12773-0, 2022/15727-2 and 2023/06617-1. M.F. Stapenhorst thanks the University of Technology Berlin for the kind hospitality during a research stay in November 2024. P. Winkert thanks the University of the National Education Commission, Krakow, for the kind hospitality during a research stay in September 2024. M.F. Stapenhorst and P. Winkert were financially supported by TU Berlin-FAPESP Mobility Promotion.

Author Contributions The authors contributed equally to this work.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Ethical Approval Not applicable.

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Amoroso, E., Crespo-Blanco, Á., Pucci, P., Winkert, P.: Superlinear elliptic equations with unbalanced growth and nonlinear boundary condition. Bull. Sci. Math. **197**, 103534 (2024)
2. Arora, R., Crespo-Blanco, Á., Winkert, P.: On logarithmic double phase problems. J. Differ. Equ. **433**, 113247 (2025)
3. Bahrouni, A., Rădulescu, V.D., Repovš, D.D.: Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves. Nonlinearity **32**(7), 2481–2495 (2019)
4. Baroni, P., Colombo, M., Mingione, G.: Harnack inequalities for double phase functionals. Nonlinear Anal. **121**, 206–222 (2015)
5. Baroni, P., Colombo, M., Mingione, G.: Non-autonomous functionals, borderline cases and related function classes. St. Petersburg Math. J. **27**, 347–379 (2016)
6. Baroni, P., Colombo, M., Mingione, G.: Regularity for general functionals with double phase. Calc. Var. Part. Differ. Equ. **57**(2), 62 (2018)
7. Beck, L., Mingione, G.: Lipschitz bounds and nonuniform ellipticity. Comm. Pure Appl. Math. **73**(5), 944–1034 (2020)
8. Benci, V., D'Avenia, P., Fortunato, D., Pisani, L.: Solitons in several space dimensions: Derrick's problem and infinitely many solutions. Arch. Ration. Mech. Anal. **154**(4), 297–324 (2000)
9. Borer, F., Pimenta, M.T.O., Winkert, P.: Degenerate Kirchhoff problems with nonlinear Neumann boundary condition. J. Funct. Anal. **289**(4), 110933 (2025)
10. Borowski, M., Chlebicka, I., De Filippis, F., Miasojedow, B.: Absence and presence of Lavrentiev's phenomenon for double phase functionals upon every choice of exponents. Calc. Var. Partial. Differ. Equ. **63**(2), 35 (2024)
11. Cen, J., Lu, Y., Vetro, C., Zeng, S.: Existence results of variable exponent double-phase multivalued elliptic inequalities with logarithmic perturbation and convections. Adv. Nonlinear Anal. **14**(1), 20240066 (2025)
12. Chen, S., Qin, D., Rădulescu, V.D., Tang, X.: Ground states for quasilinear equations of N-Laplacian type with critical exponential growth and lack of compactness. Sci China Math **68**(6), 1323–1354 (2025)

13. Cherfils, L., Il'yasov, Y.: On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian. Commun. Pure Appl. Anal. **4**(1), 9–22 (2005)
14. Colombo, M., Mingione, G.: Bounded minimisers of double phase variational integrals. Arch. Ration. Mech. Anal. **218**(1), 219–273 (2015)
15. Colombo, M., Mingione, G.: Regularity for double phase variational problems. Arch. Ration. Mech. Anal. **215**(2), 443–496 (2015)
16. Crespo-Blanco, Á., Gasiński, L., Harjulehto, P., Winkert, P.: A new class of double phase variable exponent problems: existence and uniqueness. J. Differ. Equ. **323**, 182–228 (2022)
17. De Filippis, C., Mingione, G.: Lipschitz bounds and nonautonomous integrals. Arch. Ration. Mech. Anal. **242**, 973–1057 (2021)
18. De Filippis, C., Mingione, G.: Regularity for double phase problems at nearly linear growth. Arch. Ration. Mech. Anal. **247**(5), 85 (2023)
19. Dinca, G., Mawhin, J.: Brouwer Degree—The Core of Nonlinear Analysis. Birkhäuser/ Springer, Cham (2021)
20. Diening, L., Harjulehto, P., Hästö, P., Ružička, M., M.: Lebesgue and Sobolev Spaces with Variable Exponents. Springer, Heidelberg (2011)
21. El Manouni, S., Marino, G., Winkert, P.: Existence results for double phase problems depending on Robin and Steklov eigenvalues for the p-Laplacian. Adv. Nonlinear Anal. **11**(1), 304–320 (2022)
22. Fan, X.: Boundary trace embedding theorems for variable exponent Sobolev spaces. J. Math. Anal. Appl. **339**(2), 1395–1412 (2008)
23. Fan, X., Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. **263**(2), 424–446 (2001)
24. Fang, Y., Rădulescu, V.D., Zhang, C.: Equivalence of weak and viscosity solutions for the nonhomogeneous double phase equation. Math. Ann. **388**(3), 2519–2559 (2024)
25. Farkas, C., Fiscella, A., Winkert, P.: Singular Finsler double phase problems with nonlinear boundary condition. Adv. Nonlinear Stud. **21**(4), 809–825 (2021)
26. Fiscella, A., Marino, G., Pinamonti, A., Verzellesi, S.: Multiple solutions for nonlinear boundary value problems of Kirchhoff type on a double phase setting. Rev. Mat. Complut. **37**(1), 205–236 (2024)
27. Fuchs, M., Mingione, G.: Full $C^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth. Manuscripta Math. **102**(2), 227–250 (2000)
28. Fuchs, M., Seregin, G.: Variational Methods for Problems from Plasticity Theory and for Generalized Newtonian Fluids. Springer-Verlag, Berlin (2000)
29. Gasiński, L., Winkert, P.: Sign changing solution for a double phase problem with nonlinear boundary condition via the Nehari manifold. J. Differ. Equ. **274**, 1037–1066 (2021)
30. Harjulehto, P., Hästö, P.: Orlicz Spaces and Generalized Orlicz Spaces. Springer, Cham (2019)
31. Ho, K., Kim, Y.-H., Winkert, P., Zhang, C.: The boundedness and Hölder continuity of weak solutions to elliptic equations involving variable exponents and critical growth. J. Differ. Equ. **313**, 503–532 (2022)
32. Ho, K., Winkert, P.: New embedding results for double phase problems with variable exponents and a priori bounds for corresponding generalized double phase problems. Calc. Var. Partial. Differ. Equ. **62**(8), 227 (2023)
33. Liu, J., Pucci, P.: Existence of solutions for a double-phase variable exponent equation without the Ambrosetti-Rabinowitz condition. Adv. Nonlinear Anal. **12**(1), 20220292 (2023)
34. Lu, Y., Vetro, C., Zeng, S.: A class of double phase variable exponent energy functionals with different power growth and logarithmic perturbation. Discrete Contin. Dyn. Syst. Ser. S (2024). https://doi.org/10.3934/dcdss.2024143
35. Marcellini, P.: Regularity and existence of solutions of elliptic equations with p,q-growth conditions. J. Differ. Equ. **90**(1), 1–30 (1991)
36. Marcellini, P.: Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. Arch. Rational Mech. Anal. **105**(3), 267–284 (1989)
37. Mingione, G., Rădulescu, V.D.: Recent developments in problems with nonstandard growth and nonuniform ellipticity. J. Math. Anal. Appl. **501**(1), 125197 (2021)
38. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Existence and multiplicity of solutions for double-phase Robin problems. Bull. Lond. Math. Soc. **52**(3), 546–560 (2020)
39. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Nonlinear Analysis—Theory and Methods. Springer, Cham (2019)
40. Papageorgiou, N.S., Rădulescu, V.D., Zhang, Y.: Multiple solutions for superlinear double phase Neumann problems. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116**(1), 14 (2022)
41. Papageorgiou, N.S., Rădulescu, V.D., Sun, X.: Positive solutions for nonparametric anisotropic singular solutions. Opuscula Math. **44**(3), 409–423 (2024)

42. Papageorgiou, N.S., Vetro, C., Vetro, F.: Solutions for parametric double phase Robin problems. Asymptot. Anal. **121**(2), 159–170 (2021)
43. Papageorgiou, N.S., Winkert, P.: Applied Nonlinear Functional Analysis, 2nd edn. De Gruyter, Berlin (2024)
44. Papageorgiou, N.S., Zhang, C.: Multiple ground-state solutions with sign information for double-phase Robin problems. Israel J. Math. **253**(1), 419–443 (2023)
45. Seregin, G.A., Frehse, J.: Regularity of solutions to variational problems of the deformation theory of plasticity with logarithmic hardening. In: Proceedings of the St. Petersburg Mathematical Society, Vol. V, Amer. Math. Soc. in: Providence, RI **193**, 127–152 (1999)
46. Vetro, F., Winkert, P.: Logarithmic double phase problems with convection: existence and uniqueness results. Commun. Pure Appl. Anal. **23**(9), 1325–1339 (2024)
47. Vetro, C., Zeng, S.: Regularity and Dirichlet problem for double-phase energy functionals of different power growth. J. Geom. Anal. **34**(4), 105 (2024)
48. Willem, M.: Minimax Theorems. Birkhäuser Boston Inc, Boston, MA (1996)
49. Winkert, P., Zacher, R.: A priori bounds for weak solutions to elliptic equations with nonstandard growth. Discrete Contin. Dyn. Syst. Ser. S **5**(4), 865–878 (2012)
50. Winkert, P., Zacher, R.: Corrigendum to A priori bounds for weak solutions to elliptic equations with nonstandard growth [Discrete Contin. Dyn. Syst. Ser. S 5: 865–878]. Discrete Contin. Dyn. Syst. Ser. S **2015**, 1–3 (2012)
51. Zeng, S., Bai, Y., Gasiński, L., Winkert, P.: Existence results for double phase implicit obstacle problems involving multivalued operators. Calc. Var. Partial. Differ. Equ. **59**(5), 18 (2020)
52. Zeng, S., Rădulescu, V.D., Winkert, P.: Double phase implicit obstacle problems with convection and multivalued mixed boundary value conditions. SIAM J. Math. Anal. **54**(2), 1898–1926 (2022)
53. Zeng, S., Rădulescu, V.D., Winkert, P.: Nonlocal double phase implicit obstacle problems with multivalued boundary conditions. SIAM J. Math. Anal. **56**(1), 877–912 (2024)
54. Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory. Izv. Akad. Nauk SSSR Ser. Mat. **50**(4), 675–710 (1986)
55. Zhikov, V.V.: On variational problems and nonlinear elliptic equations with nonstandard growth conditions. J. Math. Sci. **173**(5), 463–570 (2011)