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# Multiplicity results for logarithmic double phase problems via Morse theory

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**Abstract**

In this paper, we study elliptic equations of the form

$$-\operatorname{div}\mathcal{L}(u) = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\operatorname{div}\mathcal{L}$ is the logarithmic double phase operator given by

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)\left(\log(e+|\nabla u|) + \frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\right),$$

$e$ is Euler's number, $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $1<p<N$, $p<q<q+\varkappa<p^*=\frac{Np}{N-p}$ with $\varkappa = e/(e+t_0)$, $t_0 = e\log(e+t_0)$ and $0 \leqslant \mu(\cdot) \in L^\infty(\Omega)$. Under mild assumptions on the nonlinearity $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ we prove multiplicity results for the problem above and get

two constant sign solutions and another third nontrivial solution. This third solution is obtained by using the theory of critical groups. As a result of independent interest, we show that every weak solution of the problem above is essentially bounded.

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## 1 | INTRODUCTION AND NOTATION

In recent years, double phase problems have been intensely studied. These problems usually involve an operator of the form

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right),$$

which is associated with the functional given by

$$u\to\int_\Omega\left(\frac{|\nabla u|^p}{p}+\mu(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x. \tag{1.1}$$

Such type of functionals appeared for the first time in the work of Zhikov [43] and are useful in the context of homogenization and elasticity theory. In this setting, the coefficient $\mu$ is associated with the geometry of composites made of two materials of hardness $p$ and $q$. Functionals of the form (1.1) can be seen as special cases of the pioneering works by Marcellini [30, 31] which deal with problems with nonstandard growth and $p,q$-growth conditions. Indeed, the regularity theory in [30] applies to double phase integrals of the form (1.1) as well, see also the more recent papers by Cupini–Marcellini–Mascolo [16] and Marcellini [28, 29]. Later, the results of Marcellini in the setting of double phase integrals have been improved by the groundbreaking papers by Baroni–Colombo–Mingione [7–9] and Colombo–Mingione [13, 14]. We also point out that double phase problems describe several interesting applications, see the works by Bahrouni–Rădulescu–Repovš [6] on transonic flows, Benci–D'Avenia–Fortunato–Pisani [10] on quantum physics, Cherfils–Il'yasov [11] for reaction diffusion systems and Zhikov [44, 45] on the Lavrentiev gap phenomenon, the thermistor problem, and the duality theory. For the main properties of the related function space and the double phase operator, we refer to the papers by Colasuonno–Squassina [12], Crespo-Blanco–Gasiński–Harjulehto–Winkert [15], Ho–Winkert [25], Liu–Dai [26], and Perera–Squassina [36].

Recently, Arora–Crespo-Blanco–Winkert [5] studied the properties of the functional

$$u\to\int_\Omega(|\nabla u|^p+\mu(x)|\nabla u|^q\log(e+|\nabla u|))dx, \tag{1.2}$$

and the related so-called logarithmic double phase operator

$$\operatorname{div}\mathcal{L}(u)=\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u +\mu(x)\left(\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right)|\nabla u|^{q-2}\nabla u\Big), \tag{1.3}$$

where $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and

$$\mathcal{H}_{\log}(x,t)=t^p+\mu(x)t^q\log(e+t)\quad \text{for all } (x,t)\in\overline{\Omega}\times[0,\infty),$$

for $1<p<N$, $p<q$ and $0\leqslant\mu(\cdot)\in L^\infty(\Omega)$. Functionals of the form (1.2) have been studied for special cases in several works. Baroni–Colombo–Mingione [8] studied (1.2) in case $p=q$, that is,

$$u\mapsto\int_\Omega[|\nabla u|^p+\mu(x)|\nabla u|^p\log(e+|\nabla u|)]\,dx, \tag{1.4}$$

and proved local Hölder continuity of the gradient of local minimizers of (1.4) whenever $1<p<\infty$ and $0\leqslant\mu(\cdot)\in C^{0,\alpha}(\overline{\Omega})$. In a recent work by De Filippis–Mingione [17], the local Hölder continuity of the gradients of local minimizers of the functional

$$u\mapsto\int_\Omega[|\nabla u|\log(1+|\nabla u|)+\mu(x)|\nabla u|^q]\,dx, \tag{1.5}$$

has been shown provided $0\leqslant\mu(\cdot)\in C^{0,\alpha}(\overline{\Omega})$ and $1<q<1+\frac{\alpha}{n}$. Functionals of the shape (1.5) have their origin in functionals with nearly linear growth of the form

$$u\mapsto\int_\Omega|\nabla u|\log(1+|\nabla u|)\,dx, \tag{1.6}$$

see the works by Fuchs–Mingione [20] and Marcellini–Papi [32]. Note that (1.6) appears in the theory of plasticity with logarithmic hardening, see, for example, Seregin–Frehse [37] and the monograph by Fuchs–Seregin [21]. In this direction, we also mention the functional

$$u\mapsto\int_\Omega(1+|\nabla u|^2)^{\frac{p}{2}}\log(1+|\nabla u|)\,dx,$$

which has been studied by Marcellini [30].

In this paper, we are interested in the weak solvability of Dirichlet problems of the form

$$-\operatorname{div}\mathcal{L}(u)=f(x,u)\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega, \tag{1.7}$$

where $\operatorname{div}\mathcal{L}$ is the logarithmic double phase operator given in (1.3) while $\Omega\subset\mathbb{R}^N$, $N\geqslant 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$. Throughout this paper, we denote by $\varkappa$ the constant given by

$$\varkappa=\frac{e}{e+t_0}, \tag{1.8}$$

where $e$ is Euler's number and $t_0$ is the positive number that satisfies $t_0=e\log(e+t_0)$. We suppose the following hypotheses on the data:

(H$_1$) $1 < p < N$, $p < q < q + \kappa < p^*$ and $0 \leqslant \mu(\cdot) \in L^\infty(\Omega)$

(H$_2$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(x,0) = 0$ for a.a. $x \in \Omega$ and $F(x,s) = \int_0^s f(x,t)\,dt$ such that the following holds:

(i) there exist $\eta \in (q+\kappa, p^*)$ and $C > 0$ such that

$$|f(x,s)| \leqslant C(1 + |s|^{\eta-1}) \tag{1.9}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$;

(ii) there exist $\tau > \kappa$ and $c > 0$ such that

$$(q+\tau)F(x,s) - f(x,s)s \leqslant c, \tag{1.10}$$

$$(q+\kappa)F(x,s) - f(x,s)s \leqslant c \tag{1.11}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$;

(iii)

$$\limsup_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = 0 \quad \text{uniformly for a.a. } x \in \Omega; \tag{1.12}$$

(iv)

$$\lim_{s\to\pm\infty} \frac{F(x,s)}{|s|^q \log(e + |s|)} = \infty \quad \text{uniformly for a.a. } x \in \Omega; \tag{1.13}$$

(v) for all intervals $I \subset \mathbb{R}$ there exists $C_I > 0$ such that

$$|F(x,s) - F(x,t)| \leqslant C_I |s - t| \tag{1.14}$$

for a.a. $x \in \Omega$ and for all $s, t \in I$ and that there exists $0 < \beta < \min\{1, p^* - 1\}$ such that

$$|f(x,s) - f(x,t)| \leqslant C_I |s - t|^\beta \tag{1.15}$$

for a.a. $x \in \Omega$ and for all $s, t \in I$.

*Remark* 1.1. There are examples of $f$ satisfying (1.9) and (1.10). Let us test the polynomial functions of the form

$$f(s) = \begin{cases} |s|^\alpha & \text{for } s \geqslant 0 \\ -|s|^\alpha & \text{for } s < 0 \end{cases}$$

with certain exponent $\alpha < p^* - 1$. Observe that

$$(q+\tau)F(x,s) - f(x,s)s = \frac{q+\tau}{1+\alpha}|s|^{1+\alpha} - |s|^{1+\alpha} = \left(\frac{q+\tau}{1+\alpha} - 1\right)|s|^{1+\alpha},$$

for all $s \in \mathbb{R}$. Therefore, condition (1.10) holds provided

$$\frac{q+\tau}{1+\alpha} - 1 \leqslant 0 \quad \text{if and only if} \quad 1 + \alpha \geqslant q + \tau.$$

Thus, if we take $\varkappa < \tau \leqslant 1 + \alpha - q$, then condition (1.10) is fulfilled. In order words, $\alpha$ must be strictly greater than $q - 1 + \varkappa$. Since $\alpha < p^* - 1$, this yields the admissibility conditions $q + \varkappa < p^*$ and $q + \varkappa - 1 < \alpha < p^* - 1$. This corresponds more or less to the polynomial functions satisfying $(f_5)$ and $(f_2)$ in the work by Arora–Crespo-Blanco–Winkert [5], since there it is supposed that

$$\lim_{|s|\to\infty} \frac{(q+\tau)F(x,s) - f(x,s)|s|}{|s|^{1+\alpha}} \leqslant 0.$$

**Example 1.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(s) = \begin{cases} (\varkappa+\alpha)s^{\varkappa+\alpha-1}\log s + s^{\varkappa+\alpha-1} & \text{for } s \geqslant 0 \\ -(\varkappa+\alpha)|s|^{\varkappa+\alpha-1}\log|s| - |s|^{\varkappa+\alpha-1} & \text{for } s < 0, \end{cases}$$

with $q < \alpha$ and $\varkappa + \alpha < p^*$. This function satisfies the assumptions in $(H_2)$. Let us check (1.10). Here, we have

$$F(s) = |s|^{\varkappa+\alpha}\log|s| \quad \text{for all } s \in \mathbb{R}$$

and so

$$(q+\tau)F(s) - f(s)s = |s|^{\varkappa+\alpha}\log|s|(q+\tau-(\varkappa+\alpha)) - |s|^{\varkappa+\alpha}.$$

Consequently, condition (1.10) holds for $f$ if and only if $q + \tau < \varkappa + \alpha$. Thus, we need $\varkappa < \tau < \varkappa + \alpha - q$ which gives $q < \alpha$ as above.

Our main result is the following one.

**Theorem 1.3.** *Let hypotheses* $(H_1)$ *and* $(H_2)$ *be satisfied. Then, problem* (1.7) *admits three nontrivial distinct solutions* $u_0, v_0, y_0 \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$v_0 \leqslant 0 \leqslant u_0 \quad \text{in } \Omega,$$

*whereby* $u_0$ *and* $v_0$ *have positive energy.*

The proof of Theorem 1.3 is based on truncation and comparison techniques along with the mountain-pass geometry of problem (1.7). The third solution $y_0$ is obtained by using the Morse theory in terms of critical groups. Our result should be compared with the one in Arora–Crespo-Blanco–Winkert [5], where the authors obtain similar results with different conditions on $f$. In contrast to [5] we do not make any assumptions on the sign of $f$. Also, there is no assumption on the behavior of $f$ at infinity apart from the one in (1.13). Moreover, when comparing our results with the work by Papageorgiou–Qin [34], we point out that we do not require $\mu$ to be Lipschitz continuous, Hölder continuity suffices. Another feature of this work is the fact that we do not require $f$ to satisfy the Ambrosetti–Rabinowitz condition.

Since the operator (1.3) has been introduced very recently, only few works concerning existence results involving such logarithmic operator exist. The first work has been done by Arora–Crespo-Blanco–Winkert [5] who studied the problem

$$-\operatorname{div}\mathcal{L}(u) = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.16}$$

where $\mathcal{L}$ is as in (1.3) but with variable exponents and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth which satisfies appropriate conditions. Based on the Nehari manifold, the existence of a sign-changing solution of (1.16) has been shown under the more strict assumption that $q + 1 < p^*$, see also the recent work by the same authors [4] related to more general embeddings and existence results based on the concentration compactness principle. Furthermore, Lu–Vetro–Zeng [27] studied existence and uniqueness of equations involving the operator

$$u \mapsto \Delta_{\mathcal{H}_L} u = \operatorname{div}\left(\frac{\mathcal{H}_L'(x, |\nabla u|)}{|\nabla u|}\nabla u\right), \quad u \in W^{1,\mathcal{H}_L}(\Omega), \tag{1.17}$$

where $\mathcal{H}_L : \Omega \times \times [0, \infty) \to [0, \infty)$ is given by

$$\mathcal{H}_L(x, t) = [t^{p(x)} + \mu(x)t^{q(x)}] \log(e + \alpha t),$$

with $\alpha \geqslant 0$, see also Vetro–Zeng [42]. We point out that the operator (1.17) is different from the one in (1.3). Another work dealing with the logarithmic double phase operator has been published by Vetro–Winkert [41] who obtained the existence of a solution to the logarithmic problem with convection term of the form

$$-\operatorname{div} \mathcal{L}(u) = f(x, u, \nabla u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.18}$$

where $\mathcal{L}$ is as in (1.3) but with variable exponents and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying certain growth and coercivitiy conditions. The authors prove the boundedness, closedness, and compactness of the corresponding solution set to (1.18). We stress that the operator in Vetro–Winkert [41] is also involved in a nonlocal context by Vetro [40] who considered related Kirchhoff-type equations involving the logarithmic double phase operator as in (1.3) with variable exponents. We also mention some papers who study logarithmic terms on the right-hand side for Schrödinger equations or $p$-Laplace problems. Montenegro–de Queiroz [33] considered nonlinear elliptic problems

$$-\Delta u = \chi_{u>0}(\log(u) + \lambda f(x, u)) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.19}$$

where $f : \Omega \times [0, \infty) \to [0, \infty)$ is nondecreasing, sublinear and $f_u$ is continuous and proved that (1.19) has a maximal solution $u_\lambda \geqslant 0$ of type $C^{1,\gamma}(\overline{\Omega})$, see also Figueiredo–Montenegro–Stapenhorst [18, 19] where a similar problem was studied in planar domains with $f$ being of exponential growth. Squassina–Szulkin [39] studied logarithmic Schrödinger equations given by

$$-\Delta u + V(x)u = Q(x)u \log(u^2) \quad \text{in } \mathbb{R}^N \tag{1.20}$$

and proved that (1.20) has infinitely many solutions, whereby $V$ and $Q$ are 1-periodic functions of the variables $x_1, \dots, x_N$ and $Q \in C^1(\mathbb{R}^N)$. Further results for logarithmic Schrödinger equations can be found in the works of Alves–de Morais Filho [2], Alves–Ji [3], and Shuai [38], see also Alves–da Silva [1] about logarithmic Schrödinger equations on exterior domains.

As a result of independent interest, we prove the boundedness of weak solutions to more general equations than (1.7) of the form

$$-\operatorname{div} \mathcal{L}(u) = \mathcal{R}(x, u, \nabla u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.21}$$

where $\mathcal{R}: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function depending on the gradient of the solutions which may have critical growth with respect to the second argument. Finally, we also give some comments on parametric problems given by

$$-\operatorname{div} \mathcal{L}(u) = \lambda f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.22}$$

where div $\mathcal{L}$ is as in (1.3) and $\lambda > 0$. For $\lambda > 0$ large enough, (1.22) has at least two constant-sign solutions, whereby one is positive and the other one negative.

This work is structured as follows. In Section 2, we present some properties of the logarithmic Musielak–Orlicz Sobolev spaces and the related logarithmic double phase operator, while Section 3 is devoted to a priori bounds of equations of the form (1.21). In Section 4, we prove the existence of constant-sign solutions by showing the mountain-pass geometry of problem (1.7) and in Section 5 we use critical groups to show an additional nontrivial solution of (1.7).

## 2 | PRELIMINARIES

In this section, we recall some basic facts about logarithmic Musielak–Orlicz Sobolev spaces and the related logarithmic double phase operator given in (1.3). Most of the results are taken from the paper by Arora–Crespo-Blanco–Winkert [5]. To this end, we denote by $L^r(\Omega)$ the usual Lebesgue space with norm $\|\cdot\|_r$ for $1 \leqslant r \leqslant \infty$ while $W_0^{1,r}(\Omega)$ is the related Sobolev space with zero trace equipped with the equivalent norm $\|\nabla \cdot\|_r$ for $1 < r < \infty$. Suppose now hypothesis $(\mathrm{H}_1)$ and consider the map $\mathcal{H}_{\log}: \overline{\Omega} \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{H}_{\log}(x,t) = t^p + \mu(x) t^q \log(e+t).$$

Let $L^0(\Omega)$ be the space of all measurable functions on $\Omega$. We define

$$L^{\mathcal{H}_{\log}}(\Omega) = \left\{ u \in L^0(\Omega) : \rho_{\mathcal{H}_{\log}}(u) := \int_\Omega \mathcal{H}_{\log}(x,|u|)\,dx < \infty \right\},$$

where $\rho_{\mathcal{H}_{\log}}$ is the modular function corresponding to $\mathcal{H}_{\log}$. We equip $L^{\mathcal{H}_{\log}}(\Omega)$ with the Luxemburg norm $\|\cdot\|_{\mathcal{H}_{\log}}$ defined by

$$\|u\|_{\mathcal{H}_{\log}} := \inf \left\{ \lambda > 0 : \rho_{\mathcal{H}_{\log}}\left(\frac{u}{\lambda}\right) \leqslant 1 \right\} \quad \text{for } u \in L^{\mathcal{H}_{\log}}(\Omega).$$

With this norm, $L^{\mathcal{H}_{\log}}(\Omega)$ becomes a Banach space which is separable and reflexive. Next, we introduce the related Musielak–Orlicz Sobolev space given by

$$W^{1,\mathcal{H}_{\log}}(\Omega) = \left\{ u \in L^{\mathcal{H}_{\log}}(\Omega) : |\nabla u| \in L^{\mathcal{H}_{\log}}(\Omega) \right\},$$

and endow it with the norm

$$\|u\|_{1,\mathcal{H}_{\log}} := \|u\|_{\mathcal{H}_{\log}} + \|\nabla u\|_{\mathcal{H}_{\log}}.$$

Also, we set

$$W_0^{1,\mathcal{H}_{\log}}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{H}_{\log}}}.$$

Both $W^{1,\mathcal{H}_{\log}}(\Omega)$ and $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ are separable, reflexive Banach spaces. Moreover, on $W_0^{1,\mathcal{H}_{\log}}(\Omega)$, the Poincaré inequality holds, that is, we can find $c > 0$ such that

$$\|u\|_{\mathcal{H}_{\log}} \leqslant c\|\nabla u\|_{\mathcal{H}_{\log}} \quad \text{for } u \in W_0^{1,\mathcal{H}_{\log}}(\Omega).$$

Therefore, we can consider on $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ the equivalent norm $\|\cdot\|$ defined by

$$\|u\| = \|\nabla u\|_{\mathcal{H}_{\log}} \quad \text{for all } u \in W_0^{1,\mathcal{H}_{\log}}(\Omega).$$

We have the following embedding results, see Arora–Crespo-Blanco–Winkert [5, Proposition 3.7].

**Proposition 2.1.** *Let hypotheses* $(\mathrm{H}_1)$ *be satisfied. Then, the following holds:*

(i) $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ *is continuous;*
(ii) $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ *is continuous;*
(iii) $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{r}(\Omega)$ *is continuous and compact for all* $1 \leqslant r < p^*$.

Also, there is a close relation between the norm $\|\cdot\|$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ and the modular function $\rho_{\mathcal{H}_{\log}}$, see Arora–Crespo-Blanco–Winkert [5, Proposition 3.6].

**Proposition 2.2.** *Let hypotheses* $(\mathrm{H}_1)$ *be satisfied,* $\lambda > 0$, $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$, *and* $\kappa$ *as in (1.8). Then, the following holds:*

(i) $\|u\| = \lambda$ *if and only if* $\rho_{\mathcal{H}_{\log}}\left(\frac{\nabla u}{\lambda}\right) = 1$;
(ii) $\|u\| < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho_{\mathcal{H}_{\log}}(\nabla u) < 1$ *(resp.* $= 1, > 1$*);*
(iii) *if* $\|u\| < 1$ *then* $\|u\|^{q+\kappa} \leqslant \rho_{\mathcal{H}_{\log}}(\nabla u) \leqslant \|u\|^{p}$;
(iv) *if* $\|u\| > 1$ *then* $\|u\|^{p} \leqslant \rho_{\mathcal{H}_{\log}}(\nabla u) \leqslant \|u\|^{q+\kappa}$;
(v) $\|u_n\| \to 0$ *if and only if* $\rho_{\mathcal{H}_{\log}}(\nabla u_n) \to 0$ *as* $n \to \infty$.

Next, consider the nonlinear map $A : W_0^{1,\mathcal{H}_{\log}}(\Omega) \to \left(W_0^{1,\mathcal{H}_{\log}}(\Omega)\right)^*$ defined by

$$\begin{aligned} \langle A(u), v\rangle = &\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x \\ &+ \int_\Omega \mu(x)\left(\log(e + |\nabla u|) + \frac{|\nabla u|}{q(e + |\nabla u|)}\right)|\nabla u|^{q-2}\nabla u \cdot \nabla v \,\mathrm{d}x. \end{aligned} \tag{2.1}$$

This operator has the following properties, see Arora–Crespo-Blanco–Winkert [5, Theorem 4.4].

**Theorem 2.3.** *Let hypotheses* $(\mathrm{H}_1)$ *be satisfied and* $A$ *be given as in (2.1). Then,* $A$ *is bounded, continuous, strictly monotone, coercive, a homeomorphism and satisfies the* $(\mathrm{S}_+)$*-property, that is, any sequence* $\{u_n\}_{n\in\mathbb{N}}$ *in* $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ *and* $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$ *converges strongly to* $u$ *in* $W_0^{1,\mathcal{H}_{\log}}(\Omega)$.

Also from Arora–Crespo-Blanco–Winkert [5, Lemma 5.1], we know that if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(x, 0) = 0$ for a.a. $x \in \Omega$ satisfying $(H_2)$(i) then the functional

$$I_f(u) = \int_\Omega F(x,u)\,dx$$

and its derivative

$$\langle I_f'(u), v\rangle = \int_\Omega f(x,u)v\,dx,$$

are strongly continuous in the sense that if $u_n \rightharpoonup u$ weakly in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ then $I_f(u_n) \to I_f(u)$ in $\mathbb{R}$ and $I_f'(u_n) \to I_f'(u)$ in $\left(W_0^{1,\mathcal{H}_{\log}}(\Omega)\right)^*$.

Moreover, for $u \in L^0(\Omega)$, we define $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Then, we have

$$u = u^+ - u^-, |u| = u^+ + u^- \text{ and if } u \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \text{ then } u^\pm \in W_0^{1,\mathcal{H}_{\log}}(\Omega).$$

Next, consider the functionals

$$I(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e + |\nabla u|)\,dx - \int_\Omega F(x,u)\,dx,$$

$$I_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e + |\nabla u|)\,dx - \int_\Omega F(x,u^+)\,dx, \tag{2.2}$$

$$I_-(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e + |\nabla u|)\,dx - \int_\Omega F(x,-u^-)\,dx. \tag{2.3}$$

Then, we know that $I, I_+$ and $I_-$ are of class $C^1$ with derivatives

$$I'(u)(v) = \langle A(u), v\rangle - \int_\Omega f(x,u)v\,dx,$$

$$I_+'(u)(v) = \langle A(u), v\rangle - \int_\Omega f(x,u^+)v\,dx,$$

$$I_-'(u)(v) = \langle A(u), v\rangle - \int_\Omega f(x,-u^-)v\,dx,$$

where $A$ is given in (2.1), see Arora–Crespo-Blanco–Winkert [5, Theorem 4.1].

We recall some results from calculus of variations. Let $X$ be a Banach space. We say that a functional $\varphi : X \to \mathbb{R}$ satisfies the Cerami condition or C-condition if for every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and it also satisfies

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{as } n \to \infty,$$

then it contains a strongly convergent subsequence. Furthermore, we say that it satisfies the Cerami condition at the level $c \in \mathbb{R}$ or the $C_c$-condition if it holds for all the sequences such that $\varphi(u_n) \to c$ as $n \to \infty$ instead of for all the bounded sequences.

The proof of the following mountain-pass theorem can be found in the book by Papageorgiou–Rădulescu–Repovš [35, Theorem 5.4.6].

**Theorem 2.4** (Mountain-pass theorem). *Let $X$ be a Banach space and suppose $\varphi \in C^1(X)$, $u_0, u_1 \in X$ with $\|u_1 - u_0\| > \delta > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} \leqslant \inf\{\varphi(u) : \|u - u_0\| = \delta\} = m_\delta,$$

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \varphi(\gamma(t)) \text{ with } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$$

*and $\varphi$ satisfies the $\mathrm{C}_c$-condition. Then, $c \geqslant m_\delta$ and $c$ is a critical value of $\varphi$.*

Next, we recall some results from the theory of critical groups. For this purpose, if $Y_2 \subset Y_1 \subset X$ then by $H_k(Y_1, Y_2)$ we denote the $k$th singular homology group with integer coefficients for the pair $(Y_1, Y_2)$ with $k \in \mathbb{N}_0$. Let $\varphi \in C^1(X)$. Then, $K_\varphi$ is the critical set of $\varphi$, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

For $c \in \mathbb{R}$, we define

$$\varphi^c = \{u \in X : \varphi(u) \leqslant c\}.$$

Let $u \in K_\varphi$ be an isolated critical point with $\varphi(u) = c$. Then, the critical groups of $\varphi$ at $u$ are given by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where $U$ is a neighborhood of $u$ such that $\varphi^c \cap K_\varphi \cap U = \{u\}$. The excision property of singular homology implies that this definition is independent of the choice of the isolating neighborhood $U$. If $\varphi$ fulfills the C-condition and if $-\infty < \inf \varphi(K_\varphi)$ we define the critical groups of $\varphi$ at infinity by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0,$$

where $c \in \mathbb{R}$ is such that $c < \inf \varphi(K_\varphi)$. This definition is independent of the choice of $c$. Suppose that $K_\varphi$ is finite. Then, we define

$$M(t, u) = \sum_{k \in \mathbb{N}_0} \operatorname{rank} C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R} \text{ and for all } u \in K_\varphi,$$

$$P(t, \infty) = \sum_{k \in \mathbb{N}_0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.4}$$

where $Q(t) = \sum_{k \in \mathbb{N}_0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

## 3 | A PRIORI BOUNDS

In this section, we are going to prove that every weak solution of problems of type (1.7) is essentially bounded. We present the result for more general problems and study the equation

$$-\operatorname{div}\mathcal{L}(u)=\mathcal{R}(x,u,\nabla u)\quad \text{in } \Omega, \qquad u=0\quad \text{on } \partial\Omega, \tag{3.1}$$

where div $\mathcal{L}$ is the logarithmic double phase operator given in (1.3). A weak solution of Equation (3.1) is a function $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$\begin{aligned}&\int_\Omega\left(|\nabla u|^{p-2}\nabla u+\mu(x)\left[\log(e+|\nabla u|)+\frac{|\nabla u|}{q(e+|\nabla u|)}\right]|\nabla u|^{q-2}\nabla u\right)\cdot\nabla v\,dx\\&=\int_\Omega\mathcal{R}(x,u,\nabla u)v\,dx\end{aligned}$$

is fulfilled for all $v\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. We also include the critical case for problem (3.1) and suppose the following assumptions.

($\mathrm{H}_3$) $\mathcal{R}:\ \Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function and there exists $\ell\in(1,p^*]$ such that

$$|\mathcal{R}(x,t,\xi)|\leqslant b\left[|\xi|^{\frac{p}{\ell'}}+|t|^{\ell-1}+1\right],$$

for a.a. $x\in\Omega$, for all $t\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^N$ with a positive constant $b$.

We have the following result.

**Theorem 3.1.** *Let hypotheses* ($\mathrm{H}_1$) *and* ($\mathrm{H}_3$) *be satisfied and let* $u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ *be a weak solution of problem (3.1). Then,* $u\in L^\infty(\Omega)$.

*Proof.* From Proposition 2.1 (i) we know that $W_0^{1,\mathcal{H}_{\log}}(\Omega)\hookrightarrow W_0^{1,p}(\Omega)$ continuously. Since

$$\mathcal{L}(\xi)\cdot\xi\geqslant|\xi|^p$$

for all $\xi\in\mathbb{R}^N$, the result follows from Ho–Kim–Winkert–Zhang [24, Theorem 3.1]. □

## 4 | EXISTENCE OF TWO SOLUTIONS

In this section, we are going to prove that problem (1.7) admits two nontrivial bounded weak solutions with constant sign. To this end, we first show that the truncated functionals $I_+$ and $I_-$ given by (2.2) and (2.3) satisfy the Cerami condition. Before, we recall the following lemma, see Arora–Crespo-Blanco–Winkert [5, Lemma 5.4]

**Lemma 4.1.** *Let* $q>1$ *and*

$$h(t)=\frac{t}{q(e+t)\log(e+t)},\qquad t>0.$$

*Then,* $h$ *attains its maximum value at* $t_0$ *and its value is* $\frac{\kappa}{q}$*, where* $t_0$ *and* $\kappa$ *are given by (1.8).*

**Proposition 4.2.** *Let hypotheses* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$*(i)–(iv) be satisfied. Then, the functionals* $I_+$ *and* $I_-$ *satisfy the Cerami condition.*

*Proof.* We only show the assertion of the proposition for $I_+$, the proof for $I_-$ is very similar. To this end, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that

$$|I_+(u_n)| \leqslant c_1, \tag{4.1}$$

$$(1+\|u_n\|)I'_+(u_n) \to 0 \quad \text{in } \left(W_0^{1,\mathcal{H}_{\log}}(\Omega)\right)^*. \tag{4.2}$$

Relation (4.2) implies the existence of a sequence $\varepsilon_n \to 0$ such that

$$\begin{aligned}
&\bigg|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla v\,dx\\
&+\int_\Omega \mu(x)\left(\log(e+|\nabla u_n|)+\frac{|\nabla u_n|}{q(e+|\nabla u_n|)}\right)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla v\,dx\\
&-\int_\Omega f(x,u_n^+)v\,dx\bigg| \leqslant \frac{\varepsilon_n\|v\|}{1+\|u_n\|},
\end{aligned} \tag{4.3}$$

for all $n\in\mathbb{N}$ and for all $v\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Taking $v=-u_n^-\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ in (4.3) and using the fact that $f(x,u_n^+)u_n^-=0$ for a.a. $x\in\Omega$ (since $f(x,0)=0$ for a.a. $x\in\Omega$), we obtain

$$\begin{aligned}
&\rho_{\mathcal{H}_{\log}}(\nabla u_n^-)\\
&\leqslant \int_\Omega\left(|\nabla u_n^-|^p+\mu(x)\left[\log(e+|\nabla u_n^-|)+\frac{|\nabla u_n^-|}{q(e+|\nabla u_n^-|)}\right]|\nabla u_n^-|^q\right)dx\\
&\leqslant \varepsilon_n \quad \text{for all } n\in\mathbb{N},
\end{aligned}$$

because $\frac{|\nabla u_n^-|}{q(e+|\nabla u_n^-|)}>0$. From Proposition 2.2 (v), we then conclude that $u_n^-\to 0$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. We now prove that $u_n^+$ is bounded in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Choosing $v=u_n^+\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ as a test function in (4.3) gives

$$\begin{aligned}
&\int_\Omega f(x,u_n^+)u_n^+dx-\|\nabla u_n^+\|_p^p\\
&-\int_\Omega \mu(x)\left(\log(e+|\nabla u_n^+|)+\frac{|\nabla u_n^+|}{q(e+|\nabla u_n^+|)}\right)|\nabla u_n^+|^q dx\leqslant\varepsilon_n.
\end{aligned}$$

Applying Lemma 4.1 leads to

$$\begin{aligned}
&\int_\Omega f(x,u_n^+)u_n^+\,dx-\|\nabla u_n^+\|_p^p\\
&-\left(1+\frac{\kappa}{q}\right)\int_\Omega \mu(x)\log(e+|\nabla u_n^+|)|\nabla u_n^+|^q\,dx\leqslant\varepsilon_n.
\end{aligned} \tag{4.4}$$

On the other hand, (4.1) and the fact that $u_n^- \to 0$ in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ imply that

$$\frac{1}{p}\|\nabla u_n^+\|_p^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla u_n^+|^q \log(e+|\nabla u_n^+|)\,dx - \int_\Omega F(x,u_n^+)\,dx \leqslant c_2, \tag{4.5}$$

for some $c_2 > 0$, which implies

$$\begin{aligned}&\frac{q+\kappa}{p}\|\nabla u_n^+\|_p^p + \left(1+\frac{\kappa}{q}\right)\int_\Omega \mu(x)|\nabla u_n^+|^q \log(e+|\nabla u_n^+|)\,dx\\ &-\int_\Omega (q+\kappa)F(x,u_n^+)\,dx \leqslant (q+\kappa)c_2.\end{aligned}$$

Summing with (4.4), we get

$$\left(\frac{q+\kappa}{p}-1\right)\|\nabla u_n^+\|_p^p + \int_\Omega \big(f(x,u_n^+)u_n^+ - (q+\kappa)F(x,u_n^+)\big)\,dx \leqslant (q+\kappa)c_2 + \varepsilon_n.$$

Using (1.11), we obtain

$$\left(\frac{q+\kappa}{p}-1\right)\|\nabla u_n^+\|_p^p \leqslant (q+\kappa)c_2 + \varepsilon_n + c.$$

We conclude that $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. Going back to (4.5), we get

$$\frac{1}{q}\int_\Omega \mu(x)|\nabla u_n^+|^q \log(e+|\nabla u_n^+|)\,dx \leqslant c_3 + \int_\Omega F(x,u_n^+)\,dx$$

for some $c_3 > 0$. Using (1.10), one can estimate the right-hand side of this inequality, thus obtaining

$$\frac{1}{q}\int_\Omega \mu(x)|\nabla u_n^+|^q \log(e+|\nabla u_n^+|)\,dx \leqslant c_4 + \frac{1}{q+\tau}\int_\Omega f(x,u_n^+)u_n^+\,dx$$

for $c_4 > 0$. But from (4.4) and the fact that $\{u_n^+\}_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,p}(\Omega)$, we get, for $c_5 > 0$, that

$$\begin{aligned}&\frac{1}{q}\int_\Omega \mu(x)|\nabla u_n^+|^q \log(e+|\nabla u_n^+|)\,dx\\ &\leqslant c_5 + \left(\frac{q+\kappa}{q+\tau}\right)\frac{1}{q}\int_\Omega \mu(x)\log(e+|\nabla u_n^+|)|\nabla u_n^+|^q\,dx.\end{aligned}$$

Consequently,

$$\int_\Omega \mu(x)|\nabla u_n^+|^q \log(e+|\nabla u_n^+|)\,dx \leqslant c_6 \quad \text{for all } n\in\mathbb{N}.$$

since $\tau > \kappa$. We conclude that $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$, and hence one can find a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ and an element $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that $u_{n_k} \rightharpoonup u$ weakly. Choosing $u_{n_k} - u$ as a test function in (4.3) and letting $k \to \infty$, we arrive at

$$\lim_{k\to\infty} \langle I'_+(u_{n_k}), u_{n_k} - u\rangle = 0.$$

On the other hand, the strong continuity of $I_f$ (see Section 2) implies that

$$\lim_{k\to\infty} \int_\Omega f(x, u^+_{n_k})(u_{n_k} - u)\, dx = 0.$$

Taking these two limits together yields (see again Section 2)

$$\lim_{k\to\infty} \langle A(u_{n_k}), u_{n_k} - u\rangle = 0,$$

whereby the operator $A$ given by (2.1) satisfies the $(\mathrm{S}_+)$-property, see Theorem 2.3. Thus, $u_n \to u$ strongly in $W_0^{1,\mathcal{H}_{\log}}(\Omega)$. This proves that $I_+$ satisfies the Cerami condition. □

Next, we show that zero is a local minimizer for the functionals $I$ and $I_\pm$.

**Proposition 4.3.** *Let hypotheses* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$*(i)–(iv) be satisfied. Then, zero is a local minimizer of* $I$ *and* $I_\pm$.

*Proof.* We only prove it for $I$, the proofs are similar for $I_+$ and $I_-$. From (1.9) and (1.12), we know that for each $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(x,s)| \leqslant \frac{\varepsilon}{p}|s|^p + c_\varepsilon |s|^\eta \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{4.6}$$

In (4.6), we choose $\varepsilon = \frac{1}{2c_1}$ with $c_1 > 0$ being an embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. Then, from this and for $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ with $\|u\| \leqslant 1$ by using the embeddings $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ as well as $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^\eta(\Omega)$ (see Proposition 2.1(iii)) and Proposition 2.2 (iii), we have

$$\begin{aligned}
I(u) &\geqslant \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e + |\nabla u|)\, dx - \frac{\varepsilon}{p}\|u\|_p^p - c_\varepsilon \|u\|_\eta^\eta \\
&\geqslant \frac{1}{p}(1 - c_1\varepsilon)\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e + |\nabla u|)\, dx - c_2\|u\|^\eta \\
&\geqslant \min\left\{\frac{1 - c_1\varepsilon}{p}, \frac{1}{q}\right\}\rho_{\mathcal{H}_{\log}}(|\nabla u|) - c_2\|u\|^\eta \\
&\geqslant \frac{1}{2q}\|u\|^{q+\kappa} - c_2\|u\|^\eta,
\end{aligned} \tag{4.7}$$

for some $c_2 > 0$. Since $q + \kappa < \eta$ by $(\mathrm{H}_2)$ (i) the result of the proposition follows. □

Now, we study the energy level of the functionals $I_+$ and $I_-$.

**Proposition 4.4.** *Let hypotheses* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$*(i)–(iv) be satisfied. Then, it holds* $I_+(tu) \to -\infty$ *as* $t \to \infty$ *for all* $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\}$ *such that* $u \geqslant 0$ *a.e. in* $\Omega$*. Similarly,* $I_-(tu) \to -\infty$ *as* $t \to \infty$ *for all* $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\}$ *such that* $u \leqslant 0$ *a.e. in* $\Omega$, $u \not\equiv 0$.

*Proof.* We show the result only for $I_+$, it can be shown in a similar way for $I_-$. Let $M > 0$ be a positive number. From $(\mathrm{H}_2)$ (iv) (see (1.13)), we know that there exists a number $s_M > 0$ such that

$$F(x,s) \geqslant Ms^q \log(e+s) \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geqslant s_M.$$

On the other hand, from the continuity there exists $c_M > 0$ such that

$$|F(x,s)| \leqslant c_M \quad \text{for a.a. } x \in \Omega \text{ and for all } s \leqslant s_M.$$

Thus, we have

$$\begin{aligned} F(x,s) &\geqslant -c_M + Ms^q \log(e+s) - Ms^q \log(e+s) \\ &\geqslant Ms^q \log(e+s) - (c_M + Ms_M^q \log(e+s_M)) \\ &= Ms^q \log(e+s) - \widetilde{c}_M \quad \text{for a.a. } x \in \Omega \text{ and for all } 0 \leqslant s \leqslant s_M. \end{aligned}$$

Taking everything together, we conclude that

$$F(x,s) \geqslant Ms^q \log(e+s) - \widetilde{c}_M \quad \text{for a.a. } x \in \Omega \text{ and for all } s \geqslant 0.$$

Consequently, for $t > 0$, it follows that, for all $u \in W_0^{1,\mathcal{H}_{\log}}(\Omega) \setminus \{0\}$ such that $u \geqslant 0$ a.e. in $\Omega$,

$$\begin{aligned} I_+(tu) = \frac{t^p}{p}\|\nabla u\|_p^p &+ \frac{t^q}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e+t|\nabla u|)\,dx \\ &- Mt^q \int_\Omega u^q \log(e+tu)\,dx - \widetilde{c}_M|\Omega|. \end{aligned}$$

Since $\log(e+xy) \leqslant \log(e+x) + \log(e+y)$ for all $x, y > 0$, we have

$$\begin{aligned} I_+(tu) \leqslant \frac{t^p}{p}\|\nabla u\|_p^p &+ \frac{t^q \log(e+t)}{q}\int_\Omega \mu(x)|\nabla u|^q\,dx \\ &+ \frac{t^q}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e+|\nabla u|)\,dx \\ &- Mt^q \int_\Omega u^q \log(e+tu)\,dx - \widetilde{c}_M|\Omega|. \end{aligned} \tag{4.8}$$

From the inequality $e + tu \geqslant e + t$ for $t \geqslant 0$ and $u \geqslant 1$ along with the monotonicity of log, we derive that

$$\int_{\{x\in\Omega\,:\ u\geqslant 1\}} u^q \log(e+tu)\,dx \geqslant \log(e+t)\int_{\{x\in\Omega\,:\ u\geqslant 1\}} u^q\,dx$$

and

$$\begin{aligned} \int_{\{x\in\Omega\,:\ 0<u<1\}} u^q \log(e+tu)\,dx &= \int_{\{x\in\Omega\,:\ 0<u<1\}} u^{q+1}\frac{1}{u}\log(e+tu)\,dx \\ &\geqslant \log(e+t)\int_{\{x\in\Omega\,:\ 0<u<1\}} u^{q+1}\,dx, \end{aligned}$$

where in the last inequality we used that $C\log(e+s) \geqslant \log(e+Cs)$ for all $C>1$ and $s \geqslant 0$. Using this in (4.8) gives

$$\begin{aligned} &I_+(tu)\\ &\leqslant \frac{t^p}{p}\|\nabla u\|_p^p + t^q\log(e+t)\left(\frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q\,dx - M\int_{\{x\in\Omega\,:\,u\geqslant 1\}} u^q\,dx\right.\\ &\quad\left. -M\int_{\{x\in\Omega\,:\,0<u<1\}} u^{q+1}\,dx\right)\\ &\quad +\frac{t^q}{q}\int_\Omega \mu(x)|\nabla u|^q\log(e+|\nabla u|)\,dx - \widetilde{c}_M|\Omega|. \end{aligned} \tag{4.9}$$

Choosing $M>0$ such that

$$\frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q\,dx < M\left(\int_{\{x\in\Omega\,:\,u\geqslant 1\}} u^q\,dx + \int_{\{x\in\Omega\,:\,0<u<1\}} u^{q+1}\,dx\right)$$

yields that $\lim_{t\to\infty} I_+(tu) = -\infty$. This proves the result. □

Now, we can prove the existence of two constant sign solutions to problem (1.7).

**Proposition 4.5.** *Let hypotheses* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$*(i)–(iv) be satisfied. Then, problem (1.7) admits two nontrivial weak solutions* $u_0, v_0 \in W_0^{1,\mathcal{H}_{\log}}(\Omega)\cap L^\infty(\Omega)$ *such that* $v_0 \leqslant 0 \leqslant u_0$ *in* $\Omega$ *as well as* $I(u_0)>0$ *and* $I(v_0)>0$.

*Proof.* From Proposition 4.3, we know that zero is a local minimum of $I$ and that there exist $\rho, m > 0$ such that

$$I_+(u) \geqslant m \quad \text{for all } u\in W_0^{1,\mathcal{H}_{\log}}(\Omega) \text{ with } \|u\| = \rho.$$

Taking Proposition 4.4 and Theorem 2.4 into account, we can find an element $w \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ such that $\|w\|>\rho$ and $I(w)<0=I(0)$. Thus,

$$c := \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} I_+(\gamma(t)),$$

with

$$\Gamma = \left\{\gamma\in C\left([0,1], W_0^{1,\mathcal{H}_{\log}}(\Omega)\right)\,:\,\gamma(0)=0, \gamma(1)=w\right\},$$

is a critical point for $I_+$. Since $c>m$ and because critical points of $I_+$ are all nonnegative, we conclude that there exists a nonnegative weak solution $u_0\in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ of problem (1.7) such that $I(u_0)\geqslant m$. Similarly, we can show the assertion for $I_-$ getting a nonnegative weak solution $v_0 \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$. Using Theorem 3.1 gives the desired results. □

*Remark* 4.6. Note that condition $(\mathrm{H}_2)$(V) was not needed in the proof of Proposition 4.5.

## 5 | THIRD SOLUTION VIA CRITICAL GROUPS

In this section, we are going to prove the existence of a third nontrivial solutions by using tools from critical groups.

**Proposition 5.1.** *Let hypotheses* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *be satisfied. Then,* $C_k(I_\pm,\infty)=C(I,\infty)=0$ *for all* $k\in\mathbb{N}_0$.

*Proof.* We show the proof only for $I_+$, it works in a similar way for $I_-$ and $I$. Let $u\in\partial B_1^+ := \{u\in W_0^{1,\mathcal{H}_{\log}}(\Omega):\ \|u\|=1 \text{ and } u^+\not\equiv 0\}$. First, observe that, due to (4.9), we get

$$\lim_{t\to\infty}\frac{I_+(tu)}{t^q}=-\infty \quad \text{for all } u\in W_0^{1,\mathcal{H}_{\log}}(\Omega)\setminus\{0\}. \tag{5.1}$$

Furthermore, due to Lemma 4.1, we have

$$\begin{aligned}
&\frac{d}{dt}I_+(tu)=I_+'(tu)(u)=\frac{1}{t}I_+'(tu)(tu)\\
&=\frac{1}{t}\bigg(t^p\|\nabla u\|_p^p+t^q\int_\Omega\mu(x)\bigg(\log(e+t|\nabla u|)+\frac{t|\nabla u|}{q(e+t|\nabla u|)}\bigg)|\nabla u|^q\,dx\\
&\qquad-\int_\Omega f(x,tu^+)tu\,dx\bigg)\\
&\leqslant\frac{1}{t}\bigg(t^p\|\nabla u\|_p^p+t^q\bigg(1+\frac{\kappa}{q}\bigg)\int_\Omega\mu(x)\log(e+t|\nabla u|)|\nabla u|^q\,dx-\int_\Omega f(x,tu^+)tu\,dx\bigg)\\
&\leqslant\frac{1}{t}\bigg(\bigg(1+\frac{\kappa}{q}\bigg)\bigg(t^p\|\nabla u\|_p^p+t^q\int_\Omega\mu(x)\log(e+t|\nabla u|)|\nabla u|^q\,dx\bigg)\\
&\qquad-\int_\Omega f(x,tu^+)tu\,dx\bigg)\\
&\leqslant\frac{1}{t}\bigg(\bigg(1+\frac{\kappa}{q}\bigg)\bigg(qI_+(tu)+\int_\Omega qF(x,tu^+)\,dx\bigg)-\int_\Omega f(x,tu^+)tu\,dx\bigg)\\
&=\frac{1}{t}\bigg(\bigg(1+\frac{\kappa}{q}\bigg)qI_+(tu)+\int_\Omega\bigg(q\bigg(1+\frac{\kappa}{q}\bigg)F(x,tu^+)-f(x,tu^+)tu\bigg)\,dx\bigg)\\
&=\frac{1}{t}\bigg((q+\kappa)I_+(tu)+\int_\Omega\big((q+\kappa)F(x,tu^+)-f(x,tu^+)tu^+\big)\,dx\bigg),
\end{aligned}$$

where in the last line we used the fact that $f(x,0)=0$ for a.a $x\in\Omega$. Consequently, using (1.11) in $(\mathrm{H}_2)$(ii), we get

$$\frac{d}{dt}I_+(tu)\leqslant\frac{1}{t}\big((q+\kappa)I_+(tu)+c|\Omega|\big).$$

We conclude that

$$\text{if } I_+(tu)<-\frac{c|\Omega|}{q+\kappa}:=-\nu_0,\ \text{then } \frac{d}{dt}I_+(tu)<0. \tag{5.2}$$

Now, let $\nu$ be a positive number such that $-\nu < \min\{-\nu_0, \inf_{\partial B_1^+} I_+\}$. From (5.1) we know that for all $u \in \partial B_1^+$, there exists a number $t_u > 1$ such that $I_+(t_u u) = -\nu$. Property (5.2) implies that $t_u$ is unique. Thus, the function $\eta : \partial B_1^+ \to \mathbb{R}$ given by $\eta(u) = t_u$ is well defined and satisfies $I_+(\eta(u)u) = -\nu$ for all $u \in \partial B_1^+$. Moreover, the implicit function theorem implies that $\eta$ is continuous.

Let $E^+ = \{tu : t \geqslant 1, u \in \partial B_1^+\}$. We may extend $\eta$ to $E^+$ by setting

$$\eta_0(u) = \frac{1}{\|u\|}\eta\left(\frac{u}{\|u\|}\right).$$

Clearly, $\eta_0 \in C(E^+)$. Furthermore,

$$I_+(\eta_0(u)u) = I_+\left(\eta\left(\frac{u}{\|u\|}\right)\frac{u}{\|u\|}\right) = -\nu \quad \text{for all } u \in E^+.$$

Moreover, if $I_+(u) = -\nu$, then $\eta(\frac{u}{\|u\|}) = \|u\|$ and thus $\eta_0(u) = 1$ by using (5.2). Consequently, if we define $h : [0,1] \times E^+ \to E^+$ by $h(t,u) = (1-t)u + t\eta_0(u)u$, we get

$$h(0,u) = u, \quad h(1,u) = \eta_0(u)u \in I_+^\mu \quad \text{for all } u \in E^+$$

and

$$h(t,\cdot)\big|_{I_+^\nu} = \mathrm{id}\big|_{I_+^\nu}.$$

Thus, $I_+^\nu$ is a strong deformation retract of $E^+$. Using the radial retraction, we obtain that $\partial B_1^+$ is a deformation retract of $E^+$, and thus $I_+^\nu$ and $\partial B_1^+$ are homotopically equivalent. Due to Corollary 6.1.24 by Papageorgiou–Rădulescu–Repovš [35, p 468], we conclude that

$$H_k\left(W_0^{1,\mathcal{H}_{\log}}(\Omega), I_+^\nu\right) = H_k\left(W_0^{1,\mathcal{H}_{\log}}(\Omega), \partial B_1^+\right) \quad \text{for all } k \in \mathbb{N}_0.$$

Since $W_0^{1,\mathcal{H}_{\log}}(\Omega)$ is infinite dimensional, we know that $\partial B_1^+$ is contractible in itself. Thus, by Granas–Dugundji [23, p. 389], we know that

$$H_k\left(W_0^{1,\mathcal{H}_{\log}}(\Omega), \partial B_1^+\right) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Hence, if $|\nu|$ is large enough, we get

$$C_k(I_+, \infty) = H_k\left(W_0^{1,\mathcal{H}_{\log}}(\Omega), I_+^\nu\right) = 0 \quad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

**Proposition 5.2.** *Let hypotheses* (H$_1$) *and* (H$_2$) *be satisfied. Then, we have*

$$C_k(I, u_0) = C_k(I_+, u_0) \quad \textit{for all } k \in \mathbb{N}_0.$$

*Proof.* Let $M > \|u_0\|_{L^\infty(\Omega)}$, see Theorem 3.1 and consider the following truncation of $f(x,\cdot)$:

$$f^M(x,s) = \begin{cases} f(x,-M) & \text{if } s < -M, \\ f(x,s) & \text{if } -M \leqslant s \leqslant M \\ f(x,M) & \text{if } s > M. \end{cases}$$

Let the positive truncation of $f^M(x\cdot)$ be the function

$$f_+^M(x,s) := f^M(x,s^+).$$

We set $F^M(x,s) = \int_0^s f^M(x,t)\,dt$ and $F_+^M(x,s) = \int_0^s f_+^M(x,t)\,dt$ and consider the $C^1$-functionals $I^{\sim M}$ and $I_+^{\sim M}$ defined by

$$I^{\sim M}(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e+|\nabla u|)\,dx - \int_\Omega F^M(x,u)\,dx,$$
$$I_+^{\sim M}(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \mu(x)|\nabla u|^q \log(e+|\nabla u|)\,dx - \int_\Omega F_+^M(x,u)\,dx.$$

Since $F$ satisfies (1.14) in hypothesis $(\mathrm{H}_2)$ (v), we know that there exists a global constant $C > 0$ such that

$$|F^M(x,s) - F^M(x,t)| \leqslant C|s-t| \quad \text{and} \quad |F_+^M(x,s) - F_+^M(x,t)| \leqslant C|s-t| \tag{5.3}$$

for all $s,t \in \mathbb{R}$. Using (5.3) and the embedding $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^1(\Omega)$, we have

$$\begin{aligned}
&|I^{\sim M}(u) - I_+^{\sim M}(u)|\\
&\leqslant \int_\Omega |F^M(x,u) - F_+^M(x,u)|\,dx\\
&\leqslant \int_\Omega |F^M(x,u) - F^M(x,u_0)|\,dx + \int_\Omega |F^M(x,u_0) - F_+^M(x,u)|\,dx\\
&= \int_\Omega |F^M(x,u) - F^M(x,u_0)|\,dx + \int_\Omega |F_+^M(x,u_0) - F_+^M(x,u)|\,dx\\
&\leqslant 2C\int_\Omega |u-u_0|\,dx\\
&\leqslant \widetilde{C}\|u-u_0\|,
\end{aligned}$$

with $\widetilde{C} > 0$.

On the other hand, since $f$ satisfies (1.15), we know that $f^M$ and $f_+^M$ also satisfy (1.15) with a global constant $C$. Thus, using Hölder's inequality and the embeddings $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ as well as $W_0^{1,\mathcal{H}_{\log}}(\Omega) \hookrightarrow L^{\frac{p^*}{p^*-\beta}}(\Omega)$ (see Proposition 2.1 (ii), (iii)), we get for $h \in W_0^{1,\mathcal{H}_{\log}}(\Omega)$ that

$$\begin{aligned}
&|\langle (I^{\sim M})'(u) - (I_+^{\sim M})'(u), h\rangle|\\
&\leqslant \int_\Omega |f^M(x,u) - f_+^M(x,u)||h|\,dx\\
&\leqslant \int_\Omega |f^M(x,u) - f^M(x,u_0)||h|\,dx + \int_\Omega |f_+^M(x,u_0) - f^M(x,u)||h|\,dx\\
&\leqslant 2C\int_\Omega |u-u_0|^\beta |h|\,dx
\end{aligned}$$

$$\leqslant 2C\,\|u-u_0\|_{p^*}^{\beta}\,\|h\|_{\frac{p^*}{p^*-\beta}}$$

$$\leqslant \widetilde{C}\|u-u_0\|^{\beta}\,\|h\|,$$

because $\frac{p^*}{p^*-\beta} < p^*$, since $\beta < p^*-1$. Consequently, given $\varepsilon > 0$, it is possible to find $\delta > 0$ such that

$$\|I^{\sim M} - I_+^{\sim M}\|_{C^1(\overline{B}_\delta(u_0))} < \varepsilon.$$

Using the $C^1$-continuity property of critical groups, see Gasiński–Papageorgiou [22, Theorem 5.126], we have

$$C_k(I^{\sim M}, u_0) = C_k(I_+^{\sim M}, u_0) \quad \text{for all } k \in \mathbb{N}_0.$$

Thus, we may let $M \to \infty$ and use Granas–Dugundji [23, Theorem D.6, p. 615] to conclude that

$$C_k(I, u_0) = C_k(I_+, u_0) \qquad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

**Proposition 5.3.** *Let hypotheses* ($H_1$) *and* ($H_2$) *be satisfied. Then, we have*

$$C_k(I_+, u_0) = \delta_{k,1}\mathbb{Z} \quad \textit{for all } k \in \mathbb{N}_0.$$

*Proof.* Assume that $K_{I_+} = \{0, u_0\}$. Otherwise we would have already had a third solution. From Proposition 4.3 and (4.7), we can find $\rho_+ > 0$ such that

$$m_+ := \inf\left\{I_+(u) : \|u\| = \rho_+\right\} > 0.$$

Let $\nu_-$ and $\nu_+$ be constants such that

$$\nu_- < 0 < \nu_+ < m_+ \leqslant I_+(u_0).$$

We have

$$I_+^{\nu_-} \subset I_+^{\nu_+} \subset W_0^{1,\mathcal{H}_{\log}}(\Omega) = X.$$

Let $i$ be the embedding of the map $I_+^{\nu_-}$ into $I_+^{\nu_+}$, and consider the corresponding long exact sequence of singular homology groups, see, for example, Papageorgiou–Rădulescu–Repovš [35, Proposition 6.1.23 p. 466],

$$\cdots \longrightarrow H_k(X, I_+^{\nu_-}) \xrightarrow{i_*} H_k(X, I_+^{\nu_+}) \xrightarrow{\partial_*} H_{k-1}(I_+^{\nu_+}, I_+^{\nu_-}) \longrightarrow \cdots, \tag{5.4}$$

where $i^*$ is the group homomorphism related to the embedding $i$ and $\partial_*$ is the boundary homomorphism, see Papageorgiou–Rădulescu–Repovš [35] for more details. Since $K_{I_+} = \{0, u_0\}$ and using Proposition 5.1, we get

$$H_k\left(X, I_+^{\nu_-}\right) = C_k(I_+, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

On the other hand, since

$$\nu_- < 0 = I_+(0) < \nu_+ < m_+ \leqslant I_+(u_0),$$

it follows that

$$H_k(X, I_+^{\nu_+}) = C_k(I_+, u_0) \quad \text{for all } k \in \mathbb{N}_0, \tag{5.5}$$

and (see Papageorgiou–Rădulescu–Repovš [35, Proposition 6.2.16, p. 486]).

$$H_{k-1}(I_+^{\nu_+}, I_+^{\nu_-}) = C_{k-1}(I_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{5.6}$$

In the last sequence of equalities, we used the fact that $\delta_{k-1,0}\mathbb{Z}$ is the $k-1$th critical group of $I_+$ at zero whenever zero is a local minimum, see Papageorgiou–Rădulescu–Repovš [35, Proposition 6.2.3, p. 477]. Furthermore, we know that $u_0$ is a critical point of mountain-pass type, and thus (see Papageorgiou–Rădulescu–Repovš [35, Theorem 6.5.8, p. 527])

$$C_1(I_+, u_0) \neq 0.$$

From this and (5.5), we conclude that

$$H_1(X, I_+^{\nu_+}) \neq 0.$$

However, using the exactness of the sequence (5.4), leads to

$$\begin{aligned} \operatorname{rank} H_1(X, I_+^{\nu_+}) &= \operatorname{rank} \ker \partial_* + \operatorname{rank} \operatorname{im} \partial_* \\ &= \operatorname{rank} \operatorname{im} i_* + \operatorname{rank} \operatorname{im} \partial_* \leqslant 1, \end{aligned}$$

because $H_1(X, I_+^{\nu_-}) = 0$. Thus $\operatorname{im} i_* = \{0\}$ and because $H_0(I_+^{\nu_+}, I_+^{\nu_-}) = \mathbb{Z}$, see (5.6). We thus obtain

$$C_1(I_+, u_0) = H_1(X, I_+^{\nu_+}) = \mathbb{Z}.$$

On the other hand, for $k > 1$, we know that

$$H_k(I_+^{\nu_+}, I_+^{\nu_-}) = 0 \quad \text{and} \quad H_k(X, I_+^{\nu_-}) = 0.$$

Consequently, the exactness of the sequence yields

$$H_k(X, I_+^{\nu_+}) = 0.$$

We have thus shown that $C_k(X, I_+^{\nu_+}) = \delta_{k,1}\mathbb{Z}$. This proves the result. □

We have analogously results for the functional $I_-$ and $v_0$.

**Proposition 5.4.** *Let hypotheses* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *be satisfied. Then, we have*

$$C_k(I, v_0) = C_k(I_-, v_0) \quad \textit{for all } k \in \mathbb{N}_0,$$

*and*

$$C_k(I_-, v_0) = \delta_{k,1}\mathbb{Z} \quad \textit{for all } k \in \mathbb{N}_0.$$

*Proof.* The proof is similar to the proofs of Propositions 5.2 and 5.3. □

**Proposition 5.5.** *Let hypotheses* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *be satisfied. Then, problem (1.7) has three nontrivial bounded weak solutions* $v_0, y_0$ *and* $u_0$ *such that* $v_0 \leqslant 0 \leqslant u_0$.

*Proof.* From Proposition 4.5, we know that $v_0$ and $u_0$ are bounded nontrivial weak solutions of (1.7) such that $v_0 \leqslant 0 \leqslant u_0$. Suppose that $K_I = \{0, v_0, u_0\}$. Since zero is a local minimum, we know that

$$C_k(I, 0) = \delta_{k,0}\mathbb{Z}.$$

From Proposition 5.1, we also know that

$$C_k(I, \infty) = 0.$$

Moreover, Propositions 5.2, 5.3, and 5.4 yield that

$$C_k(I, u_0) = \delta_{k,1}\mathbb{Z} = C_k(I, v_0).$$

Then, from the Morse relation stated in (2.4), we conclude that

$$(-1)^0 + 2(-1)^0 = 0,$$

which is a contradiction. Hence, there exists $y_0 \in K_I$ such that $y_0 \notin \{0, u_0, v_0\}$. This proves the result. □

Theorem 1.3 follows now from Propositions 4.5 and 5.5.

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