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Multiple sign-changing solutions for superlinear (p, q)-equations in symmetrical expanding domains

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In this paper we study quasilinear elliptic equations defined on symmetrical expanding domains driven by the (p, q)-Laplacian and with a superlinear right-hand side. Based on the Lusternik-Schnirelmann category we prove the existence of at least $\gamma(\Omega_\lambda \setminus \{0\})$ pairs $(\pm u)$ of odd weak solutions with precisely two nodal domains, where γ stands for the genus.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $\Omega_\lambda := \lambda\Omega$ be an expanding domain, where λ is a positive parameter. In this paper we consider the following problem

$$\begin{aligned} -\Delta_p u - \mu\Delta_q u &= f(u) - |u|^{p-2}u && \text{in } \Omega_\lambda, \\ u &= 0 && \text{on } \partial\Omega_\lambda, \\ u(-x) &= -u(x) && \text{for a.a. } x \in \Omega_\lambda, \end{aligned} \tag{1.1}$$

where we suppose the following assumptions:

(H1) $\mu > 0$ and $1 < q < p < N$.
(H2) $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous and odd function with primitive $F(s) = \int_0^s f(t)\,\mathrm{d}t$ satisfying the following conditions:
  (i) there exist $r \in (p, p^*)$ and a constant $C > 0$ such that

$$|f(s)| \leq C\left(1 + |s|^{r-1}\right) \quad \text{for all } s \in \mathbb{R},$$

  where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent to p;
  (ii) $\displaystyle\lim_{s\to 0} \frac{f(s)}{|s|^{q-2}s} = 0$;
  (iii) $\displaystyle\lim_{|s|\to+\infty} \frac{F(s)}{|s|^p} = +\infty$;
  (iv) $\displaystyle\frac{f(s)}{|s|^{p-1}}$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

A function $u \in W_0^{1,p}(\Omega_\lambda)$ is said to be a weak solution of problem (1.1) if $u(-x) = -u(x)$ for a.a. $x \in \Omega_\lambda$ and if

$$\int_{\Omega_\lambda} \left(|\nabla u|^{p-2}\nabla u + \mu|\nabla u|^{q-2}\nabla u\right) \cdot \nabla v \,\mathrm{d}x = \int_{\Omega_\lambda} \left(f(u) - |u|^{p-2}u\right) v \,\mathrm{d}x$$

is satisfied for all $v \in W_0^{1,p}(\Omega_\lambda)$. The corresponding energy functional $J_\lambda\colon W_0^{1,p}(\Omega_\lambda) \to \mathbb{R}$ for problem (1.1) is given by

$$J_\lambda(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{\mu}{q}\|\nabla u\|_q^q - \int_{\Omega_\lambda} F(u)\,\mathrm{d}x \quad \text{for all } u \in W_0^{1,p}(\Omega_\lambda). \tag{1.2}$$

Under the assumptions in (H1) and (H2), it is clear that J_λ is well-defined and of class C^1.

The following theorem is our main result.

Theorem 1.1. *Let hypotheses* (H1) *and* (H2) *be satisfied and let* Ω *be symmetric with respect to the origin, that is,* $\Omega = -\Omega$*. Then there exists* $\lambda^* > 0$ *such that, for any* $\lambda \geqslant \lambda^*$*, problem* (1.1) *has at least* $\gamma(\Omega_\lambda \setminus \{0\})$ *pairs* $(\pm u)$ *of odd weak solutions with precisely two nodal domains, where* γ *stands for the genus.*

The proof of Theorem 1.1 relies on the Lusternik-Schnirelmann category in combination with the odd symmetry invariant Nehari submanifold. As far as we know this is the first work dealing with a superlinear (p, q)-equation in expanding domains that has multiple sign-changing solutions obtained via the Lusternik-Schnirelmann category.

A starting point in the direct application of the Lusternik-Schnirelmann category to elliptic equations was the work of Benci-Cerami [11] who studied the problem

$$\begin{aligned} -\Delta u + \lambda u = u^{p-1} \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $p \in (2, 2^*)$. It is shown that problem (1.3) has at least $\operatorname{cat}(\Omega)$ solutions when p is close to 2^*, where $\operatorname{cat}(\Omega)$ denotes the Lusternik-Schnirelmann category of Ω. Motivated by this work and its used methods, Bartsch-Wang [9] treated nonlinear Schrödinger equations of the form

$$-\Delta u + (\lambda a(x) + 1)u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

with $1 < p < 2^* - 1$ and showed the existence of at least $\operatorname{cat}(\Omega)$ solutions of (1.4) when the parameter $\lambda > 0$ is large enough, see also [8] of the same authors. Afterwards, the Lusternik-Schnirelmann category has been applied to several types of problems. We mention, for example, the works of Alves [2] for p-Laplace equations with expanding domains, Alves-Ding [3] for critical p-Laplace equations, Alves-Figueiredo-Furtado [4] for multiple solutions for nonlinear Schrödinger equations with magnetic fields, Benci-Bonanno-Micheletti [10] for elliptic equations on Riemannian manifolds, Cingolani [16] for nonlinear Schrödinger equations with an external magnetic field, Cingolani-Lazzo [17] for nonlinear Schrödinger equations, Figueiredo-Pimenta-Siciliano [20] for fractional Laplacian in expanding domains, Figueiredo-Siciliano [21] for fractional Schrödinger equations in $\mathbb{R}^N$ and Wang-Tian-Xu-Zhang [26] for Kirchhoff type problems, see also the references therein. All these works are dealing with constant sign solutions.

For sign-changing solutions via the Lusternik-Schnirelmann category we refer to the paper of Castro-Clapp [14] in which the problem

$$\begin{aligned} \Delta u + \lambda u + |u|^{2^*-2}u = 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \\ u(\tau x) = -u(x) \quad & \text{for all } x \in \Omega \end{aligned} \tag{1.5}$$

was studied where τ is a nontrivial orthogonal involution. For $\lambda > 0$ to be small, the existence of pairs of sign-changing solutions which change the sign exactly once has been shown for problem (1.5). These results have been improved by Cano-Clapp [13]. Finally, we mention some results concerning problems with expanding domains, see, for example the papers of Ackermann-Clapp-Pacella [1] for alternating sign multibump solutions in expanding tubular domains, Alves-Figueiredo-Furtado [5] for complex equations, Bartsch-Clapp-Grossi-Pacella [7] for asymptotically radial solutions in expanding domains, Byeon-Tanaka [12] for multibump positive solutions in expanding tubular domains, Catrina-Wang [15] for Dirichlet Laplace problems in an expanding annulus, Dancer-Yan [18] for multibump solutions and Feireisl-Nečasová-Sun [19] for inviscid incompressible limits on expanding domains.

The paper is organized as follows. In Section 2 we recall some basic definitions and investigate the relation between the unit sphere and the odd symmetry invariant Nehari manifold. Section 3 is devoted to the (PS)-condition property and some needed estimates and in Section 4 we prove Theorem 1.1. Our results are combining ideas from the work of Alves [2], Castro-Clapp [14] and Catrina-Wang [15].

2. The mapping between $\mathcal{S}_{\pm}^{\circ}$ and $\mathcal{N}_{\pm}^{\circ}$

We denote by $L^s(\Omega)$ (resp. $L^s(\Omega;\mathbb{R}^N)$) and $L^s(\Omega_\lambda)$ (resp. $L^s(\Omega_\lambda;\mathbb{R}^N)$) the usual Lebesgue spaces equipped with the norm $\|\cdot\|_s$ for every $1 \le s < \infty$. For $1 < s < \infty$, $W^{1,s}(\Omega)$ and $W_0^{1,s}(\Omega_\lambda)$ stand for the Sobolev spaces endowed with the norm $\|\cdot\|_{1,s}$.

Let X be a Banach space and let $\mathcal{A}$ be the class of all closed subsets B of $X \setminus \{0\}$ which are symmetric, that is, $u \in B$ implies $-u \in B$.

Definition 2.1. Let $B \in \mathcal{A}$. The genus $\gamma(B)$ of B is defined as the least integer n such that there exists $\varphi \in C(X,\mathbb{R}^n)$ such that φ is odd and $\varphi(x) \neq 0$ for all $x \in B$. We set $\gamma(B) = +\infty$ if there are no integers with the above property and $\gamma(\emptyset) = 0$.

Remark 2.2. An equivalent way to define $\gamma(B)$ is to take the minimal integer n such that there exists an odd map $\varphi \in C(B,\mathbb{R}^n \setminus \{0\})$.

For a function u, from now on, we denote by u^+ (resp. u^-) the positive (resp. negative) part of u, that is

$$u^+ = \max(u,0), \quad u^- = \min(u,0). \tag{2.1}$$

Let

$$W_0^{1,p}(\Omega_\lambda)^\circ := \left\{ u \in W_0^{1,p}(\Omega_\lambda) : \ u(-x) = -u(x) \right\}.$$

We denote the Nehari manifold corresponding to (1.1) by

$$\mathcal{N}_\lambda := \left\{ u \in W_0^{1,p}(\Omega_\lambda) \setminus \{0\} : \langle J_\lambda'(u), u \rangle = 0 \right\}$$

and the odd symmetry invariant Nehari submanifold by

$$\mathcal{N}_\lambda^\circ := \{ u \in \mathcal{N}_\lambda : \ u(-x) = -u(x) \} .$$

It is clear that

$$\mathcal{N}_\lambda^\circ = \mathcal{N}_\lambda \cap W_0^{1,p}(\Omega_\lambda)^\circ .$$

Note that $J_\lambda \colon W_0^{1,p}(\Omega_\lambda)^\circ \to \mathbb{R}$ is an even functional with $(J_\lambda(-u))' = -J_\lambda'(u)$. Therefore, if $J_\lambda \in C^2$, then the nontrivial solutions of (1.1) are the critical points of the restriction of J_λ to the odd symmetry invariant Nehari submanifold $\mathcal{N}_\lambda^\circ$. However, we only assume that f is continuous. This leads to $J_\lambda \in C^1$ and the non-differentiability of $\mathcal{N}_\lambda^\circ$. To overcome these difficulties, we need the following two lemmas.

We write

$$\mathcal{S}^\circ = \left\{ u \in W_0^{1,p}(\Omega_\lambda)^\circ \ : \ \|u\|_{1,p} = 1 \right\}, \ \mathcal{S}_\pm^\circ = \left\{ u^\pm : u \in \mathcal{S}^\circ \right\} \text{ and } \mathcal{N}_\pm^\circ = \left\{ u^\pm : u \in \mathcal{N}_\lambda^\circ \right\}.$$

Then we can set up a one-to-one correspondence between $\mathcal{S}_\pm^\circ$ and $\mathcal{N}_\pm^\circ$ as follows.

Lemma 2.3. *Let hypotheses* (H1) *and* (H2) *be satisfied.*

(i) *For each $w \in W_0^{1,p}(\Omega_\lambda)^\circ \setminus \{0\}$, set $h_{w^\pm}(t) = J_\lambda(tw^\pm)$ for $t \geq 0$. Then there exists a unique $t_{w^\pm} > 0$ such that $h_{w^\pm}'(t) > 0$ if $0 < t < t_{w^\pm}$ and $h_{w^\pm}'(t) < 0$ if $t > t_{w^\pm}$, that is, $\max_{t \in [0,+\infty)} h_{w^\pm}(t)$ is achieved at $t = t_{w^\pm}$ and $t_{w^\pm} w^\pm \in \mathcal{N}_\pm^\circ$.*

(ii) *There exists $\delta > 0$ such that $t_{w^\pm} \geqslant \delta$ for $w \in \mathcal{S}_\pm^\circ$ and for each compact subset $\mathcal{W}^\circ \subseteq \mathcal{S}_\pm^\circ$ there exists a constant $C_{\mathcal{W}^\circ}$ such that $t_{w^\pm} \leqslant C_{\mathcal{W}^\circ}$ for all $w \in \mathcal{W}^\circ$.*

Proof. (i) Let $w \in W_0^{1,p}(\Omega_\lambda)^\circ \setminus \{0\}$ be fixed and define $h_{w^\pm}(t) = J_\lambda(tw^\pm)$ on $[0, \infty)$. It is clear that $h_{w^\pm}(0) = 0$. From (H2)(i) and (H2)(ii) we know that for given $\varepsilon > 0$ we can find $C_\varepsilon > 0$ such that

$$|F(s)| \leq \varepsilon |s|^q + C_\varepsilon |s|^r \quad \text{for a. a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{2.2}$$

Using (2.2) and the embedding $W_0^{1,q}(\Omega_\lambda) \to L^q(\Omega_\lambda)$ with embedding constant $C_q > 0$ we get for $t > 0$

$$\begin{aligned} h_{w^\pm}(t) = J_\lambda(tw^\pm) &= \frac{t^p}{p} \|w^\pm\|_{1,p}^p + \frac{\mu t^q}{q} \|\nabla w^\pm\|_q^q - \int_{\Omega_\lambda} F(tw^\pm)\, \mathrm{d}x \\ &\geq \frac{t^p}{p} \|w^\pm\|_{1,p}^p + \frac{\mu t^q}{q} \|\nabla w^\pm\|_q^q - \int_{\Omega_\lambda} \left(\varepsilon t^q |w^\pm|^q + C_\varepsilon t^r |w^\pm|^r \right) \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{t^p}{p}\|w^\pm\|_{1,p}^p + \left(\frac{\mu}{q} - C_q^q\varepsilon\right) t^q \|\nabla w^\pm\|_q^q - C_\varepsilon t^r \|w^\pm\|_r^r \\
&= C_1 t^p + C_2 t^q - C_3 t^r \quad \text{for } 0 < \varepsilon < \frac{\mu}{q C_q^q}
\end{aligned}
$$

with $C_1, C_2, C_3 > 0$. Hence, for $t > 0$ small enough we see that $h_{w^\pm}(t) > 0$ due to $q < p < r$.

From hypothesis (H2)(iii) there exists for any $M > 0$ a number $T_M > 0$ such that

$$
F(s) \geq M|s|^p \quad \text{for a. a. } x \in \Omega \text{ and for all } |s| > T_M. \tag{2.3}
$$

Taking (2.3) into account, we have for $t > 0$ large

$$
\begin{aligned}
h_{w^\pm}(t) = J_\lambda(tw^\pm) &\leq \frac{t^p}{p}\|w^\pm\|_{1,p}^p + \frac{\mu t^q}{q}\|\nabla w^\pm\|_q^q - M \int_{\Omega_\lambda} t^p |w^\pm|^p \,\mathrm{d}x \\
&= C_1 t^p + C_2 t^q - C_3 M t^p \\
&\leqslant -C_4 t^p + C_2 t^q \quad \text{for } M > \frac{C_1}{C_3},
\end{aligned}
$$

with $C_1, C_2, C_3, C_4 > 0$. This implies that $h_{w^\pm}(t) < 0$ for t large enough. Hence there exists $t_{w^\pm} > 0$ such that $h'_{w^\pm}(t_{w^\pm}) = 0$. Note that

$$
0 = h'_{w^\pm}(t) = t^{p-1}\|w^\pm\|_{1,p}^p + \mu t^{q-1}\|\nabla w^\pm\|_q^q - \int_{\Omega_\lambda} f(tw^\pm) w^\pm \,\mathrm{d}x
$$

implies $tw^\pm \in \mathcal{N}_\pm^\circ$ and

$$
\begin{aligned}
\|w^\pm\|_{1,p}^p &= \int_{\Omega_\lambda} \frac{f(tw^\pm)w^\pm}{t^{p-1}} \,\mathrm{d}x - \frac{\mu}{t^{p-q}}\|\nabla w^\pm\|_q^q \\
&= \begin{cases} \displaystyle\int_{\Omega_\lambda^>} \frac{f(tw^+)w^+}{t^{p-1}} \,\mathrm{d}x - \frac{\mu}{t^{p-q}}\|\nabla w^\pm\|_q^q, \\ \displaystyle\int_{\Omega_\lambda^<} \frac{f(tw^-)w^-}{t^{p-1}} \,\mathrm{d}x - \frac{\mu}{t^{p-q}}\|\nabla w^\pm\|_q^q, \end{cases}
\end{aligned} \tag{2.4}
$$

where

$$
\begin{aligned}
\Omega_\lambda^> &= \{x \in \Omega_\lambda : w(x) > 0\}, \\
\Omega_\lambda^< &= \{x \in \Omega_\lambda : w(x) < 0\}
\end{aligned}
$$

and w^+ (resp. w^-) is the positive (resp. negative) part of w, given in (2.1). By (H2)(iv), the right-hand side of (2.4) is a strictly increasing function in t. It follows that $h_{w^\pm}(t)$

has a unique critical point. Therefore $\max_{t\in[0,+\infty)} h_{w^\pm}(t)$ is achieved at the unique point $t = t_{w^\pm} > 0$ so that $h'_{w^\pm}(t_{w^\pm}) = 0$ and $t_{w^\pm} w^\pm \in \mathcal{N}^\circ_\pm$.

(ii) First, we prove that there exists $\delta > 0$ such that $t_{w^\pm} > \delta$ for any $w \in \mathcal{S}^\circ_\pm$. From (H2)(i) and (H2)(ii) we know that for given $\varepsilon > 0$ we can find $C_\varepsilon > 0$ such that

$$|f(s)| \le \varepsilon |s|^{q-1} + C_\varepsilon |s|^{r-1} \quad \text{for a. a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{2.5}$$

Let $w^\pm \in \mathcal{S}^\circ_\pm$. Using $t_{w^\pm} w^\pm \in \mathcal{N}^\circ_\pm$, (2.5) and the embeddings $W^{1,q}_0(\Omega_\lambda) \to L^q(\Omega_\lambda)$, $W^{1,p}_0(\Omega_\lambda) \to L^r(\Omega_\lambda)$ with embedding constants $C_q, C_p > 0$ we obtain

$$\begin{aligned} t^p_{w^\pm} \|w^\pm\|^p_{1,p} + \mu t^q_{w^\pm} \|\nabla w^\pm\|^q_q &= \int_{\Omega_\lambda} f(t_{w^\pm} w^\pm) t_{w^\pm} w^\pm \, \mathrm{d}x \\ &\le \varepsilon t^q_{w^\pm} \int_{\Omega_\lambda} |w^\pm|^q \, \mathrm{d}x + C_\varepsilon t^r_{w^\pm} \int_{\Omega_\lambda} |w^\pm|^r \, \mathrm{d}x \\ &\le C^q_q \varepsilon t^q_{w^\pm} \|\nabla w^\pm\|^q_q + C^r_p C_\varepsilon t^r_{w^\pm} \|w^\pm\|^r_{1,p}. \end{aligned}$$

Choosing $\varepsilon \in (0, \frac{\mu}{C^q_q})$ and using the fact that $\|w^\pm\|_{1,p} = 1/2$, it follows that

$$\frac{t^p_{w^\pm}}{2^p} \le t^p_w \|w\|^p_{1,p} + \left(\mu - C^q_q \varepsilon\right) t^q_w \|\nabla w\|^q_q \le C^r_p C_\varepsilon \frac{t^r_{w^\pm}}{2^r}.$$

We take $\delta = 2\left(\frac{1}{C^r_p C_\varepsilon}\right)^{\frac{1}{r-p}} > 0$ in order to get the desired assertion.

Next, let $\mathcal{W}^\circ \subseteq \mathcal{S}^\circ_\pm$ be compact. Suppose by contradiction that there is a sequence $\{w^\pm_n\}_{n\in\mathbb{N}} \subseteq \mathcal{W}^\circ$ such that $t_n := t_{w^\pm_n} \to +\infty$. By (i), we know that $J_\lambda(t_n w^\pm_n) = \max_{t\in[0,+\infty)} J_\lambda(t w^\pm_n) \geqslant 0$.

Using $\|\cdot\|^q_{1,q} \le C_{pq} \|\cdot\|^q_{1,p}$ along with (H2)(iii), we deduce that

$$0 \leqslant \frac{J_\lambda(t_n w^\pm_n)}{t^p_n} \leqslant \frac{1}{p} + \frac{\mu C_{pq}}{q} - \int_{\Omega_\lambda} \frac{F(t_n w^\pm_n)}{t^p_n} \, \mathrm{d}x \to -\infty \quad \text{as } n \to \infty,$$

which yields a contradiction. Thus there exists $C_{\mathcal{W}^\circ}$ such that $t_{w^\pm} \leqslant C_{\mathcal{W}^\circ}$. □

We define

$$\hat{m}_\pm : \left\{ w^\pm : w \in W^{1,p}_0(\Omega_\lambda)^\circ \setminus \{0\} \right\} \to \mathcal{N}^\circ_\pm, \quad w^\pm \mapsto \hat{m}_\pm(w^\pm) := t_{w^\pm} w^\pm,$$

where $t_{w^\pm}$ is defined in Lemma 2.3. For simplification we write $m_\pm := \hat{m}_\pm|_{\mathcal{S}^\circ_\pm}$. Next, we are going to prove that $m_\pm$ is a one-to-one correspondence between $\mathcal{S}^\circ_\pm$ and $\mathcal{N}^\circ_\pm$.

Lemma 2.4. *Let hypotheses* (H1) *and* (H2) *be satisfied.*

(i) *The mapping $\hat{m}_\pm$ is continuous.*
(ii) *The mapping $m_\pm$ is a homeomorphism between $\mathcal{S}_\pm^\circ$ and $\mathcal{N}_\pm^\circ$ and the inverse of $m_\pm$ is given by*

$$m_\pm^{-1}(u^\pm) = \frac{u^\pm}{\|u^\pm\|_{1,p}} \quad \textit{for all } u \in \mathcal{N}_\pm^\circ$$

Proof. (i) Assume that $w_n^\pm \to w^\pm$. From Lemma 2.3 (ii) it follows that $\{t_{w_n^\pm}\}_{n\in\mathbb{N}}$ is uniformly bounded. Hence, there exists a subsequence of $\{t_{w_n^\pm}\}_{n\in\mathbb{N}}$, not relabeled, which converges to a limit t_0. From (2.4) we conclude that $t_0 = t_{w^\pm}$. But then $t_{w_n^\pm} \to t_{w^\pm}$. Thus $\hat{m}_\pm$ is continuous.

(ii) From (i) we know that $m_\pm(\mathcal{S}_\pm^\circ)$ is a bounded set in $W_0^{1,p}(\Omega_\lambda)$ and for any $u^\pm \in m_\pm(\mathcal{S}_\pm^\circ) \subseteq \mathcal{N}_\pm^\circ$, there exists $\delta > 0$ such that $\|u^\pm\|_{1,p} \geq \delta$. Indeed, similar to the proof of Lemma 2.3 (i), by using $u \in \mathcal{N}_\pm^\circ \subseteq \mathcal{N}_\lambda$, (2.3) and the embeddings $W_0^{1,q}(\Omega_\lambda) \to L^q(\Omega_\lambda)$, $W_0^{1,p}(\Omega_\lambda) \to L^r(\Omega_\lambda)$ with embedding constants $C_q, C_p > 0$ we have

$$\begin{aligned}\|u^\pm\|_{1,p}^p + \mu\|\nabla u^\pm\|_q^q = \int_{\Omega_\lambda} f(u^\pm)u^\pm \,\mathrm{d}x \leq \varepsilon \int_{\Omega_\lambda} |u^\pm|^q \,\mathrm{d}x + C_\varepsilon \int_{\Omega_\lambda} |u^\pm|^r \,\mathrm{d}x \\ \leq C_q^q \varepsilon \|\nabla u^\pm\|_q^q + C_p^r C_\varepsilon \|u^\pm\|_{1,p}^r.\end{aligned}$$

Choosing $\varepsilon > 0$ small enough, we obtain from this

$$\|u^\pm\|_{1,p}^p \leq \|u^\pm\|_{1,p}^p + \left(\mu - C_q^q\varepsilon\right)\|\nabla u^\pm\|_q^q \leq C_p^r C_\varepsilon \|u^\pm\|_{1,p}^r.$$

Taking $\delta = 2\left(\frac{1}{C_p^r C_\varepsilon}\right)^{\frac{1}{r-p}} > 0$ we have $\|u^\pm\|_{1,p} \geq \delta$. From the continuity of $\hat{m}_\pm$ and its definition, we know that the map $m_\pm \colon \mathcal{S}_\pm^\circ \to \mathcal{N}_\pm^\circ$ is continuous and one-to-one. It is clear that the inverse function of $m_\pm$ is given by $m_\pm^{-1}(u^\pm) = \frac{u^\pm}{\|u^\pm\|_{1,p}}$ for any $u^\pm \in \mathcal{N}_\pm^\circ$. To reach the desired conclusion, it is enough to show that $m_\pm^{-1}$ is continuous. Indeed, we have

$$\begin{aligned}\left\|m_\pm^{-1}(u^\pm) - m_\pm^{-1}(v^\pm)\right\|_{1,p} &= \left\|\frac{u^\pm}{\|u^\pm\|_{1,p}} - \frac{v^\pm}{\|v^\pm\|_{1,p}}\right\|_{1,p} \\ &= \left\|\frac{u^\pm - v^\pm}{\|u\|_{1,p}} + \frac{v^\pm\left(\|v^\pm\|_{1,p} - \|u^\pm\|_{1,p}\right)}{\|u^\pm\|_{1,p}\|v^\pm\|_{1,p}}\right\|_{1,p} \\ &\leq \frac{2\|u^\pm - v^\pm\|_{1,p}}{\|u^\pm\|_{1,p}} \leq \frac{2}{\delta}\|u^\pm - v^\pm\|_{1,p},\end{aligned}$$

that is, $m_\pm^{-1}$ is Lipschitz continuous. □

We write $\hat{\Psi}(w^\pm) := J_\lambda(\hat{m}_\pm(w^\pm))$. In the next lemma, we are going to show that the problem of finding critical points of $\hat{\Psi}|_{\mathcal{S}_\pm^\circ}$ is equivalent to the problem of finding critical points of $J_\lambda|_{\mathcal{N}_\pm^\circ}$. Recall that a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ is called a $(\mathrm{PS})_c$-sequence if $J(u_n) \to c$ and $J'(u_n) \to 0$. We say that J_λ satisfies the (PS)-condition on $\mathcal{M}$, if every $(\mathrm{PS})_c$-sequence has a converging subsequence.

Lemma 2.5. *Let hypotheses* (H1) *and* (H2) *be satisfied.*

(i) $\hat{\Psi} \in C^1\left(\left\{w^\pm : w \in W_0^{1,p}(\Omega_\lambda)^\circ \setminus \{0\}\right\}, \mathbb{R}\right)$ *and*

$$\left\langle \hat{\Psi}'(w^\pm), z \right\rangle = \left\langle J_\lambda'(m_\pm(w^\pm)), \|m_\pm(w^\pm)\|_{1,p} z \right\rangle$$
$$\text{for all } w^\pm \in \mathcal{S}_\pm^\circ \text{ and for all } z \in T_{w^\pm}(\mathcal{S}_\pm^\circ),$$

where $T_{w^\pm}(\mathcal{S}_\pm^\circ)$ denote the tangent space to $\mathcal{S}_\pm^\circ$ at $w^\pm$.

(ii) *If $\{w_n^\pm\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_\pm^\circ$ is a $(\mathrm{PS})_c$-sequence for $\hat{\Psi}$, then $\{m_\pm(w_n^\pm)\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\pm^\circ$ is a $(\mathrm{PS})_c$-sequence for J_λ. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}_\pm^\circ$ is a bounded $(\mathrm{PS})_c$-sequence for J_λ, then $\{m_\pm^{-1}(u_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_\pm^\circ$ is a $(\mathrm{PS})_c$-sequence for $\hat{\Psi}$.*

(iii) *$w^\pm \in \mathcal{S}_\pm^\circ$ is a critical point of $\hat{\Psi}$ if and only if $m_\pm(w^\pm) \in \mathcal{N}_\pm^\circ$ is a nontrivial critical point of J_λ. Moreover, $\inf_{\mathcal{S}_\pm^\circ} \hat{\Psi} = \inf_{\mathcal{N}_\pm^\circ} J_\lambda$.*

(iv) *If J_λ is even, then so is $\hat{\Psi}$.*

Proof. The lemma follows from Szulkin-Weth [25, Proposition 9 and Corollary 10] and Lemmas 2.3 and 2.4. We omit the details. □

Remark 2.6.

(i) Set

$$c^\circ(\Omega_\lambda) = \inf_{u\in\mathcal{N}_\lambda^\circ} J_\lambda(u).$$

Then it follows from Lemma 2.5 (iii) that

$$c^\circ(\Omega_\lambda) = \inf_{w\in\mathcal{S}^\circ} \hat{\Psi}(w).$$

From Lemmas 2.3 and 2.4 it is easy to see that $c^\circ(\Omega_\lambda)$ has the following minimax characterization:

$$c^\circ(\Omega_\lambda) = \inf_{w\in W_0^{1,p}(\Omega_\lambda)^\circ\setminus\{0\}} \max_{t>0} J_\lambda(tw) = \inf_{w\in\mathcal{S}^\circ} \max_{t>0} J_\lambda(tw).$$

We know from the proof of Lemma 2.3 that there exists a unique $t_w > 0$ such that $\max_{t>0} J_\lambda(tw) = J(t_w w)$ for $w \in \mathcal{S}^\circ$. Lemma 2.3 (ii) implies that there exists $\delta > 0$ such that $t_w \geqslant \delta$ uniformly for $w \in \mathcal{S}^\circ$. Thus, for any $w \in \mathcal{S}^\circ$, we have

$$J(t_w w) = \max_{t>0} J_\lambda(tw) \geqslant \sigma,$$

for some $\sigma > 0$ independent of w and consequently

$$\inf_{w\in\mathcal{S}^\circ} \max_{t>0} J_\lambda(tw) \geqslant \sigma,$$

that is

$$c^\circ(\Omega_\lambda) \geqslant \sigma > 0.$$

(ii) Set

$$c(\Omega_\lambda) = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u). \tag{2.6}$$

By an argument similar to that of (i), we can show that $c(\Omega_\lambda) > 0$. We can also show that $c^\circ(\Omega_\lambda) \geq 2c(\Omega_\lambda)$. It is similar to the proof of Lemma 3.2 and we omit it.

3. (PS)-condition and some estimates

Our first result is that $\hat{\Psi}$ satisfies the (PS)-condition on $\mathcal{S}^\circ_\pm$. We set

$$I_\lambda(u) = \frac{1}{p}\|u\|^p_{1,p} + \frac{\mu}{q}\|\nabla u\|^q_q \quad \text{and} \quad K_\lambda(u) = \int_{\Omega_\lambda} F(u)\,\mathrm{d}x.$$

Then $J_\lambda(u) = I_\lambda(u) - K_\lambda(u)$. We denote the derivative operator of I_λ in the weak sense by A_λ. It is well known that the operator A_λ is of type (S_+). We also denote by $\partial\mathcal{S}^\circ_\pm$ the boundary of $\mathcal{S}^\circ_\pm$.

Lemma 3.1. *Let hypotheses* (H1) *and* (H2) *be satisfied.*

(i) *Let $\{w_n^\pm\}_{n\in\mathbb{N}} \subseteq \mathcal{S}^\circ_\pm$ be a sequence such that $\operatorname{dist}(w_n^\pm, \partial\mathcal{S}^\circ_\pm) \to 0$ as $n \to +\infty$. Then $\|m(w_n^\pm)\| \to +\infty$ and $\hat{\Psi}(w_n^\pm) \to +\infty$ as $n \to +\infty$.*
(ii) *For any $\lambda > 0$, $\hat{\Psi}$ satisfies the* (PS)*-condition on $\mathcal{S}^\circ_\pm$.*

Proof. (i) Recall that we denote u^+ (resp. u^-) the positive (resp. negative) part of u, given in (2.1) and write

$$\mathcal{S}^\circ_\pm = \left\{u^\pm : u \in \mathcal{S}^\circ\right\}.$$

Let $w \in \mathcal{S}^\circ_\pm$ and $\gamma \in [1, p^*]$. By the embedding theorem, we have

$$\begin{aligned}\|w^+\|_{L^\gamma(\Omega_\lambda)} &= \inf_{v\in\overline{\mathcal{S}^\circ_\pm}} \|w-v\|_{L^\gamma(\Omega_\lambda)} \le \inf_{v\in\partial\mathcal{S}^\circ_\pm} \|w-v\|_{L^\gamma(\Omega_\lambda)} \\ &\le C_\gamma \inf_{v\in\partial\mathcal{S}^\circ_\pm} \|w-v\|_{1,p} = C_\gamma \operatorname{dist}\left(w,\partial\mathcal{S}^\circ_\pm\right).\end{aligned}$$

Here we denote by $\overline{\mathcal{S}^\circ_\pm}$ the closure of $\mathcal{S}^\circ_\pm$.

Similarly, it holds

$$\|w^-\|_{L^\gamma(\Omega_\lambda)} \le C_\gamma \operatorname{dist}\left(w,\partial\mathcal{S}^\circ_\pm\right).$$

Let $\{w_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}^\circ_\pm$ be a sequence such that $\operatorname{dist}(w_n,\partial\mathcal{S}^\circ_\pm) \to 0$ as $n\to+\infty$ and let

$$\begin{aligned}\Omega^>_\lambda &= \{x\in\Omega_\lambda : w_n(x) > 0\},\\ \Omega^<_\lambda &= \{x\in\Omega_\lambda : w_n(x) < 0\},\\ \Omega^=_\lambda &= \{x\in\Omega_\lambda : w_n(x) = 0\}.\end{aligned}$$

For every $t>0$, using (2.2), we have

$$\begin{aligned}|K_\lambda(tw_n)| &= \left|\int_{\Omega^<_\lambda} F(tw_n)\,\mathrm{d}x + \int_{\Omega^>_\lambda} F(tw_n)\,\mathrm{d}x + \int_{\Omega^=_\lambda} F(tw_n)\,\mathrm{d}x\right| \\ &= \left|\int_{\Omega_\lambda} F(tw_n^+)\,\mathrm{d}x + \int_{\Omega_\lambda} F(tw_n^-)\,\mathrm{d}x\right| \\ &\le \varepsilon t^q \left(\|w_n^+\|^q_{L^q(\Omega_\lambda)} + \|w_n^-\|^q_{L^q(\Omega_\lambda)}\right) + C_\varepsilon t^r \left(\|w_n^+\|^r_{L^r(\Omega_\lambda)} + \|w_n^-\|^r_{L^r(\Omega_\lambda)}\right) \\ &\le C\left[t^q\left(\operatorname{dist}(w_n,\partial\mathcal{S}^\circ_\pm)\right)^q + t^r\left(\operatorname{dist}(w_n,\partial\mathcal{S}^\circ_\pm)\right)^r\right] \to 0 \text{ as } n\to+\infty.\end{aligned}$$

Note that for any $t>1$,

$$\begin{aligned}\left(\frac{1}{p}+\frac{\mu C_{pq}}{q}\right)\|tw_n\|^p_{1,p} + |K_\lambda(tw_n)| \ge J_\lambda(tw_n) &\ge \frac{1}{p}\|tw_n\|^p_{1,p} - |K_\lambda(tw_n)| \\ &= \frac{t^p}{p} - |K_\lambda(tw_n)|.\end{aligned}$$

Consequently

$$\liminf_{n\to+\infty}\left(\frac{1}{p}+\frac{\mu C_{pq}}{q}\right)\|m(w_n)\|^p_{1,p} \ge \liminf_{n\to+\infty}\hat{\Psi}(w_n) \ge \liminf_{n\to+\infty} J_\lambda(tw_n) \ge \frac{t^p}{p}$$

for every $t>1$. Hence, $\|m(w_n)\|\to+\infty$ and $\hat{\Psi}(w_n)\to+\infty$ as $n\to+\infty$.

(ii) For any $c>0$, let $\{w_n^\pm\}_{n\in\mathbb{N}} \subseteq \mathcal{S}^\circ_\pm$ be a $(\mathrm{PS})_c$-sequence for $\hat{\Psi}$. Let $u_n^\pm := m_\pm(w_n^\pm)$ for all $n\in\mathbb{N}$. It follows from Lemma 2.5 that $\{u_n^\pm\}_{n\in\mathbb{N}} \subseteq \mathcal{N}^\circ_\pm$ is a $(\mathrm{PS})_c$-sequence for

J_λ. First we will prove that $\{u_n^\pm\}_{n\in\mathbb{N}}$ is bounded. Let us assume this is not the case, so there exists a subsequence (still denoted by $u_n^\pm$) such that $\|u_n^\pm\|_{1,p} \to +\infty$. We define $v_n^\pm := \frac{u_n^\pm}{\|u_n^\pm\|_{1,p}}$, then $\|v_n^\pm\|_{1,p} = 1$. Thus we may assume that

$$v_n^\pm \rightharpoonup v^\pm \quad \text{in } W_0^{1,p}(\Omega_\lambda).$$

If $v^\pm = 0$, then it follows from Lemma 2.3 and Remark 2.6 that

$$c + o(1) \geqslant J_\lambda(u_n^\pm) = J_\lambda(t_{v_n^\pm} v_n^\pm) \geqslant J_\lambda(tv_n^\pm) \quad \text{for all } t > 0.$$

Recalling that K_λ is weakly continuous, we have that

$$J_\lambda(tv_n^\pm) \geq \frac{1}{p}t^p - \int_{\Omega_\lambda} F(tv_n^\pm)\,\mathrm{d}x \to \frac{1}{p}t^p \quad \text{as } n \to +\infty.$$

Choosing $t > 2(pc)^{\frac{1}{p}}$ yields a contradiction. If $v^\pm \neq 0$, then we know from (H2)(iii) that

$$0 \leq \frac{J_\lambda(u_n^\pm)}{\|u_n^\pm\|_{1,p}^p} \leq \frac{1}{p} + \frac{\mu C_{pq}}{q} - \int_{\Omega_\lambda} \frac{F(\|u_n^\pm\|_{1,p} v_n^\pm)}{\|u_n^\pm\|_{1,p}^p}\,\mathrm{d}x \to -\infty \quad \text{as } n \to +\infty.$$

This is again a contradiction. Hence $\{u_n^\pm\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega_\lambda)$ and so there exists a subsequence of $\{u_n^\pm\}_{n\in\mathbb{N}}$ (not relabeled) such that

$$u_n^\pm \rightharpoonup u^\pm \quad \text{in } W_0^{1,p}(\Omega_\lambda).$$

It is clear that $K_\lambda'(u_n^\pm) \to K_\lambda'(u^\pm)$, see Liu-Dai [22]. Since

$$J_\lambda'(u_n^\pm) = A_\lambda(u_n^\pm) - K_\lambda'(u_n^\pm) \to 0 \quad \text{as } n \to +\infty,$$

one has

$$A_\lambda(u_n^\pm) \to K_\lambda'(u^\pm) \quad \text{as } n \to +\infty.$$

Therefore, we conclude that $u_n^\pm \to u^\pm$ since A_λ is a mapping of type (S_+). Consequently, $m_\pm^{-1}(u_n^\pm) \to m_\pm^{-1}(u^\pm)$ by Lemma 2.4, that is, $w_n^\pm \to w^\pm$. Therefore, $\hat{\Psi}$ satisfies the $(\mathrm{PS})_c$-condition on $\mathcal{S}_\pm^\circ$. $\square$

We say that u changes sign m times if the set $\{x \in \Omega_\lambda : u(x) \neq 0\}$ has $m+1$ connected components. It is clear that a solution of problem (1.1) changes sign an odd number of times. Following the ideas of Castro-Clapp [14], we can show the following energy estimate.

Lemma 3.2. *Let hypotheses* (H1) *and* (H2) *be satisfied. If u is a solution of problem* (1.1) *which changes sign $2m-1$ times, then $J_\lambda(u) \geq mc^\circ(\Omega_\lambda)$.*

Proof. From the assumptions we know that the set $\{x \in \Omega : u(x) > 0\}$ has m connect components $\Omega_1, \Omega_2, \cdots, \Omega_m$. Let

$$u_i(x) = \begin{cases} u(x), & \text{if } x \in -\Omega_i \cup \Omega_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since u is a solution of problem (1.1), it is a critical point of J_λ. This gives

$$\begin{aligned} 0 &= \langle J_\lambda'(u), u_i \rangle \\ &= \int_{\Omega_\lambda} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla u_i + |u|^{p-2} u u_i \right) \mathrm{d}x + \mu \int_{\Omega_\lambda} |\nabla u|^{q-2} \nabla u \cdot \nabla u_i \, \mathrm{d}x - \int_{\Omega_\lambda} f(u) u_i \, \mathrm{d}x \\ &= \|u_i\|_{1,p}^p + \mu \|\nabla u_i\|_{1,q}^q - \int_{\Omega_\lambda} f(u_i) u_i \, \mathrm{d}x, \end{aligned}$$

which implies that $u_i \in \mathcal{N}_\lambda^\circ$ for all $i = 1, 2, \cdots, m$. Consequently

$$J_\lambda(u) = J_\lambda(u_1) + J_\lambda(u_2) + \cdots + J_\lambda(u_m) \geqslant mc^\circ(\Omega_\lambda). \quad \square$$

We denote the limiting energy functional by

$$J_\infty(u) := \int_{\mathbb{R}^N} \left(\frac{1}{p} |\nabla u|^p + \frac{1}{p} |u|^p + \frac{\mu}{q} |\nabla u|^q - F(u) \right) \mathrm{d}x.$$

The corresponding Nehari manifold is

$$\mathcal{N}_\infty := \left\{ u \in W_r^{1,p}(\mathbb{R}^N) \setminus \{0\} : \langle J_\infty'(u), u \rangle = 0 \right\},$$

where

$$W_r^{1,p}(\mathbb{R}^N) := \left\{ u \in W^{1,p}(\mathbb{R}^N) : u \text{ is radially symmetric} \right\}.$$

The least energy level is given by

$$0 < c\left(\mathbb{R}^N\right) := \inf_{u \in \mathcal{N}_\infty} J_\infty(u).$$

Lemma 3.3. *Let hypotheses* (H1) *and* (H2) *be satisfied. Then $c(\mathbb{R}^N)$ is achieved by a positive radially symmetric function.*

Proof. We define

$$f^{+}(t)=\begin{cases}0 & \text{if } t\leqslant 0,\\ f(t) & \text{if } t>0\end{cases}$$

with primitive $F^{+}(s)=\int_0^s f^{+}(t)\,\mathrm{d}t$. We set

$$J_\infty^{+}(u):=\int\limits_{\mathbb{R}^N}\left(\frac{1}{p}|\nabla u|^p+\frac{1}{p}|u|^p+\frac{\mu}{q}|\nabla u|^q-F^{+}(u)\right)\mathrm{d}x \quad \text{for all } u\in W_r^{1,p}(\mathbb{R}^N).$$

It is clear that (H2) remain valid for f^+ and F^+. Similar to the proof of Lemma 2.3, we can define

$$\hat{m}:W_r^{1,p}(\mathbb{R}^N)\setminus\{0\}\to\mathcal{N}_\infty,\quad w\mapsto\hat{m}(w):=t_w w,$$

where t_w is similar to the definition in the proof of Lemma 2.3. We set $m:=\hat{m}|_{\mathcal{S}}$ and can show that m is a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{N}_\infty$, where

$$\mathcal{S}=\left\{w\in W_r^{1,p}(\mathbb{R}^N)\,:\,\|w\|_{1,p}=1\right\}.$$

Setting $\hat{\Psi}_\infty^{+}(w):=J_\infty^{+}(\hat{m}(w))$ we can show that $\hat{\Psi}_\infty^{+}$ satisfies the (PS)-condition on $\mathcal{S}$ as in Lemma 3.1(ii), since $W_r^{1,p}(\mathbb{R}^N)\hookrightarrow L^\gamma(\mathbb{R}^N)$ is compact for all $\gamma\in(p,p^*)$. Therefore, it follows from Theorem 1 in Szulkin-Weth [25] that $\inf\limits_{\mathcal{S}}\hat{\Psi}_\infty^{+}$ is attained by a function $w\in W_r^{1,p}(\mathbb{R}^N)$. Just like Lemma 2.5 (iii), we are able to show that $\inf\limits_{\mathcal{S}}\hat{\Psi}_\infty^{+}=\inf\limits_{\mathcal{N}_\infty}J_\infty^{+}$, that is, $\inf\limits_{\mathcal{N}_\infty}J_\infty^{+}$ is attained by $m(w)$, which is obviously radially symmetric. By an argument similar to that in the proof of Theorem 1.4 of the first two authors [23], we can also prove that $m(w)$ is positive. □

We also need the auxiliary functional which is defined as in (1.2) replacing Ω_λ by $B_R:=B_R(0)$ with $R>0$, that is,

$$J_R(u)=\int\limits_{B_R}\left(\frac{1}{p}|\nabla u|^p+\frac{1}{p}|u|^p+\frac{\mu}{q}|\nabla u|^q-F(u)\right)\mathrm{d}x.$$

The corresponding Nehari manifold is denoted by

$$\mathcal{N}_R:=\left\{u\in W_0^{1,p}(B_R)\setminus\{0\}\,:\,\langle J_R'(u),u\rangle=0\right\}.$$

We write

$$c\,(B_R):=\inf_{u\in\mathcal{N}_R}J_R(u). \tag{3.1}$$

Then $c(B_R)$ is achieved by a positive radially symmetric function Ψ_R. Indeed, similar to the proof of Lemma 3.3, we can show that $c(B_R)$ is attained by a positive function $v \in W_0^{1,p}(B_R)$.

Let v^* be the Schwartz symmetrization of v, then we have that $v^* \in W_0^{1,p}(B_R)$ and

$$\int_{B_R} \left(\frac{1}{p}|\nabla v^*|^p + \frac{\mu}{q}|\nabla v^*|^q\right) \mathrm{d}x \leq \int_{B_R} \left(\frac{1}{p}|\nabla v|^p + \frac{\mu}{q}|\nabla v|^q\right) \mathrm{d}x,$$

$$\int_{B_R} \frac{1}{p}|v^*|^p \,\mathrm{d}x = \int_{B_R} \frac{1}{p}|v|^p \,\mathrm{d}x,$$

$$\int_{B_R} F(v^*) \,\mathrm{d}x = \int_{B_R} F(v) \,\mathrm{d}x$$

are satisfied.

Just as in the proof of Lemma 2.3, we can show that there exists a unique $t_{v^*} > 0$ such that $t_{v^*}v^* \in \mathcal{N}_R$. Moreover,

$$c(B_R) \leq J_R(t_{v^*}v^*) \leq J_R(t_{v^*}v) \leq \max_{t \geqslant 0} J_R(tv) = J_R(v) = c(B_R).$$

Setting $\Psi_R := t_{v^*}v^*$, then it has all the required properties. Furthermore, we can determine the asymptotic behavior of $c(B_R)$.

Lemma 3.4. *Let hypotheses* (H1) *and* (H2) *be satisfied and let* $c(B_R)$ *and* $c(\Omega_\lambda)$ *be defined as in* (3.1) *and* (2.6), *respectively. Then it holds*

$$\lim_{R \to +\infty} c(B_R) = c\left(\mathbb{R}^N\right) \quad \text{and} \quad \lim_{\lambda \to +\infty} c(\Omega_\lambda) = c\left(\mathbb{R}^N\right).$$

Proof. We only prove the second equality, the other works very similarly.

We follow the ideas of Alves [2] who studied the p-Laplacian equation. To this end, fix $\tilde{\lambda} > 0$ and $R > 0$ such that $B_R \subseteq \Omega_{\tilde{\lambda}}$. Let $\eta_R\colon [0, +\infty) \to \mathbb{R}$ be a smooth, nonincreasing cut-off function such that

$$\eta_R(t) = 1 \quad \text{if } 0 \leq t \leq \frac{R}{2}, \quad \eta_R(t) = 0 \quad \text{if } t \geq R, \quad 0 \leq \eta_R \leq 1 \quad \text{and} \quad |\eta_R'(t)| \leq 2.$$

We write $w_R(x) = \eta_R(x)w(x)$, where $w \in \mathcal{N}_\infty$ such that $J_\infty(w) = c(\mathbb{R}^N)$. Let $t_R > 0$ be such that $t_R w_R \in \mathcal{N}_\lambda$. Then

$$c(\Omega_\lambda) \leq J_\lambda(t_R w_R) \quad \text{for all } \lambda > \tilde{\lambda}.$$

Passing to the limit as $\lambda \to +\infty$ we obtain

$$\limsup_{\lambda \to +\infty} c(\Omega_\lambda) \leq J_\infty(t_R w_R).$$

As in the proof of Lemma 2.3 we can show that $t_R \to 1$ as $R \to +\infty$. Then we have $J_\infty(t_R w_R) \to J_\infty(w) = c\left(\mathbb{R}^N\right)$ as $R \to +\infty$. Therefore,

$$\limsup_{\lambda \to +\infty} c\left(\Omega_\lambda\right) \leq c\left(\mathbb{R}^N\right). \tag{3.2}$$

On the other hand, from the definition of $c\left(\Omega_\lambda\right)$ and $c\left(\mathbb{R}^N\right)$ it follows that

$$c\left(\mathbb{R}^N\right) \leq c\left(\Omega_\lambda\right) \quad \text{for all } \lambda > 0,$$

which implies that

$$c\left(\mathbb{R}^N\right) \leq \liminf_{\lambda \to +\infty} c\left(\Omega_\lambda\right). \tag{3.3}$$

From (3.2) and (3.3) we get the assertion. □

4. Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. In what follows, without any loss of generality, we shall assume that $0 \in \Omega$. Moreover, we choose $\tilde{R} \geq \operatorname{diam}(\Omega)$ and $\tilde{R} > R > 0$ such that $B_R(0) \subseteq \Omega \subseteq B_{\tilde{R}}(0)$ and the sets

$$\Omega_R^+ := \left\{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) \leq R\right\} \quad \text{and} \quad \Omega_R^- := \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega \cup \{0\}) \geq R\right\}$$

are homotopically equivalent to Ω. For $\lambda > 0$, let $\Psi_{\lambda R} \in \mathcal{N}_{\lambda R}$ be given as in Section 3 satisfying $J_{\lambda R}(\Psi_{\lambda R}) = c(B_{\lambda R})$. We define $\Phi_\lambda \colon \lambda\Omega_R^- \to \mathcal{N}_\lambda^\circ$ by

$$\left[\Phi_\lambda(\xi)\right](x) = \begin{cases} t_\lambda \left[\Psi_{\lambda R}\left(|x-\xi|\right) - \Psi_{\lambda R}\left(|x+\xi|\right)\right], & \text{if } x \in B_{\lambda R}(\xi), \\ 0, & \text{if } x \in \Omega_\lambda \setminus B_{\lambda R}(\xi), \end{cases}$$

where $t_\lambda > 0$ is such that $\Phi_\lambda(\xi) \in \mathcal{N}_\lambda^\circ$. Note that

$$\left[\Phi_\lambda(\xi)\right](-x) = -\left[\Phi_\lambda(\xi)\right](x) \quad \text{and} \quad \Phi_\lambda(-\xi) = -\Phi_\lambda(\xi).$$

Hence $\Phi_\lambda(\xi)^\pm \in \mathcal{N}_\pm^\circ$.

Then we have the following lemma.

Lemma 4.1. *Let hypotheses* (H1) *and* (H2) *be satisfied. Then we have*

$$\lim_{\lambda \to +\infty} J_\lambda\left(\Phi_\lambda(\xi)^\pm\right) = c\left(\mathbb{R}^N\right)$$

uniformly in $\xi \in \lambda\Omega_R^-$.

Proof. For any $\xi \in \lambda\Omega_R^-$, by the definition of $\lambda\Omega_R^-$, we have $|\xi| \geq \lambda R$ and $|-\xi| \geq \lambda R$, and so $|\xi - (-\xi)| \geq 2\lambda R$. Following the same arguments as in the proofs of Lemmas 2.3 and 3.2 as well as Remark 2.6, it is easy to see that

$$c(\Omega_\lambda) \leq J_\lambda\left(\Phi_\lambda(\xi)^\pm\right) = \begin{cases} J_\lambda\left(t_\lambda \Psi_{\lambda R}(|x-\xi|)\right) \\ J_\lambda\left(-t_\lambda \Psi_{\lambda R}(|x+\xi|)\right) \end{cases}$$
$$= J_\lambda\left(t_\lambda \Psi_{\lambda R}(|x|)\right) \leq J_\lambda\left(\Psi_{\lambda R}(|x|)\right) = c(B_{\lambda R}).$$

Here we have used translation invariance of the Lebesgue integral the in second equality. From Lemma 3.4 we then deduce that

$$\lim_{\lambda\to+\infty} c(B_{\lambda R}) = \lim_{\lambda\to+\infty} c(\Omega_\lambda) = c\left(\mathbb{R}^N\right)$$

Hence the assertion of the lemma follows. □

Given $\xi \in \lambda\Omega_R^-$, we set

$$h(\lambda) := \left|J_\lambda\left(\Phi_\lambda(\xi)^\pm\right) - c\left(\mathbb{R}^N\right)\right|.$$

From Lemma 4.1 we conclude that $h(\lambda) \to 0$ as $\lambda \to +\infty$. We define the sublevel set

$$\widetilde{\mathcal{N}_\pm^\circ} = \left\{u \in \mathcal{N}_\pm^\circ \,:\, J_\lambda(u) \leqslant c\left(\mathbb{R}^N\right) + h(\lambda)\right\}.$$

It is clear that $\Phi_\lambda(\xi)^\pm \in \widetilde{\mathcal{N}_\pm^\circ}$ which implies $\widetilde{\mathcal{N}_\lambda^\circ} \neq \emptyset$ for any $\lambda > 0$.

For $u \in W^{1,p}(\mathbb{R}^N)$ with compact support in $B_{\tilde{R}}(0)$, we define the barycenter map

$$\beta_+ : W^{1,p}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N, \quad \beta_+(u) = \frac{\displaystyle\int_{\mathbb{R}^N} x|u^+(x)|^p \,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u^+(x)|^p \,\mathrm{d}x},$$
$$\beta_- : W^{1,p}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N, \quad \beta_-(u) = \frac{\displaystyle\int_{\mathbb{R}^N} x|u^-(x)|^p \,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u^-(x)|^p \,\mathrm{d}x}. \tag{4.1}$$

Proof of Theorem 1.1. From Lemmas 4.1 and 2.5 we know that

$$\lim_{\lambda\to+\infty} \hat{\Psi}\left(m^{-1}\left(\Phi_\lambda(\xi)^\pm\right)\right) = \lim_{\lambda\to+\infty} J_\lambda\left(\Phi_\lambda(\xi)^\pm\right) = c\left(\mathbb{R}^N\right)$$

uniformly in $\xi \in \lambda\Omega_R^-$. We set

$$\widetilde{\mathcal{S}_\pm^\circ} := \left\{ u \in \mathcal{S}_\pm^\circ \,:\, \hat{\Psi}(u) \le c\left(\mathbb{R}^N\right) + h(\lambda) \right\},$$

where h is given in the definition of $\widetilde{\mathcal{N}_\pm^\circ}$. It is clear that $\widetilde{\mathcal{S}_\pm^\circ} \ne \emptyset$ since $m_\pm^{-1}(\Phi_\lambda(\xi)^\pm) \in \widetilde{\mathcal{S}_\pm^\circ}$. From Lemma 3.1 and Krasnosel'skii's genus theory, see for example Ambrosetti-Malchiodi [6, Theorem 10.9], it follows that $\hat{\Psi}$ has at least $\gamma(\widetilde{\mathcal{S}_\pm^\circ})$ pairs of critical points on $\widetilde{\mathcal{S}_\pm^\circ}$.

We claim that $\gamma(\widetilde{\mathcal{S}_\pm^\circ}) \ge 2\gamma(\Omega_\lambda \setminus \{0\})$. Indeed, suppose that $\gamma(\widetilde{\mathcal{S}_\pm^\circ}) = 2n$. For a set A, we denote $A^* = \{(x,-x) \,:\, x \in A\}$. From Theorem 3.9 of Rabinowitz [24] it follows that

$$\gamma(\widetilde{\mathcal{S}_\pm^\circ}) = \operatorname{cat}_{\left(W_0^{1,p}(\Omega_\lambda)\setminus\{0\}\right)^*} \widetilde{\mathcal{S}_\pm^\circ}^*.$$

Therefore, there exists a smallest positive integer n such that

$$\widetilde{\mathcal{S}_\pm^\circ}^* \subseteq \mathcal{D}_{\pm 1}^* \cup \mathcal{D}_{\pm 2}^* \cup \cdots \cup \mathcal{D}_{\pm n}^*,$$

where $\mathcal{D}_{\pm i}^*$, $i = 1, 2, \cdots, n$ are closed and contractible in $(W_0^{1,p}(\Omega_\lambda) \setminus \{0\})^*$, that is, there exist

$$h_i^* \in C\left([0,1] \times \mathcal{D}_{\pm i}^*, \left(W_0^{1,p}(\Omega_\lambda) \setminus \{0\}\right)^*\right) \qquad \text{for } i = 1, 2, \cdots, n$$

such that

$$\begin{aligned} &h_i^*(0, u^\pm) = (u^\pm, -u^\pm) \quad \text{for all } (u^\pm, -u^\pm) \in \mathcal{D}_{\pm i}^*, \\ &h_i^*(1, u^\pm) = \left(\omega_i^\pm, -\omega_i^\pm\right) \in \left(W_0^{1,p}(\Omega_\lambda) \setminus \{0\}\right)^* \quad \text{for all } (u^\pm, -u^\pm) \in \mathcal{D}_{\pm i}^*. \end{aligned}$$

Here we have used the fact that $-u^\pm(x) = u^\mp(-x) \in \mathcal{D}_{\pm i}^*$.

Let

$$\mathcal{D}_i = \left\{ u^\pm \in W_0^{1,p}(\Omega_\lambda) \,:\, (u^\pm, -u^\pm) \in \mathcal{D}_i^* \right\}.$$

Then there exists a homotopy

$$h_i \in C\left([0,1] \times \mathcal{D}_i, \left(W_0^{1,p}(\Omega_\lambda) \setminus \{0\}\right)\right)$$

such that $h_i(0,\cdot) = \mathrm{id}$, $h_i(1,\cdot) = \omega_i^\pm$ or $-\omega_i^\pm$ and $h_i(t,u^\pm) = -h_i(t,-u^\pm)$.

We define $\Phi_\lambda^* = (\Phi_\lambda^\pm, -\Phi_\lambda^\pm)\colon \left(\lambda\Omega_R^-\right)^* \to \left(\mathcal{N}_\pm^\circ\right)^*$ by

$$\left[\Phi_\lambda^*(\xi,-\xi)\right](x) = \left(\left[\Phi_\lambda^\pm(\xi)\right](x), -\left[\Phi_\lambda^\pm(\xi)\right](x)\right) = \left(\left[\Phi_\lambda(\xi)^\pm\right](x), \left[\Phi_\lambda(-\xi)^\mp\right](x)\right).$$

Note that for any $(\xi,-\xi) \in \left(\lambda\Omega_R^-\right)^*$ we have

$$\beta_\pm\left(\Phi_\lambda(\xi)^\pm\right) = \xi \quad \text{and} \quad \beta_\mp\left(\Phi_\lambda(-\xi)^\mp\right) = -\xi,$$

that is,

$$\beta^*\left(\Phi_\lambda(\xi)^{\pm}, -\Phi_\lambda(\xi)^{\pm}\right) = \left(\beta_{\pm}\left(\Phi_\lambda(\xi)^{\pm}\right), \beta_{\mp}\left(\Phi_\lambda(-\xi)^{\mp}\right)\right) = (\xi, -\xi),$$

where $\beta^*(\cdot,\cdot) = (\beta_{\pm}(\cdot), \beta_{\mp}(\cdot))$ and $\beta_{\pm}$ is given in (4.1). We set

$$\mathcal{K}_{\pm i}^* = (\Phi_\lambda^*)^{-1}\left(m^*\left(\mathcal{D}_{\pm i}^*\right)\right),$$

where $m^*(\cdot,\cdot) = (m_{\pm}(\cdot), m_{\pm}(\cdot))$. It is clear that $\mathcal{K}_{\pm i}^*$ are closed subsets of $\left(\lambda\Omega_R^- \setminus \{0\}\right)^*$ and $\left(\lambda\Omega_R^- \setminus \{0\}\right)^* \subseteq \mathcal{K}_{\pm 1}^* \cup \cdots \cup \mathcal{K}_{\pm n}^*$. Moreover, for $i = 1, \ldots, n$, $\mathcal{K}_{\pm i}^*$ is contractible in $\left(\mathbb{R}^N \setminus \{0\}\right)^*$ by using the deformation $\mathfrak{h}_i \colon [0,1] \times \mathcal{K}_{\pm i}^* \to \left(\mathbb{R}^N \setminus \{0\}\right)^*$ defined by

$$\mathfrak{h}_i(t,x) = (\beta^* \circ h_i^*)\left(t, (m^*)^{-1}\left(\Phi_\lambda^*(\xi,-\xi)\right)\right).$$

From Lemma 4.1 and the definition of $\beta^{\pm}$ we conclude that

$$\begin{aligned}
&\mathfrak{h}_i \in C\left([0,1] \times \mathcal{K}_{\pm i}^*, \left(\mathbb{R}^N \setminus \{0\}\right)^*\right),\\
&\mathfrak{h}_i(0,x) = (\beta^* \circ h_i^*)\left(0, (m^*)^{-1}\left(\Phi_\lambda^*(\xi,-\xi)\right)\right) = (\xi,-\xi) \quad \text{for all } (\xi,-\xi) \in \mathcal{K}_{\pm i}^*,\\
&\mathfrak{h}_i(1,x) = (\beta^* \circ h_i^*)\left(1, (m^*)^{-1}\left(\Phi_\lambda^*(\xi,-\xi)\right)\right)\\
&\qquad = \beta^*\left(\omega_i^{\pm}, -\omega_i^{\pm}\right) = \left(\xi_i^0, -\xi_i^0\right) \in \left(\mathbb{R}^N \setminus \{0\}\right)^* \quad \text{for all } (\xi,-\xi) \in \mathcal{K}_{\pm i}^*.
\end{aligned}$$

Hence

$$\gamma\left(\Omega_\lambda \setminus \{0\}\right) = \mathrm{cat}_{(\mathbb{R}^N \setminus \{0\})^*}\left(\Omega_\lambda \setminus \{0\}\right)^* = \mathrm{cat}_{(\mathbb{R}^N \setminus \{0\})^*}\left(\lambda\Omega_R^- \setminus \{0\}\right)^* \leq n,$$

which implies that $\widetilde{\mathcal{S}_{\pm}^{\circ}}$ contains at least $2\gamma(\Omega_\lambda \setminus \{0\})$ pairs of critical points of $\hat{\Psi}$. Thus we conclude from Lemma 2.5 that there exist at least $2\gamma(\Omega_\lambda \setminus \{0\})$ pairs $(u^{\pm}, -u^{\pm})$ of critical points of J_λ. It is clear that $u = u^+ + u^-$ is odd, and is also the critical point of J_λ, that is, problem (1.1) has at least $\gamma(\Omega_\lambda \setminus \{0\})$ pairs of odd solutions. □

Declaration of competing interest

The authors declare that they have no competing interests.

Data availability

No data was used for the research described in the article.

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