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On (p,n)-Laplace Schrödinger equations with Stein-Weiss convolution parts in $\mathbb{R}^n$

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ABSTRACT

By using the mountain pass theorem, this article deals with the existence of positive ground state solutions to a class of (p,n)-Laplace Schrödinger equations with Stein-Weiss reaction under critical exponential growth in the sense of the Moser–Trudinger inequality in the whole $\mathbb{R}^n$.

1. Introduction and main result

Nowadays, there is a great interest in the study of nonlinear PDEs involving the (p,q)-Laplace operator, see [1–3], due to several relevant physical applications in the field of applied sciences such as physics, plasma physics, chemical reaction design, electromagnetism, electrostatics and electrodynamics, see [4–6]. Such problems are both fascinating and difficult to deal with. From an analytical point of view, there are several technical difficulties, such as the inhomogeneous nature of the (p,q)-Laplacian and the lack of compactness of Palais–Smale sequences due to the unboundedness of the domain. Concerning problems driven by the (p,n)-Laplacian, we refer to the papers [7–10], see also the references therein. Moreover, we also mention [11–13] for the study of existence and multiplicity of solutions of Choquard equations with critical exponential growth in the whole Euclidean space.

Motivated by the above mentioned works, we study the following (p,n)-Laplace Schrödinger-Choquard type equation

$$\mathcal{L}_{p,V}(u)+\mathcal{L}_{n,V}(u)=\left(\int_{\mathbb{R}^n}\frac{F(y,u)}{|x-y|^{\mu}|y|^{\beta}}\,\mathrm{d}y\right)\frac{f(x,u)}{|x|^{\beta}}\quad\text{in } \mathbb{R}^n, \tag{$\mathcal{P}$}$$

where $1<p<n$ with $n\geq 2$, $\beta>0$, $0<\mu<n$, $0<2\beta+\mu<n$, $\mathcal{L}_{m,V}(u)=-\Delta_m u+V(x)|u|^{m-2}u$ and $\Delta_m u=\operatorname{div}(|\nabla u|^{m-2}\nabla u)$ denotes the usual m-Laplacian for $m\in\{p,n\}$. The nonlinearity $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ has critical exponential growth at infinity, that is, it behaves like $\exp(\alpha|s|^{\frac{n}{n-1}})$ when $|s|\to+\infty$ for some $\alpha>0$, which means that there exists $\alpha_0>0$ such that

$$(f_0)\quad \lim_{|s|\to+\infty}|f(x,s)|\exp(-\alpha|s|^{\frac{n}{n-1}})=\begin{cases}0 & \text{if } \alpha>\alpha_0,\\ +\infty & \text{if } \alpha<\alpha_0,\end{cases}\quad \text{uniformly with respect to } x\in\mathbb{R}^n.$$

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For the scalar potential $V : \mathbb{R}^n \to \mathbb{R}$, we suppose that the following hypotheses are satisfied:

(v_1) $V \in C(\mathbb{R}^n, \mathbb{R})$ and there exists a constant $V_0 > 0$ such that $\inf_{x\in\mathbb{R}^n} V(x) \geq V_0$;
(v_2) $V(x) \to +\infty$ as $|x| \to +\infty$, or more generally, for any $M > 0$, $\mu(\{x \in \mathbb{R}^n : V(x) \leq M\}) < +\infty$, where for any $A \subset \mathbb{R}^n$, $\mu(A)$ denotes the Lebesgue measure of A in $\mathbb{R}^n$.

Next, we assume the following hypotheses on the nonlinearity $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$:

(f_1) $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(\cdot, s) = 0$ for all $s \leq 0$ and $f(\cdot, s) > 0$ for all $s > 0$; further, there holds $f(x, s) = o(|s|^{n-1})$ as $s \to 0^+$ for a.a. $x \in \mathbb{R}^n$;
(f_2) there exists $\theta > n$ such that $0 < \theta F(x, s) = \theta \int_0^s f(x, t)\,\mathrm{d}t \leq 2sf(x, s)$ for a.a. $x \in \mathbb{R}^n$ and for all $s > 0$;
(f_3) there exists $\xi > n$ and $\eta > 0$ such that $F(x, s) \geq \eta s^{\xi}$ for a.a. $x \in \mathbb{R}^n$ and $s > 0$;
(f_4) the maps $s \mapsto \frac{f(x,s)}{s^{\frac{n}{2}-1}}$ and $s \mapsto \frac{F(x,s)}{s^{\frac{n}{2}}}$ are strictly increasing for a.a. $x \in \mathbb{R}^n$ and for all $s > 0$.

Due to (f_0) and (f_1), for any $q \geq n$ and $\alpha > \alpha_0$, there exist $\varepsilon > 0$ and a constant $D_\varepsilon = D_\varepsilon(\varepsilon, \alpha, q) > 0$ depending on ε, α and q such that

$$|f(x, s)| \leq \varepsilon|s|^{n-1} + D_\varepsilon|s|^{q-1}\Phi(\alpha|s|^{n'}) \quad \text{for a.a. } x \in \mathbb{R}^n \text{ and for all } s \in \mathbb{R}, \tag{1.1}$$

where $\Phi(t) = \exp(t) - \sum_{j=0}^{n-2} \frac{t^j}{j!}$ and $n' = \frac{n}{n-1}$. Thus, one has

$$\max\{|sf(x, s)|, |F(x, s)|\} \leq \varepsilon|s|^n + D_\varepsilon|s|^q\Phi(\alpha|s|^{n'}) \quad \text{for a.a. } x \in \mathbb{R}^n \text{ and for all } s \in \mathbb{R}. \tag{1.2}$$

Note that the function $F(x, s) = \eta(s^+)^\xi \exp(\alpha_0(s^+)^{n'})$ for a.a. $x \in \mathbb{R}^n$ and for all $s \in \mathbb{R}$, where $n \geq 2$, $\eta > 0$, $\xi > n$, $0 < \alpha_0 < \alpha$, $s^+ = \max\{s, 0\}$ and $f(x, s) = \frac{\partial F(x,s)}{\partial s}$ for a.a. $x \in \mathbb{R}^n$ and for all $s \in \mathbb{R}$ satisfies hypotheses $(f_0) - (f_4)$.

Next, we define the function space $X = W_V^{1,p}(\mathbb{R}^n) \cap W_V^{1,n}(\mathbb{R}^n)$ equipped with the norm $\|\cdot\| = \|\cdot\|_{W_V^{1,p}} + \|\cdot\|_{W_V^{1,n}}$. From [10], it is known that $(X, \|\cdot\|)$ is a reflexive and separable Banach space, where by means of ($v1$) and $m \in \{p, n\}$, the weighted Sobolev space

$$W_V^{1,m}(\mathbb{R}^n) = \left\{u \in W^{1,m}(\mathbb{R}^n) : V(x)|u|^m \in L^1(\mathbb{R}^n)\right\},$$

is endowed with the norm $\|\cdot\|^m_{W_V^{1,m}} = \|\nabla\cdot\|^m_m + \|\cdot\|^m_{m,V}$. As usual, $W^{1,m}(\mathbb{R}^n)$ stands for Sobolev space equipped with the norm $\|\cdot\|^m_{W^{1,m}} = \|\nabla\cdot\|^m_m + \|\cdot\|^m_m$, while $\|\cdot\|_m$ is the usual norm of $L^m(\mathbb{R}^n)$ and $\|\cdot\|_{m,V}$ is the norm of the weighted Lebesgue space $L^m_V(\mathbb{R}^n)$, see [10] for its definition.

We say that $u \in X$ is a weak solution of problem ($\mathcal{P}$), if there holds

$$\left\langle u, \psi\right\rangle_{p,V} + \left\langle u, \psi\right\rangle_{n,V} = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{F(y, u)}{|x-y|^\mu|y|^\beta}\,\mathrm{d}y\right)\frac{f(x, u)\psi}{|x|^\beta}\,\mathrm{d}x \quad \text{for all } \psi \in X,$$

where $\left\langle\cdot,\cdot\right\rangle_{m,V}$ for $m \in \{p, n\}$ is defined by

$$\left\langle u, \psi\right\rangle_{m,V} = \int_{\mathbb{R}^n} |\nabla u|^{m-2}\nabla u \cdot \nabla\psi\,\mathrm{d}x + \int_{\mathbb{R}^n} V(x)|u|^{m-2}u\psi\,\mathrm{d}x \quad \text{for } u, \psi \in X.$$

In addition, we say that a solution $u_0 \in X$ is a ground state solution of problem ($\mathcal{P}$), if there holds

$$J(u_0) = \inf\{J(u) : u \in X \setminus \{0\} \text{ and } J'(u) = 0\},$$

where $J \in C^1(X, \mathbb{R})$ is the associated functional to problem ($\mathcal{P}$).

Our main result is given by the next theorem.

Theorem 1.1. *Let hypotheses (v_1)–(v_2) and (f_0)–(f_4) be satisfied and suppose there exists $\eta_0 > 0$ large enough such that (f_3) holds for all $\eta \geq \eta_0$. Then problem ($\mathcal{P}$) has a positive ground state solution.*

The paper is organized as follows. In Section 2, we provide some preliminary results, while Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary results

In this section, we introduce some elementary results which will be useful in the sequel.

Lemma 2.1 ([10, Corollary 2.6]). *Let (v_1) and (v_2) be satisfied. Then the embedding $X \hookrightarrow L^\tau(\mathbb{R}^n)$ is compact for any $\tau \in [p, p^*) \cup [n, +\infty)$, where $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent.*

Now, we recall the celebrated Moser–Trudinger inequality, which was initially established in [14, Lemma 1], see also [15–17] and the references therein.

Theorem 2.2 (*Moser–Trudinger Inequality*). *For all $n \geq 2$, $\alpha > 0$ and $u \in W^{1,n}(\mathbb{R}^n)$, there holds*

$$\int_{\mathbb{R}^n} \left(\exp(\alpha|u|^{n'}) - \sum_{j=0}^{n-2} \frac{\alpha^j}{j!} |u|^{n'j} \right) \mathrm{d}x < +\infty.$$

Moreover, if $\|\nabla u\|_n^n \leq 1$, $\|u\|_n \leq M < +\infty$ and $\alpha < \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$, then there exists a constant $C = C(n, M, \alpha) > 0$ which depends only on n, M and α such that

$$\int_{\mathbb{R}^n} \left(\exp(\alpha|u|^{n'}) - \sum_{j=0}^{n-2} \frac{\alpha^j}{j!} |u|^{n'j} \right) \mathrm{d}x \leq C(n, M, \alpha).$$

Lemma 2.3 (*[18, Lemma 2.1 and Lemma 2.2]*). *For any $n \geq 2$, the map $s \mapsto \exp(s) - \sum_{j=0}^{n-2} \frac{s^j}{j!}$ is increasing and convex on $[0, +\infty)$. Moreover, for all $p \geq 1$ and $s \geq 0$, there holds*

$$\left(\exp(s) - \sum_{j=0}^{n-2} \frac{s^j}{j!} \right)^p \leq \exp(ps) - \sum_{j=0}^{n-2} \frac{(ps)^j}{j!}.$$

The following doubly weighted Hardy–Littlewood–Sobolev inequality can be found in [19].

Theorem 2.4 (*Doubly Weighted Hardy–Littlewood–Sobolev Inequality*). *Let $1 < r, s < +\infty$, $0 < \mu < n$, $\alpha + \beta \geq 0$ and $0 < \alpha + \beta + \mu \leq n$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\alpha+\beta+\mu}{n} = 2$ and $1 - \frac{1}{r} - \frac{\mu}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r}$, then there exists a sharp constant $C = C(r, s, n, \alpha, \beta, \mu) > 0$ independent of $g \in L^r(\mathbb{R}^n)$ and $h \in L^s(\mathbb{R}^n)$ such that*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{g(x)h(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} \,\mathrm{d}x\,\mathrm{d}y \right| \leq C(r, s, n, \alpha, \beta, \mu) \|g\|_r \|h\|_s.$$

Further, let

$$T(h(x)) = \int_{\mathbb{R}^n} \frac{h(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} \,\mathrm{d}y.$$

Then there exists a constant $\tilde{C} = \tilde{C}(t, s, n, \alpha, \beta, \mu) > 0$ independent of $h \in L^s(\mathbb{R}^n)$ with $1 + \frac{1}{t} = \frac{1}{s} + \frac{\alpha+\beta+\mu}{n}$ and $\frac{\alpha}{n} < \frac{1}{t} < \frac{\alpha+\mu}{n}$ such that $\|T(h)\|_t \leq \tilde{C}(t, s, n, \alpha, \beta, \mu)\|h\|_s$.

Next, we define the energy functional $J : X \to \mathbb{R}$ associated with problem $(\mathcal{P})$ by

$$J(u) = \frac{1}{p}\|u\|^p_{W^{1,p}_V} + \frac{1}{n}\|u\|^n_{W^{1,n}_V} - \frac{1}{2}\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{F(y,u)}{|x-y|^\mu |y|^\beta} \,\mathrm{d}y \right) \frac{F(x,u)}{|x|^\beta} \,\mathrm{d}x \quad \text{for all } u \in X.$$

In virtue of (1.2), Lemma 2.1, Theorem 2.2, Lemma 2.3 and Hölder's inequality, we have $F(x,u) \in L^\zeta(\mathbb{R}^n)$ for all $u \in X$ and $\zeta \geq 1$. Thus, by employing Theorem 2.4 with $r = s = \frac{2n}{2n-2\beta-\mu}$ and $\alpha = \beta$, one has

$$|I(u)| \leq C(\beta, n, \mu)\|F(\cdot,u)\|_r^2, \quad \text{where} \quad I(u) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{F(y,u)}{|x-y|^\mu |y|^\beta} \,\mathrm{d}y \right) \frac{F(x,u)}{|x|^\beta} \,\mathrm{d}x. \tag{2.1}$$

By using the above inequality, it is easy to see that J is well defined and of class $C^1(X, \mathbb{R})$ with

$$\langle J'(u), \psi \rangle = \langle u, \psi \rangle_{p,V} + \langle u, \psi \rangle_{n,V} - \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{F(y,u)}{|x-y|^\mu |y|^\beta} \,\mathrm{d}y \right) \frac{f(x,u)\psi}{|x|^\beta} \,\mathrm{d}x \quad \text{for all } u, \psi \in X, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the duality pair between the dual X^* and X. It is standard to see that the critical points of J are exactly the weak solutions to problem $(\mathcal{P})$.

3. Existence of ground state solutions: Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. First, we need to check that J satisfies the geometrical hypotheses of the mountain pass theorem.

Lemma 3.1. *Let hypotheses (v_1)–(v_2) and (f_0)–(f_2) be satisfied. Then the following hold:*

(a) Any nontrivial critical point of J is nonnegative;
(b) There exist $\delta, \rho > 0$ such that $J(u) \geq \delta$ for all $u \in X$ and $\|u\| = \rho$;
(c) There exists $e \in X$ with $\|e\| > \rho$ such that $J(e) < 0$.

Proof. Let $u \in X \setminus \{0\}$ be a critical point of J. Denote $u = u^+ - u^-$, where $u^\pm = \max\{\pm u, 0\}$. Now, testing (2.2) by $u^- \in X$, we obtain from $\langle J'(u), u^- \rangle = 0$ and (f_1) that

$$\|\nabla u^-\|^p_{W^{1,p}_V} + \|\nabla u^-\|^n_{W^{1,n}_V} = 0, \text{ that is, } \|\nabla u^-\|_{W^{1,p}_V} = \|\nabla u^-\|_{W^{1,n}_V} = 0.$$

It follows that $\|u^-\| = 0$. Thus, we have $u^- = 0$ a.e. in $\mathbb{R}^n$ and $u = u^+ \geq 0$ a.e. in $\mathbb{R}^n$. This completes the proof of (a).

Let $\vartheta > 0$ be such that $2\alpha r \vartheta^{n'} < \alpha_n$ for all $u \in X$ satisfying $\|u\| \in (0, \vartheta]$. By direct calculations, one has

$$\frac{u}{\|u\|} \in W^{1,n}(\mathbb{R}^n), \quad \left\| \nabla \left(\frac{u}{\|u\|} \right) \right\|_n^n \leq 1 \quad \text{and} \quad \left\| \frac{u}{\|u\|} \right\|_n \leq \frac{1}{\sqrt[n]{V_0}} < +\infty. \tag{3.1}$$

It follows from (1.2), (2.1), (3.1), Lemma 2.1, Theorem 2.2, Lemma 2.3 and Hölder's inequality that

$$\begin{aligned}|I(u)| &\leq C\left(\varepsilon^2 \|u\|_{nr}^{2n} + D_\varepsilon^2 \|u\|_{2qr}^{2q} \left(\int_{\mathbb{R}^n} \Phi(2\alpha r |u|^{n'})\, \mathrm{d}x \right)^{\frac{1}{r}} \right) \leq C\left(\varepsilon^2 \|u\|^{2n} + D_\varepsilon^2 \|u\|^{2q} \left(\int_{\mathbb{R}^n} \Phi(2\alpha r \|u\|^{n'} |(u/\|u\|)|^{n'})\, \mathrm{d}x \right)^{\frac{1}{r}} \right) \\ &\leq C(\varepsilon^2 \|u\|^{2n} + D_\varepsilon^2 \|u\|^{2q}),\end{aligned}$$

where $C > 0$ is a suitable constant varies from step to step. Define $\rho = \min\{1, \vartheta\}$. Then, by taking $\|u\| = \rho$ for all $u \in X$, we have $J(u) \geq \|u\|^n \big(\frac{1}{2^{n-1}n} - C\big(\varepsilon^2 \|u\|^n + D_\varepsilon^2 \|u\|^{2q-n}\big)\big)$. Choosing $0 < \varepsilon < \big(\sqrt{2^{n-1}nC}\big)^{-1}$ and using $q \geq n$, we are able to find $\rho > 0$ small enough such that $\frac{1}{2^{n-1}n} - C\big(\varepsilon^2 \rho^n + D_\varepsilon^2 \rho^{2q-n}\big) > 0$. Thus, we have $J(u) \geq \rho^n \big(\frac{1}{2^{n-1}n} - C\big(\varepsilon^2 \rho^n + D_\varepsilon^2 \rho^{2q-n}\big)\big) := \delta > 0$ for $\|u\| = \rho$. This shows (b).

Next, fix $u_0 \in X \setminus \{0\}$ with $u_0 \geq 0$ and define $\Pi : (0, +\infty) \to \mathbb{R}$ by $\Pi(t) = \frac{1}{2} I\big(\frac{t u_0}{\|u_0\|}\big)$ for all $t > 0$. By using (f_2), one has $\Pi'(t) \geq \frac{\theta}{t} \Pi(t)$ for all $t > 0$. Integrating it on $[1, s_0\|u_0\|]$ with $s > \frac{1}{\|u_0\|}$, we deduce that $I(su_0) \geq s^\theta \|u_0\|^\theta I\big(\frac{u_0}{\|u_0\|}\big)$ and

$$J(su_0) \leq \frac{s^p}{p} \|u_0\|_{W_V^{1,p}}^p + \frac{s^n}{n} \|u_0\|_{W_V^{1,n}}^n - \frac{1}{2} s^\theta \|u_0\|^\theta I\left(\frac{u_0}{\|u_0\|} \right) \to -\infty \quad \text{as } s \to \infty,$$

where we have used that $p < n < \theta$. Taking $s > \frac{1}{\|u_0\|}$ large enough and $e = su_0$, we get $J(e) < 0$ and $\|e\| > \rho$. This completes the proof of (c) and consequently of Lemma 3.1. □

By employing Lemma 3.1 and the mountain pass theorem [20] without the Palais–Smale condition, one can see that there exists a $(\mathrm{PS})_c$-sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$, that is, $J(u_k) \to c$ in $\mathbb{R}$ and $J'(u_k) \to 0$ in X^* as $k \to \infty$, where c is the mountain pass level given by

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \geq \delta > 0 \quad \text{with} \quad \Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Next, we compute the following key estimate for the mountain pass minimax level c.

Lemma 3.2. *There exists $\eta_0 > 0$ large enough such that if (f_3) is satisfied for all $\eta \geq \eta_0$, then there holds*

$$0 < c < c_0 := \min\left\{ \frac{1}{2^n} \left(\frac{\theta - n}{n\theta} \right)^{\frac{n}{p}} \left(\frac{\alpha_n}{\alpha_0} \right)^{\frac{n}{n'}}, \frac{1}{2^p} \left(\frac{\theta - n}{n\theta} \right) \left(\frac{\alpha_n}{\alpha_0} \right)^{\frac{p}{n'}} \right\}. \tag{3.2}$$

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^n, [0,1])$ be a cut-off function such that $\psi(x) \equiv 1$ if $|x| \leq 1$, $\psi(x) \equiv 0$ if $|x| \geq 2$ and $|\nabla \psi(x)| \leq 1$ for all $x \in \mathbb{R}^n$. By direct computations, we obtain

(i) $$\frac{1}{m} \int_{B_2(0)} (|\nabla(s\psi)|^m + V(x)|s\psi|^m)\, \mathrm{d}x \leq \frac{2^n \omega_{n-1}(1 + \|V\|_\infty)}{nm} s^m \quad \text{for } m \in \{p, n\},$$

(ii) $$\int_{B_1(0)} \int_{B_1(0)} \frac{\mathrm{d}x\mathrm{d}y}{|x|^\beta |x-y|^\mu |y|^\beta} \geq \frac{\omega_{n-1}^2 \beta(n, n-\mu+1)}{n - \mu}, \quad \text{where} \quad \beta(z_1, z_2) = \int_0^1 t^{z_1 - 1}(1-t)^{z_2 - 1}\, \mathrm{d}t$$

for all $z_1, z_2 \in \mathbb{C}$ with $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$ is called the Euler integral of the first kind. Thus, for all $s \in [0,1]$, we obtain from (f_3) that

$$J(s\psi) \leq C_1 s^p - C_2 \eta^2 s^{2\xi} \quad \text{with} \quad C_1 = \frac{2^{n+1}\omega_{n-1}(1 + \|V\|_\infty)}{np} \quad \text{and} \quad C_2 = \frac{\omega_{n-1}^2 \beta(n, n-\mu+1)}{2(n-\mu)}.$$

Define the map $g : [0,1] \to X$ by $g(s) = s\psi$ for all $s \in [0,1]$ and choose $\eta_1 > 0$ such that $J(\psi) \leq C_1 - C_2\eta_1^2 < 0$ for all $\eta > \eta_1$. It follows that $g \in \Gamma$. Hence, one has

$$c \leq \max_{s \in [0,1]} J(g(s)) \leq \max_{s \geq 0} J(s\psi) \leq \max_{s \geq 0} \left[C_1 s^p - C_2 \eta^2 s^{2\xi} \right] = \frac{(2\xi - p)C_1}{2\xi} \left(\frac{pC_1}{2\xi \eta^2 C_2} \right)^{\frac{p}{2\xi - p}} \to 0 \quad \text{as } \eta \to \infty.$$

Therefore, we can find $\eta_0 > \eta_1$ large enough such that $c < c_0$ for all $\eta \geq \eta_0$, where c_0 is defined in (3.2). □

Lemma 3.3. *Every $(\mathrm{PS})_c$-sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ for J is bounded in X, where $c \in (0, c_0)$ and c_0 is given in Lemma 3.2. Moreover, there holds*

$$\limsup_{k \to \infty} \|u_k\|^{n'} < \frac{\alpha_n}{\alpha_0}. \tag{3.3}$$

Proof. Let $c \in (0, c_0)$ and $\{u_k\}_{k\in\mathbb{N}} \subset X$ be a $(\mathrm{PS})_c$-sequence for J. Applying (f_2) leads to

$$c + o_k(1) + o_k(1)\|u_k\| \geq \left(\frac{1}{n} - \frac{1}{\theta} \right) \left(\|u_k\|_{W_V^{1,p}}^p + \|u_k\|_{W_V^{1,n}}^n \right) \quad \text{as } k \to \infty. \tag{3.4}$$

Suppose now that $\{u_k\}_{k\in\mathbb{N}}$ is unbounded in X. We consider three cases. Case 1: Assume that $\|u_k\|_{W_V^{1,p}} \to \infty$ and $\|u_k\|_{W_V^{1,n}} \to \infty$ as $k \to \infty$. By using $1 < p < n$, one has $\|u_k\|^n_{W_V^{1,n}} \geq \|u_k\|^p_{W_V^{1,n}} > 1$ for large k. It follows from (3.4) that

$$c + o_k(1) + o_k(1)\|u_k\| \geq 2^{1-p}\left(\frac{1}{n} - \frac{1}{\theta}\right)\|u_k\|^p \quad \text{as } k \to \infty.$$

Dividing $\|u_k\|^p$ on both sides and letting $k \to \infty$, we have $0 \geq 2^{1-p}\left(\frac{1}{n} - \frac{1}{\theta}\right) > 0$, which is a contradiction. Case 2: Suppose that $\|u_k\|_{W_V^{1,p}} \to \infty$ as $k \to \infty$ and $\|u_k\|_{W_V^{1,n}}$ is bounded. Thus, we deduce from (3.4) that

$$c + o_k(1) + o_k(1)\|u_k\| \geq \left(\frac{1}{n} - \frac{1}{\theta}\right)\|u_k\|^p_{W_V^{1,p}} \quad \text{as } k \to \infty.$$

Dividing $\|u_k\|^p_{W_V^{1,p}}$ on both sides and sending $k \to \infty$, we have $0 \geq \left(\frac{1}{n} - \frac{1}{\theta}\right) > 0$, which is again a contradiction. Case 3: Suppose that $\|u_k\|_{W_V^{1,n}} \to \infty$ as $k \to \infty$ and $\|u_k\|_{W_V^{1,p}}$ is bounded, then by arguing as in Case 2, we can easily arrive at a contradiction. Thus, $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in X. Further, by using (3.4), we get

$$\limsup_{k\to\infty} \|u_k\|^n_{W_V^{1,n}} \leq \left(\frac{n\theta}{\theta - n}\right)c \quad \text{and} \quad \limsup_{k\to\infty} \|u_k\|^n_{W_V^{1,p}} \leq \left[\left(\frac{n\theta}{\theta - n}\right)c\right]^{\frac{n}{p}}. \tag{3.5}$$

It follows from (3.5) and the inequality $(a+b)^\sigma \leq 2^{\sigma-1}(a^\sigma + b^\sigma)$ for all $a, b \geq 0$ and $\sigma \in [1, +\infty)$ that

$$\limsup_{k\to\infty} \|u_k\|^{n'} \leq 2^{\frac{1}{n-1}}\left(\left[\left(\frac{n\theta}{\theta - n}\right)c\right]^{\frac{1}{n-1}} + \left[\left(\frac{n\theta}{\theta - n}\right)c\right]^{\frac{n}{(n-1)p}}\right).$$

By using the fact that $\frac{n\theta}{\theta-n} > 1$ and $\frac{1}{n-1} < \frac{n}{(n-1)p}$, we obtain from the previous inequality that

$$\limsup_{k\to\infty} \|u_k\|^{n'} \leq \left[2\left(\frac{n\theta}{\theta - n}\right)^{\frac{n}{p}}\right]^{\frac{1}{n-1}}\left(c^{\frac{1}{n-1}} + c^{\frac{n}{(n-1)p}}\right) \leq \begin{cases} \left[2^n\left(\frac{n\theta}{\theta-n}\right)^{\frac{n}{p}} c\right]^{\frac{1}{n-1}} & \text{if } c \leq 1, \\ \left[2^n\left(\frac{n\theta c}{\theta-n}\right)^{\frac{n}{p}}\right]^{\frac{1}{n-1}} & \text{if } c \geq 1. \end{cases} \tag{3.6}$$

Due to (3.2) and (3.6), one can easily obtain (3.3). This finishes the proof of the lemma. □

Lemma 3.4. *Let hypotheses (v_1)–(v_2) and (f_0)–(f_3) be satisfied. Then J satisfies the $(\mathrm{PS})_c$ compactness condition for all $c \in (0, c_0)$, where c_0 is defined in Lemma 3.2.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset X$ be a $(\mathrm{PS})_c$-sequence for J. Then, by employing Lemma 3.3, one sees that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X and (3.3) is satisfied. Hence, up to a subsequence not relabeled, there exists $u \in X$ such that $u_k \rightharpoonup u$ in X. Thus, from Lemma 2.1, we obtain $u_k \to u$ in $L^\tau(\mathbb{R}^n)$ for all $\tau \in [p, p^*) \cup [n, +\infty)$ and $u_k \to u$ a.e. in $\mathbb{R}^n$. In addition, as a consequence of (1.2), (3.3), Lemma 2.1, Theorem 2.2, Lemma 2.3 and Hölder's inequality, one can deduce that $\{F(\cdot, u_k)\}_{k\in\mathbb{N}}$ is bounded in $L^r(\mathbb{R}^n)$. Using the continuity of the map $s \mapsto F(\cdot, s)$, we obtain $F(x, u_k) \to F(x, u)$ a.e. in $\mathbb{R}^n$ as $k \to \infty$. It immediately follows that $F(x, u_k) \rightharpoonup F(x, u)$ in $L^r(\mathbb{R}^n)$ as $k \to \infty$. Set $t = \frac{2n}{2\beta+\mu}$, then by Theorem 2.4, the map

$$L^r(\mathbb{R}^n) \ni h(x) \mapsto \int_{\mathbb{R}^n} \frac{h(y)}{|x|^\beta |x-y|^\mu |y|^\beta}\,\mathrm{d}y \in L^t(\mathbb{R}^n)$$

is a linear and bounded operator. As a result, we obtain

$$\int_{\mathbb{R}^n} \frac{F(y, u_k)}{|x|^\beta |x-y|^\mu |y|^\beta}\,\mathrm{d}y \rightharpoonup \int_{\mathbb{R}^n} \frac{F(y, u)}{|x|^\beta |x-y|^\mu |y|^\beta}\,\mathrm{d}y \quad \text{in } L^t(\mathbb{R}^n) \quad \text{as } k \to \infty.$$

Therefore, the sequence $\left\{\int_{\mathbb{R}^n} \frac{F(y,u_k)}{|x|^\beta|x-y|^\mu|y|^\beta}\,\mathrm{d}y\right\}_{k\in\mathbb{N}}$ is bounded in $L^t(\mathbb{R}^n)$. Note that (3.1) holds replacing u by u_k. Since (3.3) holds and passing to a subsequence if necessary (not relabeled), we can assume that $\sup_{k\in\mathbb{N}} \|u_k\|^{n'} < \frac{\alpha_n}{\alpha_0}$. Let us fix $\bar{m} \in (\|u_k\|^{n'}, \frac{\alpha_n}{\alpha_0})$ and $\alpha > \alpha_0$ close to α_0 in such a way that $2\alpha q r \bar{m} < \alpha_n$. Now, by using (1.1), Lemma 2.1, Theorem 2.2, Lemma 2.3, Hölder's inequality and the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in X, we have

$$\begin{aligned} |\langle I'(u_k), u_k - u\rangle| &\leq C\left[\left(\int_{\mathbb{R}^n} |u_k|^{(n-1)r}|u_k - u|^r\,\mathrm{d}x\right)^{\frac{1}{r}} + \left(\int_{\mathbb{R}^n} |u_k|^{(q-1)r}|u_k - u|^r \Phi(\alpha r|u_k|^{n'})\,\mathrm{d}x\right)^{\frac{1}{r}}\right] \\ &\leq C\left[\|u_k\|^{n-1}_{nr}\|u_k - u\|_{nr} + \|u_k\|^{q-1}_{qr}\|u_k - u\|_{2qr}\left(\int_{\mathbb{R}^n} \Phi(2\alpha q r\|u_k\|^{n'}|(u_k/\|u_k\|)|^{n'})\,\mathrm{d}x\right)^{\frac{1}{2qr}}\right] = o_k(1) \end{aligned}$$

as $k \to \infty$, where $C > 0$ is a suitable positive constant. It follows that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{F(y, u_k)}{|x-y|^\mu |y|^\beta}\,\mathrm{d}y\right)\frac{f(x, u_k)(u_k - u)}{|x|^\beta}\,\mathrm{d}x = 0, \tag{3.7}$$

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{F(y,u)}{|x-y|^{\mu}|y|^{\beta}}\,\mathrm{d}y\right)\frac{f(x,u)(u_k-u)}{|x|^{\beta}}\,\mathrm{d}x=0. \tag{3.8}$$

Recall that $\langle J'(u_k)-J'(u),u_k-u\rangle=o_k(1)$ as $k\to\infty$. Hence, for $m\in\{p,n\}$, we obtain by using (3.7), (3.8), the convexity of the map $t\mapsto\frac{1}{m}|t|^m$ and (v_1) that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\left(|\nabla u_k|^{m-2}\nabla u_k-|\nabla u|^{m-2}\nabla u\right)\cdot(\nabla u_k-\nabla u)\,\mathrm{d}x=0, \tag{3.9}$$

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}V(x)\left(|u_k|^{m-2}u_k-|u|^{m-2}u\right)(u_k-u)\,\mathrm{d}x=0. \tag{3.10}$$

From [21], for all $w,z\in\mathbb{R}^d$ with $d\geq 1$, there exist two positive constants C_σ and c_σ depending only on σ such that

$$|w-z|^{\sigma}\leq\begin{cases}C_\sigma\left[\left(|w|^{\sigma-2}w-|z|^{\sigma-2}z\right)(w-z)\right]^{\frac{\sigma}{2}}\left[|w|^{\sigma}+|z|^{\sigma}\right]^{\frac{2-\sigma}{2}} & \text{if } 1<\sigma<2,\\ c_\sigma\left(|w|^{\sigma-2}w-|z|^{\sigma-2}z\right)(w-z) & \text{if } \sigma\geq 2.\end{cases} \tag{3.11}$$

Due to (3.9), (3.10), (3.11) and the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in X, we obtain for $m\geq 2$ that

$$\|\nabla u_k-\nabla u\|_m^m\leq c_m\int_{\mathbb{R}^n}\left(|\nabla u_k|^{m-2}\nabla u_k-|\nabla u|^{m-2}\nabla u\right)\cdot(\nabla u_k-\nabla u)\,\mathrm{d}x=o_k(1)\quad\text{as } k\to\infty,$$

$$\|u_k-u\|_{m,V}^m\leq c_m\int_{\mathbb{R}^n}V(x)\left(|u_k|^{m-2}u_k-|u|^{m-2}u\right)(u_k-u)\,\mathrm{d}x=o_k(1)\quad\text{as } k\to\infty.$$

Because of (3.9), (3.10), (3.11), Hölder's inequality and the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in X, we get for $1<m<2$ that

$$\begin{aligned}\|\nabla u_k-\nabla u\|_m^m&\leq C_m\int_{\mathbb{R}^n}\left[\left(|\nabla u_k|^{m-2}\nabla u_k-|\nabla u|^{m-2}\nabla u\right).(\nabla u_k-\nabla u)\right]^{\frac{m}{2}}\left[|\nabla u_k|^m+|\nabla u|^m\right]^{\frac{2-m}{2}}\mathrm{d}x\\&\leq C_m\int_{\mathbb{R}^n}\left[\left(|\nabla u_k|^{m-2}\nabla u_k-|\nabla u|^{m-2}\nabla u\right).(\nabla u_k-\nabla u)\right]^{\frac{m}{2}}\left[|\nabla u_k|^{\frac{(2-m)m}{2}}+|\nabla u|^{\frac{(2-m)m}{2}}\right]\mathrm{d}x\\&\leq C_m\left(\int_{\mathbb{R}^n}\left(|\nabla u_k|^{m-2}\nabla u_k-|\nabla u|^{m-2}\nabla u\right).(\nabla u_k-\nabla u)\,\mathrm{d}x\right)^{\frac{m}{2}}\left[\|\nabla u_k\|_m^{\frac{(2-m)m}{2}}+\|\nabla u\|_m^{\frac{(2-m)m}{2}}\right]=o_k(1)\end{aligned}$$

as $k\to\infty$ and

$$\begin{aligned}\|u_k-u\|_{m,V}^m&\leq C_m\int_{\mathbb{R}^n}V(x)\left[\left(|u_k|^{m-2}u_k-|u|^{m-2}u\right)(u_k-u)\right]^{\frac{m}{2}}\left[|u_k|^m+|u|^m\right]^{\frac{2-m}{2}}\mathrm{d}x\\&\leq C_m\int_{\mathbb{R}^n}V(x)\left[\left(|u_k|^{m-2}u_k-|u|^{m-2}u\right)(u_k-u)\right]^{\frac{m}{2}}\left[|u_k|^{\frac{(2-m)m}{2}}+|u|^{\frac{(2-m)m}{2}}\right]\mathrm{d}x\\&\leq C_m\left(\int_{\mathbb{R}^n}V(x)\left(|u_k|^{m-2}u_k-|u|^{m-2}u\right)(u_k-u)\,\mathrm{d}x\right)^{\frac{m}{2}}\left[\|u_k\|_{m,V}^{\frac{(2-m)m}{2}}+\|u\|_{m,V}^{\frac{(2-m)m}{2}}\right]=o_k(1)\quad\text{as}\quad k\to\infty.\end{aligned}$$

Hence, we deduce from the above convergences that $\nabla u_k\to\nabla u$ in $L^m(\mathbb{R}^n)$ as $k\to\infty$ and $u_k\to u$ in $L^m_V(\mathbb{R}^n)$ as $k\to\infty$ for $m\in\{p,n\}$. It follows that $u_k\to u$ in X as $k\to\infty$. This finishes the proof. □

Proof of Theorem 1.1. By using Lemma 3.4 and $J\in C^1(X,\mathbb{R})$, one has $J(u)=c>0$ and $J'(u)=0$. Thus, we obtain from Lemma 3.1 that u is a positive solution of $(\mathcal{P})$. Next, we claim that u is a ground state solution of $(\mathcal{P})$. It suffices to show $c\leq\Theta:=\inf\{J(u):u\in\mathcal{N}\}$, where $\mathcal{N}:=\{u\in X\setminus\{0\}:J'(u)=0\}$. Define the map $\pi:(0,+\infty)\to\mathbb{R}$ by $\pi(s)=J(su)$ for all $u\in\mathcal{N}$ and $s>0$. Note that $\pi'(s)=\frac{1}{s}\langle J'(su),su\rangle-s^{n-1}\langle J'(u),u\rangle$. Moreover, by direct calculations, we have

$$\begin{aligned}\pi'(s)=(s^{p-1}-s^{n-1})\|u\|^p_{W^{1,p}_V}+s^{n-1}\Bigg[&\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left(\frac{F(y,u)}{u^{\frac{n}{2}}}-\frac{F(y,su)}{(su)^{\frac{n}{2}}}\right)\frac{u^{\frac{n}{2}}}{|x-y|^{\mu}|y|^{\beta}}\,\mathrm{d}y\right\}\frac{f(x,u)}{u^{\frac{n}{2}-1}}\frac{u^{\frac{n}{2}}}{|x|^{\beta}}\,\mathrm{d}x\\&+\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{F(y,su)}{(su)^{\frac{n}{2}}}\frac{u^{\frac{n}{2}}}{|x-y|^{\mu}|y|^{\beta}}\,\mathrm{d}y\right)\left\{\frac{f(x,u)}{u^{\frac{n}{2}-1}}-\frac{f(x,su)}{(su)^{\frac{n}{2}-1}}\right\}\frac{u^{\frac{n}{2}}}{|x|^{\beta}}\,\mathrm{d}x\Bigg].\end{aligned}$$

Due to (f_4), one sees that $\pi'(s)>0$ for all $s\in(0,1)$ and $\pi'(s)<0$ for all $s\in(1,+\infty)$. It follows that 1 is the maximum point of π and thus $J(u)=\max_{s\geq 0}J(su)$. Further, let $g:[0,1]\to X$ be such that $g(s)=ss_0u$, where s_0 fulfills $J(s_0u)<0$. Hence, $g\in\Gamma$ and $c\leq\max_{s\in[0,1]}J(g(s))\leq\max_{s\geq 0}J(su)=J(u)$. Since $u\in\mathcal{N}$ is arbitrary, we have $c\leq\Theta$. □

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No data was used for the research described in the article.

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